Ref.TH.2255-CERN

GAUGE THEORIES AND MAGNETIC CHARGE

P. Goddard

DAMTP - Cambridge University

J. Nuyts

Department of Physics - University of Mons

and

D. Olive

CERN - Geneva
and
Niels Bohr Institute, Copenhagen

A B S T R A C T

If the magnetic field for an exact gauge group H (assumed compact and connected) exhibits an inverse square law behaviour at large distances then the generalized magnetic charge, appearing as the coefficient, completely determines the topological quantum number of the solution. When this magnetic charge operator is expressed as a linear combination of mutually commuting generators of H, the components are uniquely determined, up to the action of the Weyl group, and have to be weights of a new group H^v which is explicitly constructed out of H. The relation between the "electric" group H and the "magnetic" group H^v is symmetrical in the sense that $(H^v)^v = H$. The results suggest that H monopoles are H^v multiplets and vice versa and that the true symmetry group is $H \otimes H^v$. In this duality topological and Noether quantum numbers exchange rôles rather as in Sine-Gordon theory. A physical possibility is that H and H^v be the colour and weak electromagnetic gauge groups.

Ref.TH.2255-CERN
21 December 1976

1. - INTRODUCTION

Dirac's magnetic monopole [1] was associated with the U(1) electromagnetic gauge group. Recent developments [2],[3] have made it easier to consider monopoles associated with any exact, compact gauge symmetry group H rather than just U(1). There is some reason to think that the less familiar case that H be non-Abelian can be physically interesting as a means of relating strong interactions to unified theories of weak and electromagnetic interactions [4]. It appears that finite energy, singularity free models for monopoles can be found by embedding H in a larger gauge symmetry group G which is broken back to H by the vacuum. These states resemble the solutions of Sine-Gordon theory [5] and may inherit some of the interesting properties and provide viable candidates as hadrons or quarks (if H is non-Abelian).

In order to investigate this possibility we need a much better understanding of monopole theory and the aim of this paper is so directed. We shall take as our definition of magnetic monopole, a solution in which the magnetic (space-space) components of the gauge field tensor take the form

$$G_{ij} = \frac{\epsilon_{ijk} r_k}{r^3} G(\underline{r}) \qquad i,j = 1,2,3 \tag{1.1}$$

at large spatial distances from the central region of the monopole in a suitable Lorentz frame. Thus the magnetic field is asymptotically radial, and obeys a generalized inverse square law, since the magnetic charge $G(\underline{r})$ is assumed, in addition, to be covariantly constant

$$\mathcal{D}_i G(\underline{r}) \equiv \partial_i G(\underline{r}) - ie\left[W_i(\underline{r}), G(\underline{r})\right] = 0 \tag{1.2}$$

where the adjoint representation indices of W^μ_α, $G^{\mu\nu}_\alpha$, the gauge potential and gauge fields, are understood to be contracted with the generators of G. Equations (1.1) and (1.2) are gauge covariant and certainly consistent with the equations of motion and may well be implied by them for finite energy solutions within the 't Hooft-Polyakov framework [2],[3] but we do not have a satisfactory proof of this [6].

Our purpose will be to analyze the structure of $G(\underline{r})$ as much as possible for general, compact, connected H. The first result in Section 2 is that in the 't Hooft-Polyakov framework [2),3)] $G(\underline{r})$ must satisfy the "quantization condition"

$$\exp(4\pi i e G(\underline{r})) = 1 \tag{1.3}$$

Further the "topological quantum number" [3)] associated with the solution is shown to be specified by the following path, closed by virtue of (1.3), and within H since $G(\underline{r})$ is actually a generator of H :

$$J = \{ \exp(ie\Omega G(\underline{r})) ; \ 0 \leq \Omega \leq 4\pi \} \tag{1.4}$$

For example if $H = U(1)$, $eG \rightarrow gQ$ and (1.3) is the conventional Dirac quantization condition.

The first lesson then is that the magnetic charge, $G(\underline{r})$, contains all the information about the topological quantum number. However, it contains yet more gauge invariant structure which will interest us as of being of possible physical significance. In Section 3 we show that $G(\underline{r})$ can always be gauge transformed into the form

$$e\,G \rightarrow e\,G' = \sum_{i=1}^{r} \beta_i T_i \tag{1.5}$$

where the T_i are any appropriately normalized set of mutually commuting generators of H. The β_i are a set of r (= rank of H) numbers which we shall call the "magnetic weights". They are not quite uniquely determined by $G(\underline{r})$: there is a finite discrete degeneracy, characterized, we shall show, by the action of the "Weyl group" of H. These equivalence classes of values of β (under the Weyl group) are gauge invariant and hence the most economical way of characterizing the solution. Presumably they are also measurable, and hence of physical significance.

The "quantization condition" (1.3) will indeed quantize the β's. Under the assumption that H is semi-simple (as well as compact and connected) this is carried out in Sections 4 and 5, following an idea of Englert and Windey [7)]. The result is that, up to a normalization, the possible values of the $\beta_1 \dots \beta_r$ are the weights of a new group H^v, explicitly constructed out of H. [A weight of H is a possible set of eigenvalues of $(T_1 \dots T_r)$ in a single valued representation of H]. We may think of H and H^v as being "electric" and "magnetic" groups, respectively. The relation is symmetric since

$$(H^v)^v = H \tag{1.6}$$

In this work we have to use mathematical ideas which are perhaps unfamiliar to physicists but we have been impressed (and guided) by the way mathematical structures acquire a physical meaning.

In Section 6 the above result is extended by relaxing the condition that H be semi-simple, thereby incorporating the previous results of Corrigan and Olive who considered H locally like $U(1)_{EM} \otimes K_{colour}$ [8].

Section 7 illustrates the relation between H and H^v.

Section 8 discusses the results. One interpretation is that the H monopoles be H^v multiplets (and vice versa). The further work needed to establish this is discussed. If this is correct, H could be the weak and electromagnetic gauge group and H^v the colour group, with the H monopoles the H^v quarks. This situation in which there may be two equivalent formulations of the theory based alternatively on H or H^v with Noether and topological quantum numbers exchanging rôles could be the natural generalization of the Sine-Gordon Thirring model duality in two space-time dimensions.

2. - TOPOLOGICAL QUANTUM NUMBERS AND THE QUANTIZATION OF MAGNETIC CHARGE

Suppose $\psi(x)$ is a Higgs' field transforming under a representation D of the "big" gauge group G, assumed simply connected, connected and compact. Far away from the centre of a finite energy solution (taken to be at the origin) ψ must lie in M_o the set which minimizes the self-interaction of ψ. Suppose G acts transitively on M_o. Then $M_o = \{D(g)\psi_o; g \in G\}$ for any fixed $\psi_o \in M_o$. Let H_ψ be the little group of $\psi \in M_0$; $H_\psi = \{g \in G : D(g)\psi = \psi\}$. Then we may identify M_0 with the coset space G/H_ψ. We shall assume H_ψ is connected ; it is necessarily compact but not in general simply connected. Also, at large distances,

$$\mathcal{D}^\mu \psi = \partial^\mu \psi + ie D(W^\mu)\psi = 0 \tag{2.1}$$

where $W^\mu = W^\mu_a L^a$, $\{L^a\}$ being the Lie algebra of G and W^μ_a the gauge potentials. By definition, if $G^{\mu\nu} = L^a G^{\mu\nu}_a$ where $G^{\mu\nu}_a$ is the gauge field tensor,

$$[\mathcal{D}^\mu, \mathcal{D}^\nu] = ie D(G^{\mu\nu}) \tag{2.2}$$

At large distances, by (2.1) $G^{\mu\nu}$ is a generator of H which is therefore the observed exact gauge group.

Now according to the theory of "topological quantum numbers" [3) the asymptotic behaviour of the Higgs fields constitutes a map from the sphere at infinity S_2 into $M_0 = G/H_\psi$, which defines a homotopy class which is an element of $\pi_2(M_0) = \pi_2(G/H_\psi)$. This is the group of topological quantum numbers [3)]. Since $\pi_1(G) = 0$

$$\pi_2(G/H_\psi) \cong \pi_1(H_\psi) \tag{2.3}$$

We shall now show by explicit construction that if the monopole exhibits the generalized inverse square law (1.1) and (1.2) then the magnetic charge $G(\underline{r})$ defined by those equations, alone determines the appropriate element of $\pi_1(H_\psi)$.

Let us parametrize S_2 with co-ordinates (s,t) by considering a map $\underline{x}(s,t)$ from the unit square in the (s,t) plane onto S_2 such that $\underline{x}(s,t) = P$, a fixed point of S_2, whenever (s,t) lies on the boundary of the unit square (either s or $t = 0$ or 1) and is otherwise one to one.

Define $g(s,t)$ by

$$g(s,0) = 1 \tag{2.4}$$

It is always possible to choose linear combinations of the T_i's so that g_{ij} is the unit matrix, and we shall do so :

$$g_{ij} = \delta_{ij} \tag{3.8}$$

There still remains a freedom to rotate the T_i's and hence the roots.

All this is well known for semi-simple groups and follows from (3.1) in the more general, compact case.

The $T_1, T_2 \ldots T_r$ generate a compact Abelian subgroup of H, called T, which is a "maximal torus". Since H is compact and connected T enjoys an important property [12]; any element of H is conjugate to at least one element of T :

$$\text{if } h \in H, \exists\, S \in H, t \in T \text{ such that } h = S t S^{-1} \tag{3.9}$$

Immediate corollaries which we shall use later on are

a) any generator of H is conjugate to at least one generator of T ;

b) any element of H which commutes with all other elements of H lies in T, i.e.,

$$Z(H) \subseteq T \tag{3.10}$$

where Z(H) denotes the "centre" of H (and is an invariant subgroup).

It follows from a) that we can always find a gauge transformation $S \in H$ such that

$$e\,G(P) = S \sum_{i=1}^{r} \beta_i T_i \; S^{-1} \tag{3.11}$$

We shall call these coefficients $\beta_1 \ldots \beta_r$ the magnetic weights of the monopole. In terms of them the quantization condition (2.12) reads

$$\exp(4\pi i \sum_{i=1}^{r} \beta_i T_i) = 1 \qquad (3.12)$$

while the element of $\pi_1(H)$ specifying the topological quantum number is

$$J_0 = \{ \exp i\Omega \sum_{1}^{r} \beta_i T_i \;;\; 0 \leq \Omega \leq 2\pi \} \qquad (3.13)$$

since gauge transformed paths are homotopic if H is connected by Eq. (2.13).

Are these magnetic weights in (3.11) uniquely determined ? The answer is clearly <u>no</u> because if α is a root the gauge transformation S_α

$$S_\alpha = \exp[i\pi(E_\alpha + E_{-\alpha})/\sqrt{2\alpha^2}] \in H \qquad (3.14)$$

has the effect

$$S_\alpha (\sum_{1}^{r} \beta_i T_i) S_\alpha^{-1} = \sum_{1}^{r} \beta_i' T_i \qquad (3.15)$$

where

$$\beta' = \sigma_\alpha(\beta) \equiv \beta - 2\alpha \; \alpha\cdot\beta/\alpha^2 \qquad (3.16)$$

In these formulae the scalar products $\alpha.\beta$ denote $\sum_{i=1}^{\ell} \alpha_i \beta_i$ corresponding to our choice (3.8), and so only involve the first ℓ components. The linear transformation (3.16) is called a "Weyl reflection", and consists of a reflection in the hyperplane $\alpha.x = 0$, perpendicular to the root α. It can be realized by a gauge transformation within H by (3.14). The operators

$$\{ (E_\alpha + E_{-\alpha})/\sqrt{2\alpha^2} \,,\, (E_\alpha - E_{-\alpha})/i\sqrt{2\alpha^2} \,,\, \alpha\cdot T/\alpha^2 \} \qquad (3.17)$$

generate an SO(3) subalgebra of $\mathcal{L}(H)$ associated with each root α, and (3.14) is a rotation through π within the corresponding group which reverses $\alpha.T$.

Notice that the Abelian magnetic weights $\beta_{\ell+1} \ldots \beta_r$ are invariant under (3.16) (since the corresponding components of the root vanish). The ambiguity (3.16) is a new phenomenon associated with the non-Abelian components $\beta_1 \ldots \beta_\ell$. Let us illustrate it when $H = SO(3)$. Then $T_1 = t_3$ and the possible roots are ± 1. The corresponding Weyl reflection (3.16) is

$$\beta_1 \to \beta_1' = -\beta_1$$

corresponding to a gauge rotation through π about the two-axis.

So, if β is a possible magnetic weight corresponding to a given magnetic charge $G(P)$, then so is $\sigma_{\alpha_1}(\beta)$, $\sigma_{\alpha_1}(\sigma_{\alpha_2}(\beta))$ and so on. Since the Weyl group so generated from the Weyl reflections, is finite there are a finite number of possibilities for the magnetic weight, β. Are these the only possibilities ? I.e., if $S\beta.T\,S^{-1} = \beta'.T \quad S \in H$, is β' obtained from β only by a finite sequence of Weyl reflections (3.16) ? Then the answer is yes, and is proved in Appendix A. The proof is simple but depends on notions developed in the following sections.

The equivalence class of magnetic weights β, related by the action of the Weyl group and obtained from the magnetic charge G is therefore a gauge invariant entity which contains all the detectable information carried asymptotically by the magnetic field, and therefore, we claim, is the correct object to think about.

Let us mention that this class is independent of $\underline{r}$, for a given $G(\underline{r})$ satisfying (1.2), as well as of the gauge frame of reference and the choice of T.

Finally, let us note that the set of roots, denoted $\Phi(H)$ of the algebra of H, enjoy the following properties [13] :

$$\Phi(H) \text{ is finite, spans } R_\ell \text{ and does not contain } 0 \tag{3.18a}$$

$$\text{if } \alpha \in \Phi(H), \quad n\alpha \in \Phi(H) \text{ only if } n = \pm 1 \tag{3.18b}$$

$$\text{if } \alpha, \beta \in \Phi(H) \quad 2(\alpha\cdot\beta)/\alpha^2 \in \mathbb{Z} \tag{3.18c}$$

$$\text{if } \alpha \in \Phi(H) \quad \sigma_\alpha(\Phi) = \Phi \tag{3.18d}$$

where $\sigma_\alpha(\beta)$ is the Weyl reflection (3.16). Equation (3.18c) is a consequence of the two facts that $\alpha T/\alpha^2$ is a component of angular momentum (3.17) and so always has eigenvalues which are integers or half-integers, and that $\alpha.\beta/\alpha^2$ is such an eigenvalue. Equation (3.18d) follows from the fact that $S_\alpha^{-1}\ E_\beta\ S_\alpha$ is the step operator for the eigenvalue $\sigma_\alpha(\beta)$ [see (3.15)].

Let us call a finite set of points simple if it cannot be split into two mutually orthogonal subspaces. In Appendix B we shall show that a simple set of points Φ satisfying (3.18a) and (3.18d) must satisfy

$$\sum_{\alpha\in\Phi} \alpha_i \alpha_j = \mu^2 \delta_{ij} \tag{3.19}$$

Then by rescaling $\alpha \to \mu\alpha$ (3.18a) and (3.18d) remain true while (3.8) becomes valid. If the set is not simple $\sum_{\alpha\in\Phi} \alpha_i\alpha_j$ is a diagonal positive definite matrix M^2, say, with eigenvalues constant in each simple subspace. Then the set $\{M\alpha, \alpha\in\Phi\}$ satisfies (3.18a), (3.18d) and (3.8).

Given a set of points Φ, satisfying (3.18), we can define a new set satisfying both (3.18) and (3.8), and write down the Lie algebra (3.2), (3.3) (taking $r = \ell$). It is a fundamental theorem that this exists and that there is therefore a correspondence between semi-simple Lie algebras and root systems satisfying (3.8) and (3.18). The problem of constructing all possible semi-simple Lie algebras is a geometric one of constructing all possible root systems (3.18). We shall return to this later.

4. - WEIGHT LATTICES AND THE STRUCTURE OF THE CENTRE OF H (H SEMI-SIMPLE)

We wish to find the magnetic weights β satisfying the quantization condition (3.12) :

$$\exp\left[4\pi i \sum_1^r \beta_i T_i\right] = 1 \tag{4.1}$$

For the present we shall assume H is semi-simple as well as compact and connected. Then the roots span a space whose dimension is the rank of H, (r).

The global structure of H will be relevant, and is specified by

$$H = \widetilde{H}/k(H) \tag{4.2}$$

where $\widetilde{H}$ is the "universal covering group" of H. It is simply connected and is uniquely determined by the algebra, $\mathcal{L}(H)$. $k(H)$, the kernel of the homomorphism $\widetilde{H} \to H$, is a subgroup of $Z(\widetilde{H})$, the centre of $\widetilde{H}$, the set of elements commuting with all elements of $\widetilde{H}$. Since $\widetilde{H}$ is semi-simple and compact its centre is a finite Abelian group and the number of possibilities for $k(H)$ is finite [14].

Another, more physical way of specifying the global structure of H is to specify the "weights" of single valued representations of H. A weight $(w_1, w_2, \ldots, w_r)$ is the eigenvalue of $(T_1, T_2, \ldots, T_r)$ corresponding to one common eigenvector in a single valued representation of H. The set of these weights will be denoted $\Lambda(H)$. $\Lambda(H)$ must be a subset of $\mathbb{R}^r$, and be closed under addition and negation since these correspond to the physical operations of assembling two particles and antiparticle conjugation.

For example $k(SU(2)) = 1$, $k(SO(3)) = Z_2 = \{1,-1\}$

$$\Lambda(SU(2)) = \{0, \pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \ldots\}$$

$$\Lambda(SO(3)) = \Lambda(SU(2)/Z_2) = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

[the normalization $T_1 = t_3$ is not the standard one (3.8) since $\Phi = \{1,-1\}$ and $\sum_\Phi \alpha_i \alpha_j = 2$].

Since by (3.17) $\alpha.T/\alpha^2$ is the generator of an angular momentum subgroup of H, $2\alpha.w/\alpha^2$ must be an integer for any weight w and any root α. It turns out that this condition is also sufficient for w to be a weight of $\widetilde{H}$ (see Humphreys [13], section 13) ; so

$$\Lambda(\widetilde{H}) = \{w ; 2w.\alpha/\alpha^2 \in \mathbb{Z} ; \alpha \in \Phi(H)\} \tag{4.3}$$

and

$$\Lambda(H) \subseteq \Lambda(\widetilde{H}) \tag{4.4}$$

The topological quantum number (3.13) is an element of

$$\pi_1(H) \cong k(H) \qquad (4.5)$$

where the isomorphism follows by general theorems [14].

In solving the quantization condition (4.1) we shall generalize the analysis of Englert and Windey [7] who considered H to be simple and $k(H)$ to be either 1 or $Z(\widetilde{H})$. At the same time our treatment will be more self-contained.

Now let us consider the quantization condition (4.1) in $\widetilde{H}$ rather than in H, and denote $\widetilde{\exp}$ the exponential mapping in $\widetilde{H}$ (i.e., for arbitrary representations of $\widetilde{H}$ which can be multivalued representations of H). Equation (4.1) becomes

$$\widetilde{\exp}\; 4\pi i \underline{\beta}.\underline{I} \;\in k(H) \subseteq Z(\widetilde{H}) \qquad (4.6)$$

As a preliminary step we shall solve the problem of finding all β such that

$$\widetilde{\exp}\; 4\pi i \underline{\beta}.\underline{I} \;\in Z(\widetilde{H}) \qquad (4.7)$$

Note that all elements of $Z(\widetilde{H})$ must be so expressible by (3.10).

Now any element of $Z(H)$ must commute with all the generators of $\mathcal{L}(H)$ and in particular with the step operators E_α [see (3.2)]. Commuting E_α with (4.7) and using (3.2) yields

$$\exp\; 4\pi i \underline{\beta}.\underline{\alpha} = 1 \qquad \alpha \in \Phi(H) \qquad (4.8)$$

where $\Phi(H) = \Phi(\widetilde{H})$ denotes the set of roots of H. Equivalently,

$$2\,\underline{\alpha}.\underline{\beta} \in \mathbb{Z} \qquad \alpha \in \Phi(H) \qquad (4.9)$$

This condition resembles (4.3). Now the set $\{\alpha/\alpha^2; \alpha \in \Phi(H)\}$ constitutes a system of roots in the sense of satisfying (3.18) but not the standard normalization condition (3.8). As shown in Appendix B they can always be rescaled to obtain a "dual root system" satisfying both (3.8) and (3.18) :

$$\Phi^{\vee}(H) = \{\alpha^{\vee} = N^{-1}\alpha/\alpha^2 \,;\, \alpha \in \Phi(H)\} \qquad (4.10)$$

If H is a simple group N is a number, calculated in Section 7. If H is semi-simple, N is a diagonal matrix, assuming these numerical values in each simple subspace.

By the standard contruction theorems it is possible to construct a simply connected Lie group $\widetilde{H}^{V}$ such that

$$\Phi(\widetilde{H^{\vee}}) = \Phi^{\vee}(H) \qquad (4.11)$$

The algebra $\mathcal{L}(\widetilde{H}^{V})$ is specified by replacing α by α^{V} in (3.2), (3.3), etc. Notice that

$$(\alpha^{\vee})^{\vee} = \alpha \qquad (4.12)$$

Therefore

$$\begin{aligned}\Lambda(\widetilde{H^{\vee}}) &= \{w \,;\, 2w\cdot\alpha^{\vee}/(\alpha^{\vee})^2 \in \mathbb{Z} \,,\, \alpha^{\vee} \in \Phi(\widetilde{H}^{\vee})\} \\ &= \{w \,;\, 2wN\alpha \in \mathbb{Z} \,,\, \alpha \in \Phi(H)\} \qquad (4.13)\end{aligned}$$

Hence the most general solution to (4.9) and hence (4.7) is

$$\beta = Nw \quad ; \; w \in \Lambda(\widetilde{H^{\vee}}) \qquad (4.14)$$

and the magnetic weight is indeed a weight (in an unconventional normalization), but not all $N^{-1}\beta \in \Lambda(\widetilde{H}^{V})$ satisfy (4.6) in addition to (4.7). We shall return to this in the next section, but first we must analyze in more detail the relation between $Z(\widetilde{H})$ and $\Lambda(\widetilde{H}^{V})$.

Now $\Lambda(H)$, $\Lambda(\tilde{H})$ and $\Lambda(\tilde{H^\vee})$ are all examples of what we shall call a "lattice" Λ : a set of points spanning $\mathbb{R}^r$ such that

$$\alpha, \beta \in \Lambda \Rightarrow \alpha + \beta \in \Lambda \tag{4.15a}$$

$$\alpha \in \Lambda \Rightarrow -\alpha \in \Lambda \tag{4.15b}$$

Λ has no point of accumulation (4.15c)

In Appendix C we shall prove that any lattice Λ has a basis $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_r$ in the sense that

$$\Lambda = \left\{ \sum_1^r n_i \underline{e}_i \;;\; n_i \in \mathbb{Z} \right\} \tag{4.16}$$

A convenient basis for $\Lambda(\tilde{H})$ is $\{\underline{\lambda}_1, \underline{\lambda}_2 \ldots \underline{\lambda}_r\}$, the set of "fundamental weights" of H, defined by [13)]

$$2 \underline{\lambda}_i \cdot \underline{\alpha}_j / \alpha_j^2 = \delta_{ij} \;;\; \underline{\alpha}_j \in \Delta(H) \tag{4.17}$$

where $\Delta(H)$ is a "basis of r simple roots" ; so that

$$\Delta(H) \text{ spans } \mathbb{R}^r \tag{4.18a}$$

$$\underline{\beta} \in \Phi(H) \Rightarrow \underline{\beta} = \sum n_\alpha \underline{\alpha} \;;\; \underline{\alpha} \in \Delta(H) \tag{4.18b}$$

with the coefficients n_α integers either all ≥ 0, or all ≤ 0. Correspondingly we can define $\Delta(\tilde{H^\vee})$

$$\Delta(\tilde{H^\vee}) = \left\{ \underline{\alpha}_i^\vee = N^{-1} \underline{\alpha}_i / \alpha_i^2 \;;\; \alpha_i \in \Delta(H) \right\} \tag{4.19}$$

and fundamental weights

$$\underline{\lambda}_i^\vee = N^{-1} \underline{\lambda}_i / \alpha_i^2 \tag{4.20}$$

Notice that

$$2\underline{\alpha}_i N \underline{\lambda}_j^\vee = \delta_{ij} = 2\underline{\alpha}_i^\vee N \underline{\lambda}_j \qquad (4.21)$$

The results up to now tell us that Eq. (4.7) furnishes a mapping from $\Lambda(\widetilde{H^\vee})$ onto $Z(\widetilde{H})$ which is a homomorphism between the group structure of addition for the weights in $\Lambda(\widetilde{H^\vee})$ and the group structure of multiplication in $Z(\widetilde{H})$. What is the kernel ? I.e., what subset of $\Lambda(\widetilde{H^\vee})$ is mapped onto the unit element of $Z(\widetilde{H})$? Clearly it consists of those $\underline{\beta}$ for which

$$2\underline{\beta}\cdot\underline{w} \in \mathbb{Z} \text{ , for all } w \in \Lambda(\widetilde{H}) \qquad (4.22)$$

To solve, expand

$$\underline{\beta} = \sum_{\Delta(H)} m_i N \underline{\alpha}_i^\vee$$

Choosing $\underline{w} = \underline{\lambda}_1, \underline{\lambda}_2 \cdots \underline{\lambda}_r$ in turn in (4.22) we find that the m_i must be integers, i.e.,

$$2\underline{\beta}\cdot\underline{w} \in \mathbb{Z} \iff N^{-1}\underline{\beta} \in \Lambda_r(\widetilde{H^\vee}) \qquad (4.23)$$

where $\Lambda_r(\widetilde{H})$ is the "root lattice" whose basis is $\Delta(H)$, and is obviously a subset of $\Lambda(\widetilde{H})$. In fact we have proved

$$\Lambda(\widetilde{H^\vee})/\Lambda_r(\widetilde{H^\vee}) \cong Z(\widetilde{H}) \qquad (4.24)$$

This is our first main result.

An element of $\Lambda(\widetilde{H^\vee})/\Lambda_r(\widetilde{H^\vee})$ is an equivalence class of points of $\Lambda(\widetilde{H^\vee})$ under the equivalence relation

$$\alpha \sim \beta \quad ; \quad \alpha, \beta \in \Lambda(\widetilde{H}) \quad \alpha - \beta \in \Lambda_r(H) \qquad (4.25)$$

We shall call these equivalence classes "sublattices", understanding that $\Lambda_r(\widetilde{H})$ itself is the only sublattice which is also a lattice (4.15). The isomorphism in (4.24) equates the addition of sublattices to multiplication in $Z(\widetilde{H})$.

Note that every vector of an irreducible representation can be obtained from one eigenvector corresponding to weight w by applying step operators E_α. Hence by (3.2), all weights of an irreducible representation of H belong to only one sublattice of $\Lambda(\widetilde{H})$.

5. - THE DUAL OR MAGNETIC GROUP H^v (H SEMI-SIMPLE)

Now we can return to solving the quantization condition (3.12), (4.1) or (4.6) which is equivalent to

$$2\underline{\beta}.\underline{w} \in \mathbb{Z} \quad \text{for all} \quad w \in \Lambda(H) \tag{5.1}$$

Let $\Lambda(H)^*$ denote the set of points $(N^{-1}\beta)$ satisfying (5.1).

Now $\Lambda(H)$ is a lattice in the sense (4.15) since it satisfies (4.15a,b) and by (4.4) has no accumulation point. We prove in Appendix C that if Λ is a lattice then so is Λ^* and that

$$(\Lambda^*)^* = \Lambda \tag{5.2}$$

So $\Lambda(H)^*$ is a lattice which by the results of the previous section is contained within $\Lambda(\widetilde{H^v})$. We shall now show that $\Lambda(H)^*$ is the weight lattice of a specific group H^v, locally isomorphic to $\widetilde{H^v}$ by constructing H^v explicitly.

Consider the subgroup of $Z(\widetilde{H^v})$:

$$k(H^v) = \{\exp 4\pi i \underline{w} N \underline{T}^v ; w \in \Lambda(H)\} \tag{5.3}$$

$$\cong \Lambda(H)/\Lambda_r(H) \tag{5.4}$$

by the isomorphism theorem (4.24) applied to a subgroup. Define

$$H^{v} = \widetilde{H}^{v}/k(H^{v}) \tag{5.5}$$

Then since $k(H^{v}) \subset Z(\widetilde{H}^{v})$ and is the kernel of the homomorphism $\widetilde{H}^{v} \to H^{v}$

$$k(H^{v}) = \{\widetilde{\exp}\, 4\pi i \underline{w} N \underline{T}^{v} \;;\quad w \in \Lambda(H^{v})^{*}\} \tag{5.6}$$

Comparing (5.6) with (5.3) we see that

$$\Lambda(H^{v})^{*} = \Lambda(H)$$

since both are lattices. Finally by (5.2)

$$\Lambda(H)^{*} = \Lambda(H^{v}) \tag{5.7}$$

and so the most general possibility for the magnetic weight β satisfying the quantization condition (3.12) is

$$N^{-1}\beta \in \Lambda(H^{v}) \tag{5.8}$$

So up to a normalization β is the weight of the group H^{v} which we shall call the "magnetic group" in contrast to the original "electric" group, H, whose weights defined "electric" charges.

Now since $k(H)$ is the kernel of the homomorphism of $\widetilde{H}$ onto H

$$\begin{aligned} k(H) &= \{\widetilde{\exp}\, 4\pi i \underline{w} N \underline{T} \;;\; w \in \Lambda(H)^{*}\} \\ &= \{\widetilde{\exp}\, 4\pi i \underline{w} N \underline{T} \;;\; w \in \Lambda(H^{v})\} \end{aligned}$$

by (5.7). By the isomorphism theorem (4.23)

$$\cong \Lambda(H^{v})/\Lambda_{r}(H^{v}) \tag{5.9}$$

Comparing with (5.3) we see that the relation between the magnetic and electric groups $H^{\vee}$ and H is symmetrical in the sense

$$(H^{\vee})^{\vee} = H \tag{5.10}$$

These then are the main results of this paper.

How does the topological quantum number (3.13) appear in this framework ? Let us "raise" the path J_o (3.13) from H to $\widetilde{H}$. Its $\Omega = 4\pi$ endpoint becomes

$$\widetilde{\exp}\ 4\pi i\, \underline{\beta}\cdot\underline{T}$$

which by the quantization condition, lies in $k(H) \cong \pi_1(H)$, by (4.5). Equation (5.9) tells us that the particular element is determined just by the particular sublattice on which $N^{-1}\beta$ lies.

6. - EXTENSION TO THE GENERAL COMPACT CASE

In this section we extend the results of Sections 4 and 5 to the case where H is a connected compact group. We dealt first with the case of a semi-simple group because the presence of $U(1)$ subgroups in the centre $Z(H)$ of H results in the universal covering group of H, $\widetilde{H}$, being non-compact. In consequence the set of weights of $\widetilde{H}$, that is the set of eigenvalues of $(T_1, T_2, \ldots, T_r)$, is non-denumerable and so not a lattice in the sense of Section 4.

More specifically, the semi-simple algebra in Eq. (3.1) $[\mathcal{L}(H), \mathcal{L}(H)]$, generates a compact semi-simple subgroup of H, which we will call H'. This group has a compact universal covering group $\widetilde{H}'$. The universal covering group of H, $\widetilde{H}$, is isomorphic to $\mathbb{R}^{r-\ell} \times \widetilde{H}'$; roughly speaking we get a factor of $\mathbb{R}$ for each $U(1)$ in $Z(H)$. The problem is that the components of the weights of $\widetilde{H}$ corresponding to $T_{\ell+1}, \ldots, T_r$ will not be quantized.

Our strategy for circumventing this difficulty is to use a compact covering group of H,

$$\bar{H} = C(H) \otimes \tilde{H}' \tag{6.1}$$

for the purposes for which we used $\tilde{H}$ in the semi-simple case. Here $C(H)$ is the compact Abelian subgroup of H generated by the subalgebra $\mathcal{C}(H)$ occurring in Eq. (3.1). In other words, $C(H)$ is the subgroup of H generated by $T_{\ell+1} \ldots . T_r$. We may write

$$H = \bar{H} / k(H) \tag{6.2}$$

where $k(H)$ is the kernel of the homomorphism $\bar{H} \to H$ and has the structure

$$k(H) = \{ (\pi(h), h^{-1}); \ \pi(h) \in C(H) \cap H' \} \tag{6.3}$$

where π is the homomorphism $\tilde{H}' \to H'$. Using $\overline{\exp}$ to denote the exponential mapping in $\bar{H}$, the quantization condition Eq. (4.1) becomes

$$\overline{\exp}\, 4\pi i \underline{\beta} . \underline{T} \in k(H) \leq \bar{Z}(\bar{H}) \tag{6.4}$$

where

$$\bar{Z}(\bar{H}) = (C(H) \cap H') \otimes Z(\tilde{H}') \tag{6.5}$$

is not the centre of $\bar{H}$ [which is $C(H) \times Z(\tilde{H}')$ and so not a finite group] but a subgroup of the centre.

Following Section 4 we first consider the problem of determining those $\underline{\beta}$ satisfying

$$\overline{\exp}\, 4\pi i \underline{\beta} . \underline{T} \in \bar{Z}(\bar{H}) \tag{6.6}$$

Because of the direct product structure of $\bar{Z}(\bar{H})$, we write $\underline{T} = \underline{T}^{(1)}, \underline{T}^{(2)}$ where $\underline{T}^{(1)} = (T_1, \ldots, T_\ell)$ and $\underline{T}^{(2)} = (T_{\ell+1}, \ldots, T_r)$. Similarly dividing $\underline{\beta} = (\underline{\beta}^{(1)}, \underline{\beta}^{(2)})$ so that

$$\underline{\beta}.\underline{T} = \underline{\beta}^{(1)}.\underline{T}^{(1)} + \underline{\beta}^{(2)}.\underline{T}^{(2)} \tag{6.7}$$

condition (6.6) factors into [cf. Ref. 8]

$$\overline{\exp}\,(4\pi i\,\underline{\beta}^{(1)}.\underline{T}^{(1)}) \in C(H) \cap H' \tag{6.8a}$$

and

$$\overline{\exp}\,(4\pi i\,\underline{\beta}^{(2)}.\underline{T}^{(2)}) \in Z(\widetilde{H'}) \tag{6.8b}$$

Since H' is semi-simple, the general solution to condition (6.8b) was obtained in Eq. (4.14) and it may be written

$$\underline{\beta}^{(2)} = N^{(2)}\underline{w}^{(2)}; \quad \underline{w}^{2} \in \Lambda(\widetilde{H^{v}}) \tag{6.9}$$

where the normalization matrix $N^{(2)}$ is a multiple of the identity when restricted to each simple subspace.

Condition (6.8a) defines a lattice of vectors $\underline{\beta}^{(1)}$ in $\mathbb{R}^{r-\ell}$ which we will temporarily denote by $\Lambda^{(1)}$. Following Section 5, we define $\Lambda^{(1)*}$ to be the lattice of points $\underline{w}^{(1)}$ satisfying

$$2\underline{\beta}^{(1)}.\underline{w}^{(1)} \in \mathbb{Z} \quad \text{for all } \underline{\beta}^{(1)} \in \Lambda^{(1)} \tag{6.10}$$

[Clearly $\Lambda^{(1)} = \Lambda(C(H) \cap H')$, the weight lattice of $C(H) \cap H'$].

Now consider the compact Abelian group

$$C^{v}(H) = \mathbb{R}^{r-\ell}/4\pi\Lambda^{(1)*} \tag{6.11}$$

(where we identify $\underline{x} \in \mathbb{R}^{r-1}$ with $\underline{x} + 4\pi\underline{w}^{(1)}$ for all $\underline{w}^{(1)} \in \Lambda^{(1)*}$). The map $\underline{x} \to \exp(i\underline{x}.\underline{\beta}^{(1)})$ defines a representation of $C^{v}(H)$ if and only if $\beta^{(1)} \in \Lambda^{(1)}$. Thus $\Lambda^{(1)}$ is the weight lattice of $C^{v}(H)$. This leads us to write

$$\overline{H^{v}} = C^{v}(H) \otimes \widetilde{H'^{v}} \tag{6.12}$$

so that the solutions of condition (6.6) are of the form

$$\underline{\beta} = N\underline{\underline{w}} \quad \text{where} \quad \underline{\underline{w}} \in \Lambda(\overline{H^{v}}) \tag{6.13}$$

and $N\underline{w} = (\underline{w}^{(1)}, N^{(2)}\underline{w}^{(2)})$.

To derive a generalization of the isomorphism (4.24) we consider the kernel of the homomorphism from $\Lambda(\overline{H^{v}})$ onto $\overline{Z}(\overline{H})$. For $\underline{\beta}$ to be in this kernel we require

$$\overline{\exp}\,(4\pi i\, \underline{\beta}^{(1)}\cdot \underline{T}^{(1)}) = \overline{\exp}\,(4\pi i\, \underline{\beta}^{(2)}\cdot \underline{T}^{(2)}) = 1 \tag{6.14}$$

The condition on $\underline{\beta}^{(2)}$ is just $N^{(2)-1}\underline{\beta}^{(2)} \in \Lambda_r(\widetilde{H}')$. The condition on $\underline{\beta}^{(1)}$ defines a sublattice of $\Lambda^{(1)} = \Lambda(C^{v}(H))$ which extending our notation we will denote by $\Lambda_r(C^{v}(H))$. So if we define $\Lambda_r(H^{v})$ to be the direct sum of the lattices $\Lambda_r(C^{v}(H))$ and $\Lambda_r(\widetilde{H}')$, condition (6.14) is seen to be equivalent to

$$\underline{\beta} = N\underline{\underline{w}} \quad \text{where} \quad \underline{\underline{w}} \in \Lambda_r(H^{v}) \tag{6.15}$$

Consequently

$$\overline{Z}(\overline{H}) \cong \Lambda(\overline{H^{v}})/\Lambda_r(H^{v}) \tag{6.16}$$

Now we define $k(H^{v})$ by the analogue of Eq. (5.3),

$$k(H^{v}) = \{\overline{\exp}\, 4\pi i\, \underline{w}\cdot N\underline{T}^{v}\ ;\ \underline{w} \in \Lambda(H)\} \tag{6.17}$$

$$\cong \Lambda(H)/\Lambda_r(H) \tag{6.18}$$

where $\Lambda_r(H)$ is the direct sum of $\Lambda^{(1)*}$ and $\Lambda_r(H')$. As before we define

$$H^{v} = \overline{H^{v}} / k(H^{v}) \tag{6.19}$$

and by the reasoning of Section 5 see that the quantization condition becomes again (5.8).

7. - EXAMPLES OF THE DUAL RELATIONSHIP BETWEEN "ELECTRIC" AND "MAGNETIC" GROUPS H AND H^v

First we shall discuss the relationship between the local structure of H and H^v for H simple. This is essentially the relation between the root systems specified by (4.10). Assuming α and α^v are both normalized, (3.8), we can calculate the rescaling parameter N.

$$\sum_{\Phi} \alpha_i \alpha_j = \delta_{ij} = \sum_{\Phi} \alpha_i^{v} \alpha_j^{v}$$

Contracting i with j

$$\sum \alpha^2 = r = \sum \alpha^{v\,2} \tag{7.1}$$

But by (4.10)

$$(\alpha^{v})^2 \alpha^2 N^2 = 1$$

so

$$N^2 = \left(\sum 1/\alpha^2\right) \Big/ \sum \alpha^2 = \left(\sum 1/\alpha^2\right) / r \tag{7.2}$$

where r is the rank.

All possible root systems for simple groups were classified by Cartan (and are determined up to a rotation), and it turns out that the roots have at most two distinct lengths. The system is uniquely characterized by r and the number of generators, m, except in the case of $B_r = SO(2r+1)$ and $C_r = Sp(2r)$. The algebras $A_r = SU(r+1)$, $D_r = SO(2r)$ together with the exceptional ones E_6, E_7, and E_8 have roots of equal length and therefore their root systems are self-dual under (4.10). G_2 and F_4 must be self-dual up to a rotation while it turns out that the root systems of B_r and C_r transform into each other under (4.10). For A_r, D_r, E_6, E_7 and E_8, (7.1) and (7.2) tell us that

$$N = (m-r)/r \tag{7.3}$$

while (7.2) and the explicit formulae for roots imply

$$N(B_r) = N(C_r) = \sqrt{2(r+1)(2r-1)}$$

$$N(G_2) = 4\sqrt{3}$$

$$N(F_4) = 9\sqrt{2}$$

Now let us turn to the relationship between the global structure of H and H^V as given by (5.3) and (5.5). In practice it is often more convenient to use

$$k(H^v) \cong Z(H) \tag{7.4}$$

This follows from

$$Z(H) \cong Z(\widetilde{H})/k(H)$$

$$\cong \Lambda(\widetilde{H^v})/\Lambda(H^v) \tag{7.5}$$

by (4.23) and (5.9) while by (5.7)

$$\cong \Lambda_r(H)^* / \Lambda(H)^*$$

Now in Appendix C we prove that if Λ_1 and Λ_2 are lattices such that $\Lambda_1 \subseteq \Lambda_2$, then $\Lambda_2^* \subseteq \Lambda_1^*$ and $\Lambda_2/\Lambda_1 = \Lambda_1^*/\Lambda_2^*$. So

$$\cong \Lambda(H)/\Lambda_r(H)$$

$$\cong k(H^v)$$

by (5.4).

Equations (7.4) and (7.5) are useful if $Z(H)$ is cyclic because then any subgroup is cyclic and is unique, given its order [which must be a divisor of the order of $Z(H)$]. Further the quotient group is cyclic. This then applies to all simple groups, except for D_r (r even). So we have :

$$
\begin{aligned}
\left(SU(NM)/\mathbb{Z}_N\right)^{\vee} &= SU(NM)/\mathbb{Z}_M \\
\widetilde{SO(2r+1)}^{\vee} &= Sp(2r)/\mathbb{Z}_2 \\
SO(2r+1)^{\vee} &= Sp(2r) \\
G_2^{\vee} &= G_2' \\
F_4^{\vee} &= F_4' \\
\widetilde{E}_r^{\vee} &= E_r/\mathbb{Z}_{9-r}\ , \quad r=6,7,8.
\end{aligned}
$$

using the known structure of the centre. The prime denotes that the roots are rotated. For the detailed treatment of D_r see Appendix D.

For semi-simple groups it may be helpful to use

$$(A \otimes B)^{\vee} = A^{\vee} \otimes B^{\vee}$$

All other cases must be worked out explicitly from our general formulae (5.3) and (5.5). For example if $k(C)$ is the diagonal subgroup of $\tilde{C} = SU(2) \otimes SU(2)$ we find $C = \tilde{C}/k(C)$ is self-dual.

8. - DISCUSSION AND SPECULATION

We now wish to explain what we think could be the significance and meaning of the results obtained in this paper. The ideas to be presented have stimulated this research but are of necessity speculative in view of the work necessary to substantiate them.

We have seen that the magnetic weights are indeed weights of the group H^{v}. This suggests two possibilities : in some future reformulation of the theory in terms of new field operators (maybe monopole fields) it will be possible to explicitly construct the generators of H^V so that either (a) the monopoles will be eigenstates of the commuting generators T^V, with the magnetic weights as eigenvalues, or (b) the magnetic weights label not eigenvalues, but rather irreducible representations of H^V.

The fact that the gauge transformations in H induce the action of the Weyl group on the magnetic weights (Section 3) suggests to us that the latter interpretation, (b), is correct. Then H monopoles are H^V (irreducible) multiplets (and vice versa). A difficulty with this idea, that we shall return to later, is that the label for the components of the multiplet is missing, or at least hidden.

If the idea is correct it can be checked by examining the rules for the combination of two monopoles. It should be given by the Clebsch-Gordan series for H^V which tells precisely which irreducible representations can be formed out of two given ones. Thus the total magnetic weight ought to belong only to a restricted range, the particular value depending on precisely how the two constituent magnetic weights are put together.

Let $\underline{r}_1$ and $\underline{r}_2$ be the vectors from monopoles 1 and 2 to the field point in question. Suppose that if r_1 and r_2 are large enough

$$G_{ij} = \epsilon_{ijk}\left(\frac{\hat{r}_1^{\,k}\, G_1(\underline{r})}{r_1^2} + \frac{\hat{r}_2^{\,k}\, G_2(\underline{r})}{r_2^2}\right) \tag{8.1}$$

and that

$$\mathcal{D}_i \quad G_{1,2} = 0 \tag{8.2}$$

If further

$$[G_1(\underline{r}),\ G_2(\underline{r})] = 0 \tag{8.3}$$

most of the analysis of Sections 2 and 3 can be repeated, e.g., to find

$$h(s) = \exp[\,i\, G_1(\underline{r})\Omega_1 + i\, G_2(\underline{r})\Omega_2]$$

where Ω_1, Ω_2 are the solid angles subtended by the contour of integration $0 \leq t \leq 1$ for given s, at monopoles 1 and 2.

At distances large compared to the distances between the two monopoles the magnetic fields (8.1) will take the form (1.1) with

$$G(\underline{r}) = G_1(\underline{r}) + G_2(\underline{r})$$

G_1 and G_2 can be simultaneously gauge transformed to be generators of T with magnetic weights $\underline{\beta}_1$ and $\underline{\beta}_2$. So

$$\underline{\beta} = \underline{\beta}_1 + \underline{\beta}_2$$

and is automatically $\in \Lambda(H^V)$ by (4.15a). Before "putting" the two monopoles together one could gauge rotate one independently of the other and instead obtain

$$\underline{\beta} = \underline{\beta}_1 + \underline{\sigma}(\beta_2) \tag{8.4}$$

where σ is an element of the Weyl group. If $H = SU(2)$ we have the possibilities

$$\beta = |\pm \beta_1 \pm \beta_2|$$

which are indeed the extremal terms of the Clebsch-Gordan series $|\beta_1+\beta_2|, \ldots, |\beta_1-\beta_2|$. We suspect, but have not proved, that (8.4) is always a principal weight in the Clebsch-Gordan series $(\beta_1)\otimes(\beta_2)$ for any semi-simple group. We are uncertain whether the missing terms in the Clebsch-Gordan series can be found by relaxing (8.1), (8.2) or (8.3) at the present classical level of discussion.

At any rate it is interesting to see how the Weyl group partially supplies the structure that is missing owing to the lack of a label for the component of the H^V multiplet. This label should be missing since it would be connected with the quantum mechanical superposition principle for monopoles which, at the present level of discussion, are treated in a purely classical way.

The ultimate statement of our conjecture would be that the true invariance group of the theory is $H \otimes H^V$. All states would be classified according to this and their interactions would be invariant. The "topological quantum" number "now appears as the generalized triality" of the H^V multiplets and is certainly conserved, but our statement is much stronger, with further consequences, and would require much further work to prove, if true. For example, our conjecture implies that a monopole which is an H^V multiplet cannot decay into the vacuum, even if it is topologically trivial, since the Clebsch-Gordan law would be violated.

This conjecture somehow generalizes the "soliton" property of two-dimensional Sine-Gordon theory which can be reformulated in terms of the Thirring model. There the "kink" or "soliton" state of Sine-Gordon theory possesses a topological quantum number with corresponding current proportional to $\epsilon_{\mu\nu}\partial^\nu \phi$. It can equally well be described as an ordinary state of the Thirring model with an ordinary Noether charge with current $\bar{\psi}\gamma^\nu\psi$ which is proportional to $\epsilon_{\mu\nu}\partial^\nu\phi$, according to the transformation between the two theories. The secret of the relationship is quantum mechanics : ϕ has to be a quantized Sine-Gordon field. This ingredient is missing so far from our treatment of monopoles and we suspect that only when it is supplied will our conjecture be verifiable. The results so far at the classical level are just a vestigial indication of the true quantum structure.

Let us comment on possible physical applications of the idea that H and H^V both play a rôle. H could be the unified weak and electromagnetic group and H^V the colour group of strong interactions. All interactions would then be unified with the disparity in strength being nevertheless quite natural. Further the different parity transformation properties of H and H_V could be a simple consequence of the fact that monopoles violate parity. This sort of idea dates from Dirac [1] and was recently revived by Faddeev [4] and others. Faddeev [15] has recently objected that monopoles cannot be hadrons because (a) they would be too heavy and (b) they would have long range forces. Now (a) was only proved if the Higgs field was in the adjoint representation [16] which prohibits H and H^V from being semi-simple. Maybe the monopoles are quarks rather than hadrons since then the long range forces are the colour forces responsible for confinement. To illustrate, if $H = SU(3)_{colour}$ $H^V = SU(3)/Z_3$. This agrees with the proposal that leptons constitute a weak electromagnetic $SU(3)$ octet [17], but not with the quark assignment of $(3,3)$ under $H \times H^V$.

Maybe our analysis requires modifications for dyons. We admit the idea is extremely speculative and raises many new questions. For example, are weak and colour forces indeed Lorentz transformable with each other ? What about the spontaneous breaking of the weak symmetry ?

Let us conclude that the theory of monopoles opens up interesting possibilities and deserves further study, particularly with respect to the full quantization.

ACKNOWLEDGEMENTS

We should like to thank Edward Corrigan for discussion at an early stage of this work.

APPENDIX A : MAGNETIC WEIGHT AMBIGUITY

The magnetic charge is invariant with respect to gauge rotations within the Abelian part of H [generated by $\mathcal{C}(H)$ in (3.1)] and the Abelian magnetic weights $\beta_{\ell+1},\ldots,\beta_r$ are completely gauge invariant. Hence, we need only consider the effect of the semi-simple part of H on the non-Abelian magnetic weights $\beta_1,\ldots,\beta_\ell$, and so, without loss of generality can take H to be semi-simple of rank ℓ. We shall prove that if

$$S\,\underline{\beta}.\underline{T}\,S^{-1} = \underline{\beta}'.\underline{T} \quad ; \quad S \in H \tag{A.1}$$

then

$$\underline{\beta}' = \sigma(\underline{\beta}) \tag{A.2}$$

where σ is in the Weyl group $W(H)$.

The hyperplanes $\underline{\alpha}.\underline{x} = 0$, $\alpha \in \Phi(H)$, divide the ℓ dimensional space into a finite number of hypercones, called Weyl chambers, whose properties are presented in Section 10 of Humphreys [13].

Consider the Weyl chamber $K(\Delta)$ containing β (it may not be unique if $g.\alpha = 0$ for some α but this does not matter). It can be written

$$K(\Delta) = \{\underline{x}\ ;\ \underline{x}\cdot\underline{\alpha}_i \geq 0\ ;\ \underline{\alpha}_i \in \Delta(H)\} \tag{A.3}$$

where Δ is the corresponding basis of simple roots (4.18). Any one Weyl chamber can be transported into any other by a unique element of the Weyl group, $W(H)$. Hence we can find $\sigma \in W(H)$ such that

$$\beta'' = \sigma(\beta') \in K(\Delta) \tag{A.4}$$

Now let us consider (A.1) in some D dimensional irreducible representation α of the group H with weights $\underline{m}_1^\alpha, \underline{m}_2^\alpha, \ldots, \underline{m}_D^\alpha$. Now $\underline{\beta}.\underline{T}$ and $\underline{\beta}'.\underline{T}$ are $D\times D$ matrices related by a similarity transformation (A.1). They must therefore have the same sets of eigenvalues, A and A', say, where

$$A = \{ \underline{\beta}.\underline{m}_1^\alpha, \underline{\beta}.\underline{m}_2^\alpha, \ldots \quad \underline{\beta}.\underline{m}_D^\alpha \}$$

$$A' = \{ \underline{\beta}'.\underline{m}_1^\alpha, \underline{\beta}'.\underline{m}_2^\alpha, \ldots \quad \underline{\beta}'.\underline{m}_D^\alpha \}$$

But these also equal the set

$$A'' = \{ \underline{\beta}''.\underline{m}_1^\alpha, \underline{\beta}''.\underline{m}_2^\alpha, \ldots \quad \underline{\beta}''.\underline{m}_D^\alpha \}$$

since

$$\underline{\beta}''.\underline{m}_i^\alpha = \underline{\sigma}(\underline{\beta}').\underline{m}_i^\alpha = \underline{\beta}'.\underline{\sigma}(\underline{m}_i^\alpha)$$

and the Weyl group permutes the weights of an irreducible representation.

It is a property of irreducible representations that amongst the weights there is a "highest" dominant weight, $\underline{m}_1^\alpha$, let us say, $\epsilon\ K(\Delta)$, to be dominant. Highest means that any other weight must satisfy

$$\underline{m}_i^\alpha = \underline{m}_1^\alpha - \sum_{\alpha \in \Delta} n_\alpha \underline{\alpha} \qquad n_\alpha \geq 0, \; i = 1 \ldots D$$

It follows that $\underline{\beta}.\underline{m}_1^\alpha$ and $\underline{\beta}''.\underline{m}_1^\alpha$ assume the maximum values in the sets A and A" respectively, and must therefore be equal, since the sets are.

Repeating the argument for ℓ irreducible representations whose highest dominant weights $\underline{m}_1^\alpha$, $\alpha = 1 \ldots \ell$ are linearly independent. It follows that

$$\underline{\beta} = \underline{\beta}'' = \underline{\sigma}(\underline{\beta}')$$

and (A.2) is proved.

The situation can be summarized by saying that the set of eigenvalues of $e\ G(\underline{r})$ is gauge invariant. The individual eigenvalues are not gauge invariant, since they may be permuted by special gauge transformations, namely the elements of the Weyl group.

APPENDIX B : PROOF OF Eq. (3.19) FROM (3.18a) AND (3.18d) FOR SIMPLE SETS OF ROOTS

By (3.18d)

$$\sum_{\alpha \in \Phi} \sigma_\beta(\alpha)_i \, \sigma_\beta(\alpha)_j = \sum_{\alpha \in \Phi} \alpha_i \alpha_j = g_{ij}, \text{ say}$$

So

$$g_{ik} \beta_k \beta_j + g_{jk} \beta_k \beta_i = 2\beta_i \beta_j \beta_k g_{k\ell} \beta_\ell / \beta^2$$

Let γ be any vector satisfying $\gamma . \beta = 0$. Then

$$\gamma_j \, g_{jk} \beta_k = 0$$

and so

$$g_{jk} \beta_k = \mu^2(\beta) \beta_j$$

where the number $\mu^2(\beta)$ is positive but may depend on β. But

$$\alpha_j \, g_{jk} \beta_k = \mu^2(\beta) \, \alpha . \beta = \mu^2(\alpha) \, \alpha . \beta$$

Hence $\mu^2(\alpha) = \mu^2(\beta)$ if $\alpha . \beta \neq 0$. So if Φ spans R^ℓ (3.8a), and cannot be decomposed into orthogonal subspaces $g_{ij} = \delta_{ij} \mu^2$, which is the desired result.

APPENDIX C : USEFUL PROPERTIES OF LATTICES (4.15)

Theorem C1 Any lattice Λ can be written $\Lambda = \{\sum_{i=1}^{r} n_i \underline{e}_i \; ; \; n_i \in Z\}$ (the set of vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_r$ is called a basis for Λ).

The proof is by induction on the dimensions of Λ. If $r = 1$ let x be the least positive element of Λ ; such an element exists because Λ has no point of accumulation. Let Λ' be the lattice $\{nx; n \in Z\}$. Clearly $\Lambda' \subseteq \Lambda$. But if there is a $y \in \Lambda$ and $\notin \Lambda'$ the set $\{y-nx; n \in Z\} \subset \Lambda$ will contain a positive number less than x. Consequently $\Lambda = \Lambda'$ and the theorem holds for $r = 1$.

Suppose inductively that the theorem holds if the dimension of Λ does not exceed r, and that Λ has dimension $r+1$. Take r linearly independent elements of Λ. These span an r dimensional linear subspace whose intersection with Λ is a lattice Λr. For this lattice we may take a basis $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_r$. Consider any other element of Λ, linearly independent of Λ_r, $\underline{f}$ say. Consider

$$P = \left\{ \sum_{i=1}^{r} \alpha_i \underline{e}_i + \alpha_{r+1} \underline{f} \quad ; \quad 0 \leq \alpha_i \leq 1 \right\} \tag{C.1}$$

This contains a finite number of points of Λ as this set has no accumulation point. Let $\underline{e}_{r+1}$ be the point in $\Lambda \cap P$ for which α_{r+1} is the smallest positive number. Consider the lattice Λ' for which $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{r+1}$ is a basis. Given $y \in \Lambda$ write

$$\underline{y} = \sum_{i=1}^{r+1} \beta_i \underline{e}_i \tag{C.2}$$

If $\beta_{r+1} \in Z$, $\underline{y} - \beta_{r+1} \underline{e}_{r+1} \in \Lambda_r$ and so the remaining $\beta_i \in Z$. If $\beta_{r+1} \notin Z$ we may choose integers m_i so that $0 \leq \beta_i - m_i \leq 1$ and $0 < \beta_{r+1} - m_{r+1} < 1$. Then

$$\underline{y} - \sum_{i=1}^{r+1} m_i \underline{e}_i = \sum_{i=1}^{r+1} (\beta_i - m_i) \underline{e}_i \in \Lambda \cap P \tag{C.3}$$

contradicting the fact that $\underline{e}_{r+1}$ is the point in $\Lambda \cap P$ with least positive $(r+1)$ co-ordinate. Hence $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{r+1}$ is a basis for Λ.

Note there is great freedom of choice in choosing such a basis.

<u>Theorem C2</u> If Λ is a lattice and $\Lambda^* = \{\underline{y} : 2\underline{x}N\underline{y} \in Z$ for all $x \in \Lambda\}$ then (i) Λ^* is a lattice, (ii) $(\Lambda^*)^* = \Lambda$.

<u>Proof</u> Let $\underline{e}_1, \ldots, \underline{e}_r$ be a basis for Λ and define vectors $\underline{e}_1^*, \ldots, \underline{e}_r^*$ so that

$$2\,\underline{e}_i N \underline{e}_j^* = \delta_{ij} \tag{C.4}$$

Then $y = \Sigma\alpha_i\underline{e}_i^* \in \Lambda^*$ if and only if $\Sigma\alpha_i n_i \in Z$ for all $n_i \in Z$. This is equivalent to $\alpha_1 \in Z$ for each i. So Λ^* is a lattice with basis $\underline{e}_1^*, \ldots, \underline{e}_r^*$. Since $(\underline{e}_i^*)^* = \underline{e}_i$, $(\Lambda^*)^* = \Lambda$.

<u>Theorem C3</u> If Λ_1 and Λ_2 are two n dimensional lattices with $\Lambda_1 \subseteq \Lambda_2$. Then $\Lambda_1^* \supseteq \Lambda_2^*$ and Λ_2/Λ_1 is a finite Abelian group isomorphic to Λ_1^*/Λ_2^*.

<u>Proof</u> If $\underline{y} \in \Lambda_2^*$, $2\underline{x}N\underline{y} \in Z$ for all $\underline{x} \in \Lambda_2$, and since $\Lambda_2 \supseteq \Lambda_1$ for Λ_1. Only a finite number of points of Λ_2 can be in the region

$$\Sigma\,\alpha_i\,\underline{e}_i^{(1)} \qquad 0 \leqslant \alpha_i \leqslant 1 \tag{C.5}$$

But each point of Λ_2 differs from at least one such point by an element of Λ_1, and the result follows.

To show that $\Lambda_2/\Lambda_1 \cong \Lambda_1^*/\Lambda_2^*$ is less direct. For each $\underline{x} \in \mathbb{R}^r$,

$$\underline{\alpha} \rightarrow \exp(4\pi i\,\underline{x}N\underline{\alpha}) \tag{C.6}$$

defines a one-dimensional representation of Λ_2. $\underline{x}_1$ and $\underline{x}_2$ will give the same representation if and only if $\underline{x}_1 - \underline{x}_2 \in \Lambda_2^*$. (We may identify the irreducible representations of Λ_2 with $\mathbb{R}^r/\Lambda_2^*$.) Now (C.6) defines a representation of the finite group Λ_2/Λ_1 if and only if

$$2\,\underline{x}N\underline{\alpha} \in \mathbb{Z} \quad \text{for all } \alpha \in \Lambda_1 \tag{C.7}$$

that is $\underline{x} \in \Lambda_1^*$. Consequently we may identify Λ_1^*/Λ_2^* with a group of one-dimensional representations of Λ_2/Λ_1. But since every representation of Λ_2/Λ_1 induces a representation of Λ_2, Λ_1^*/Λ_2^* contains all the representations of Λ_2/Λ_1. Now the one-dimensional representations of any finite Abelian group G form a group isomorphic to G. It is not possible to set up this isomorphism in a natural way. It is necessary to appeal to the structure theorem for finite Abelian groups which states that every such group is the direct product of cyclic groups of prime power order. It is easy to check the result for a cyclic group and deduce the result by taking direct products.

APPENDIX D : EXPLICIT CALCULATION OF MAGNETIC WEIGHTS FOR THE D(n) OR SO(2n) CASE

To illustrate, we take the most complicated possibility among the simple groups, the case when (7.3) is not necessarily helpful.

The $2n(n-1)$ non-zero roots of the $D(n)$ algebra are

$$\underline{\alpha}^{(\pm n, \pm m)} = \lambda(\pm \underline{e}^n \pm \underline{e}^m) \qquad n \neq m$$

where $(\underline{e}^1, \ldots, \underline{e}^n)$ constitute an orthonormal basis in $\mathbb{R}^n$. We shall keep the Euclidean length, $\sqrt{2}\lambda$, of each root arbitrary. By the argument leading to (7.1), the Cartan metric is

$$g_{ij} = \sum \alpha_i \alpha_j = 4\lambda^2(n-1)\,\delta_{ij}$$

The "standard" normalization (3.8) would require

$$\lambda = \frac{1}{2\sqrt{n-1}} \tag{D.1}$$

Solving (4.3) we find that the weights, $\underline{w} = \sum_{i=1}^{n} m_{(i)} \underline{e}_{(i)}$, fall into four sublattices (4.25) :

$$\begin{aligned}
m_i^{[0]} &= \lambda p_i \quad ; p_i \in \mathbb{Z} \; ; \sum p_i \text{ even} \\
m_i^{[1]} &= \lambda(\tfrac{1}{2} + p_i) \; ; p_i \in \mathbb{Z} \; ; \sum p_i \text{ even} \\
m_i^{[2]} &= \lambda p_i \quad ; p_i \in \mathbb{Z} \; ; \sum p_i \text{ odd} \\
m_i^{[3]} &= \lambda(\tfrac{1}{2} + p_i) : p_i \in \mathbb{Z} \; ; \sum p_i \text{ odd}
\end{aligned} \tag{D.2}$$

The addition law for these four sublattices, $m^{[k]}$, $k = 0,1,2,3$, depends on the parity of n.

<u>If n is odd,</u> $[k] + [k'] = [k'']$, mod 4, and the composition group $\Lambda(\widetilde{SO(2n)})/\Lambda_r(\widetilde{SO(2n)}) \cong Z_4$.

If n is even, the composition group can be represented by the Table

	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

so $\Lambda(\widetilde{SO(2n)})/\Lambda_r(\widetilde{SO(2n)}) = Z_2 \times Z_2$, now, and is not cyclic.

In view of the difference between the two cases, n even or odd, we must treat them separately.

n odd

The universal covering group, corresponding to the algebra $D(n)$ $(n = 2p+1)$, is the covering $\widetilde{SO(4p+2)}$ of the orthogonal group $SO(4p+2)$. Its centre is Z_4. There are three groups with the same algebra $D(2p+1)$: $\widetilde{SO(4p+2)}$; $SO(4p+2) \cong \widetilde{SO(4p+2)}/Z_2$; $SO(4p+2)/Z_2 \cong SO(4p+2)/Z_4$ with weights

$$\Lambda(\widetilde{SO(4p+2)}) = \left\{ \sum_1^n m_i^{[k]} e_i \quad ; \quad k = 0,1,2,3 \right\}$$

$$\Lambda(SO(4p+2)) = \left\{ \sum_1^n m_i^{[k]} e_i \quad ; \quad k = 0,2 \right\}$$

$$\Lambda(SO(4p+2)/Z_2) = \left\{ \sum_1^n m_i^{[k]} e_i \quad ; \quad k = 0 \right\} \qquad (D.3)$$

Solving (5.1) explicitly for the magnetic weights β we find, comparing with (D.2) and (D.3)

$$\beta(\widetilde{SO(4p+2)}) = \frac{1}{2\lambda^2} \Lambda(\widetilde{SO(4p+2)}/Z_4)$$

$$\beta(SO(4p+2)) = \frac{1}{2\lambda^2} \Lambda(SO(4p+2))$$

$$\beta(\widetilde{SO(4p+2)}/Z_4) = \frac{1}{2\lambda^2} \Lambda(\widetilde{SO(4p+2)})$$

The results confirm our general result ; $\beta(H) = N\Lambda(H^V)$ in the standard normalization since then, by (D.1), $(1/2\lambda^2) = 2(n-1)$ which is indeed N, by (7.3). Alternatively we see that if we had chosen the roots to have unit length, $2\lambda^2 = 1$, we would get $\beta(H) = \Lambda(H^V)$. This is valid in general if all the roots have equal length.

n is even

The universal covering group corresponding to $D(n)$, $n = 2p$ is the covering $\widetilde{SO}(4p)$, of the orthogonal group $SO(4p)$. Its centre is $Z_2 \otimes Z_2$. Since there are <u>five distinct</u> subgroups of $Z_2 \otimes Z_2$, namely the identity, three Z_2 type groups and $Z_2 \otimes Z_2$ itself, there are five different quotient groups. The weights of these quotient groups are $\sum_{i=1}^{n} m_i^{(k)} e_{(i)}$, with k given by

$$\widetilde{SO}(4p) \quad : \quad k = 0, 1, 2, 3$$

$$\widetilde{SO}(4p)/Z_2 \text{ Type I} \quad : \quad k = 0, 1$$

$$\widetilde{SO}(4p)/Z_2 \text{ Type II} \quad : \quad k = 0, 2$$

$$\widetilde{SO}(4p)/Z_2 \text{ Type III} \quad : \quad k = 0, 3$$

$$\widetilde{SO}(4p)/Z_2 \otimes Z_2 \quad : \quad k = 0$$

After some computation one finds that the magnetic charges β of these groups depend on the parity of p itself. Hence there are two subcases

(b1) : $p = 2\ell \qquad n = 2p = 4\ell$

$$\beta\left(\widetilde{SO}(8\ell)\right) = \frac{1}{2\lambda^2} \Lambda\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \otimes Z_2}\right)$$

$$\beta\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \text{ Type I}}\right) = \frac{1}{2\lambda^2} \Lambda\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \text{ Type I}}\right)$$

$$\beta\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \text{ Type II}}\right) = \frac{1}{2\lambda^2} \Lambda\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \text{ Type II}}\right)$$

$$\beta\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \text{ Type III}}\right) = \frac{1}{2\lambda^2} \Lambda\left(\frac{\widetilde{SO}(8\ell)}{Z_2 \text{ Type III}}\right)$$

(b2) $p = 2\ell + 1$, $n = 2p = 4\ell + 2$

$$\beta\left(\widetilde{SO(8\ell+4)}\right) = \frac{1}{2\lambda^2}\Lambda\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \otimes Z_2}\right)$$

$$\beta\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \text{ Type I}}\right) = \frac{1}{2\lambda^2}\Lambda\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \text{ Type III}}\right)$$

$$\beta\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \text{ Type II}}\right) = \frac{1}{2\lambda^2}\Lambda\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \text{ Type II}}\right)$$

$$\beta\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \text{ Type III}}\right) = \frac{1}{2\lambda^2}\Lambda\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \text{ Type I}}\right)$$

$$\beta\left(\frac{\widetilde{SO(8\ell+4)}}{Z_2 \otimes Z_2}\right) = \frac{1}{2\lambda^2}\Lambda\left(\widetilde{SO(8\ell+4)}\right)$$

Note the subtlety of the inter-relationships.

REFERENCES

1) P.A.M. Dirac - Proc.Roy.Soc. A133, 60 (1931).

2) G. 't Hooft - Nuclear Phys. B79, 276 (1976) ;
A.M. Polyakov - JETP Letters 20, 194 (1974).

3) M.I. Monastyrsky and A.M. Perelomov - JETP Letters 21, 43 (1975) ;
Yu.S. Tyupkin, V.A. Fateev and A.S. Shvarts - JETP Letters 21, 42 (1975) ;
S. Coleman - "Classical Lumps and their Quantum Descendants", Harvard Preprint

4) L.D. Faddeev - JETP Letters 21, 54 (1975).

5) See : S. Coleman, Ref. 3), for a review.

6) If G_{io}, $\mathcal{D}_i\psi_{Higgs}$, and any other fields vanish far away (1.2) certainly follows from (1.1) and the non-Abelian Maxwell equations.

7) F. Englert and P. Windey - Phys.Rev. D14, 2728 (1976).

8) E. Corrigan and D. Olive - Nuclear Phys. B110, 237 (1976).

9) N. Christ - Phys.Rev.Letters 34, 355 (1975) ;
J. Goldstone - Lectures in Cambridge University, unpublished;
See also : L. Schlesinger - Math.Ann. 99, 413 (1927).

10) T.T. Wu and C.N. Yang - Phys.Rev. D12, 3845 (1975).

11) G.P. Hochschild - "The Structure of Lie Groups" (Holden-Day) San Francisco (1965), p. 144.

12) W. Greub, S. Halperin and R. Vanstone - "Connections, Curvature and Cohomology II ; Lie Groups, Principal Bundles and Characteristic Classes" (Academic Press) New York, London (1973), p. 92.

13) J.E. Humphreys - "Introduction to Lie Algebras and Representation Theory" (Springer Verlag), New York, Heidelberg, Berlin (1972).

14) L. Pontryagin - "Topological Groups" (Princeton University Press) (1946), Ch. 8.

15) L.D. Faddeev - "Some Comments on the Many Dimensional Solitons", CERN Preprint TH 2188 (1976).

16) E.B. Bogomolny - Soviet J.Nucl.Phys. 24, 861 (1976).

17) H. Fritzsch and P. Minkowski - "SU(3) as Gauge Group of the Vector-like Weak and Electromagnetic Interactions", CERN Preprint TH 2165 (1976).