Quantum algorithms for computing short discrete logarithms and factoring RSA integers

Martin Ekerå[1,2] Johan Håstad[1]

[1] KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden

[2] Swedish NCSA, Swedish Armed Forces, SE-107 85 Stockholm, Sweden

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SWEDISH ARMED FORCES

Introduction

Our contribution

- We modify Shor's algorithms to more efficiently solve
  - the *short* discrete logarithm problem
  - the *RSA* integer factoring problem

- The main hurdle is to exponentiate group elements. We shorten the exponents.

The integer factoring problem

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- Given an integer N compute its prime factors.

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- We focus on the RSA IFP since it is of cryptographic significance.

The discrete logarithm problem

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- Given a generator g of some group $\mathbb{G}$ and $x = g^d$ compute $d = \log_g x$.

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The short discrete logarithm problem (short DLP)

- $d \lll r$ where r is the order of $\mathbb{G}$
- r may be assumed known or unknown

Reasons for studying the short DLP

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1. The RSA IFP may be reduced to the short DLP.
2. The short DLP arises in some parameterizations of DLP-based schemes.

Reducing RSA IFP to a short DLP [HSS93]

1. Let $N = pq$ be the RSA integer to be factored.
2. Pick a random $g \in \mathbb{Z}_N^*$. Compute
$$x = g^N \equiv g^{p+q-1} \quad \text{since the order of } \mathbb{Z}_N^* \text{ is} \quad pq - p - q + 1.$$
3. Compute $d = p + q - 1$ given g and x.
4. Solve $N = pq$ and $d = p + q - 1$ for p and q.

- An RSA IFP may be reduced to a short DLP in a group of unknown order.

Domain parameters for DLP-based schemes

Group	Prime p	Order r	Exponent d	Classical security
Elliptic curve $E(\mathbb{F}_p)$	200	200	200	100
Safe-prime $\mathbb{G} \subset \mathbb{F}_p^*$	2048	2047	2047	* 100
— short d	2048	2047	200	* 100
Schnorr $\mathbb{G} \subset \mathbb{F}_p^*$	2048	200	200	* 100

* ballpark figure — various models exist for estimating these security levels

- The short DLP arises when short exponents are used with safe-prime groups.
- Important to understand quantum implications of parameterization choices.

Shor's algorithms [Shor94]

Shor's algorithms

- Shor's algorithms solve the IFP and the DLP in $\mathbb{F}_p^*$.
- May be generalized to solve the DLP in any finite cyclic group.

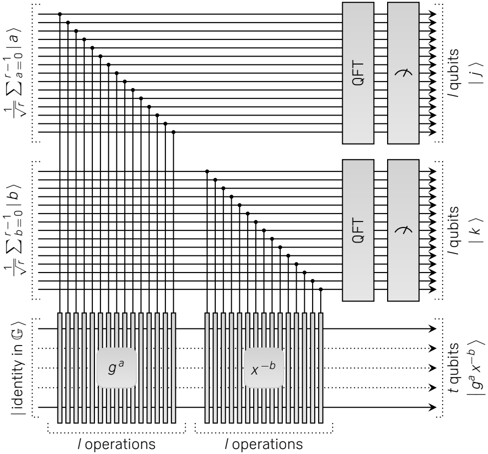
Shor's algorithm for the DLP [Shor94]

1. Compute the superposition

$$\frac{1}{r}\sum_{a=0}^{r-1}\sum_{b=0}^{r-1}\left|\,a,\,b,\,g^{a}\,x^{-b}\,\right\rangle$$

   where $\langle\, g\,\rangle = \mathbb{G}$ of order $r \sim 2^l$.

2. Compute two QFTs of size r.
3. Observe frequencies j and k.
4. Solve $dj + k \equiv 0 \pmod{r}$.



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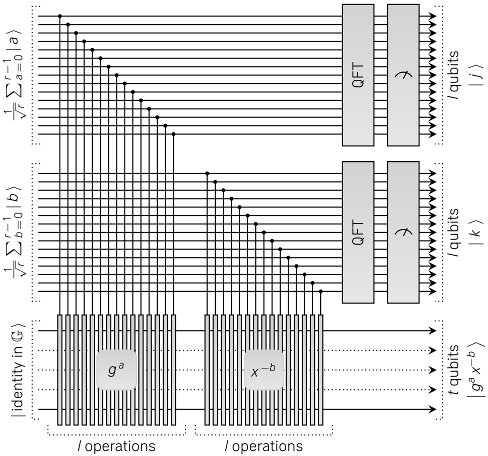

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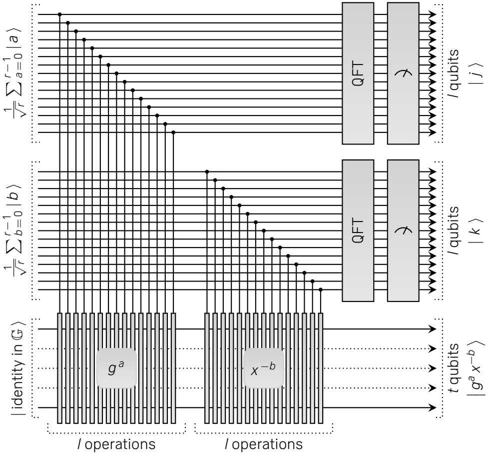
4. Solving for d yields

$$d \equiv \left\lfloor \frac{kr}{2^l} \right\rceil z^{-1} \pmod r \quad \text{where} \quad z = \frac{\{jr\}_{2^l} - jr}{2^l} \in \mathbb{Z}.$$



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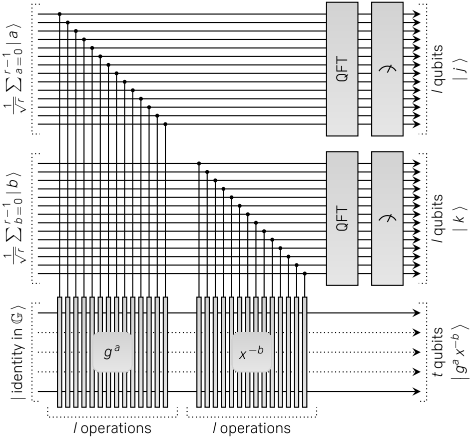

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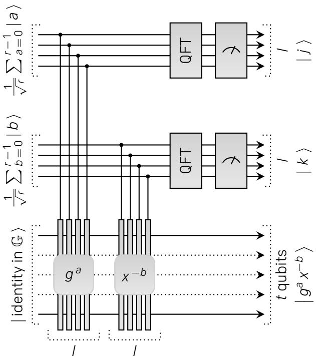

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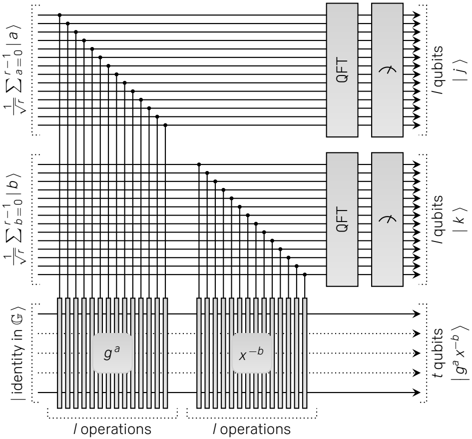

Our algorithm for the short DLP

Our improvements

1. We make the exponent length depend on d.
2. We enable tradeoffs between the exponent length and the number of runs.

- This parallels Seifert's modification [Seifert01] of Shor's order finding algorithm.

- We provide a full analysis of the algorithm and rigorous proofs.

Our algorithm for the short DLP [Ekerå16] — *single pair*

1. Compute the superposition

$$\frac{1}{\sqrt{2^{3m}}} \sum_{a=0}^{2^{2m}-1} \sum_{b=0}^{2^m-1} |\, a,\, b,\, g^a x^{-b} = g^{a-bd} \,\rangle$$

where $\langle\, g \,\rangle = \mathbb{G}$ of order r and $d < 2^m \lll r$.

2. Compute QFTs of size 2^{2m} and 2^m.

3. Observe frequencies j and k.

4. Solve $|\, \{dj + 2^m k\}_{2^{2m}} \,| \leq 2^{m-2}$ for d.

   The probability of a good pair is $\geq 1/8$.
   Need a single good pair to solve for d.
   The order r may be unknown.

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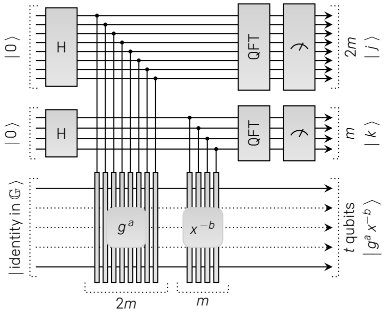

Our algorithm for the short DLP — *multiple pairs*

1. Compute the superposition

$$\frac{1}{\sqrt{2^{2\ell+m}}} \sum_{a=0}^{2^{\ell+m}-1} \sum_{b=0}^{2^{\ell}-1} \left| a,\, b,\, g^{a} x^{-b} = g^{a-bd} \right\rangle$$

where $d < 2^m \lll r$ and $\ell \approx m/s$ for small s.

2. Compute QFTs of size $2^{\ell+m}$ and 2^{ℓ}.

3. Observe frequencies j and k.

Expect $\left| \{dj + 2^m k\}_{2^{\ell+m}} \right| \leq 2^{m-2}$.

The probability of a good pair is $\geq 1/8$.
Need at least s good pairs to solve for d.
The order r may be unknown.

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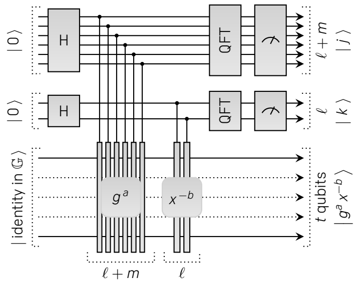
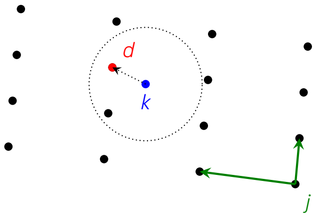


Classical post-processing



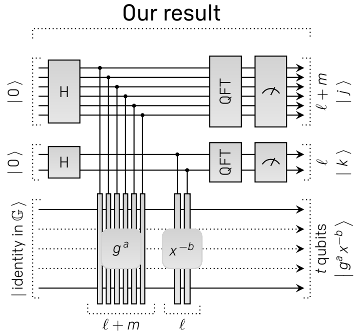
Classical post-processing

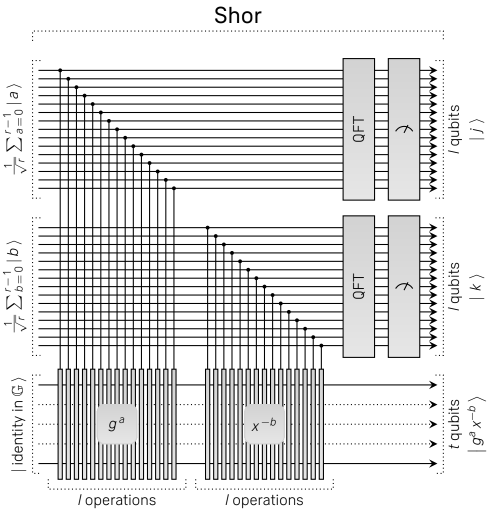
- Solve s good pairs (j, k) for d using lattice-based techniques.

- For *provable* success, execute cs times and solve all subsets of s pairs.

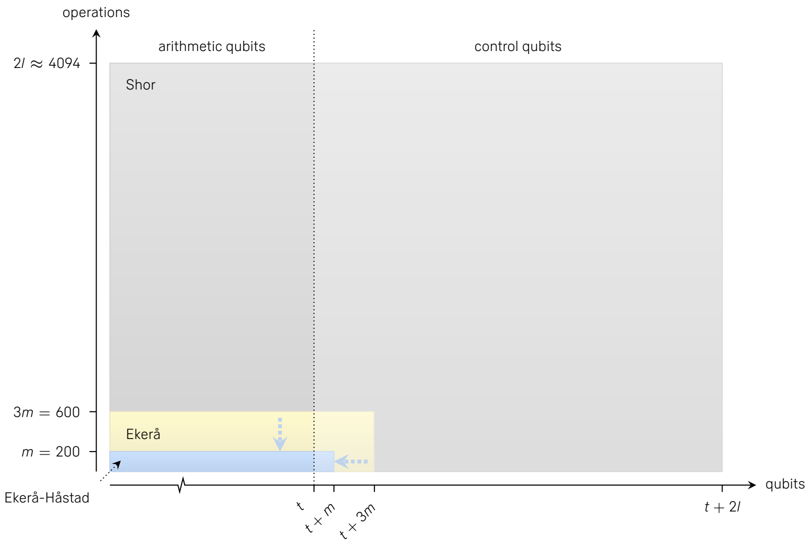
- In *practice* the condition on (j, k) may be relaxed. May trade radius for dimension.

Our advantage when solving an m bit short DLP





Short $m = 200$ bit DLP in safe-prime group $\mathbb{G} \subset \mathbb{F}_p^*$ for $p \approx 2^{2048}$



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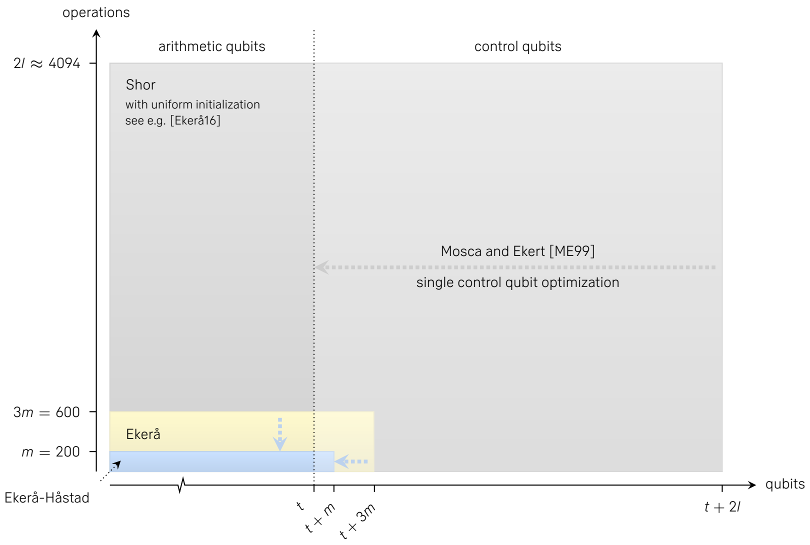


Shor's algorithm for the IFP [Shor94]

Shor's algorithm for the IFP

- Factors N by computing the order r of a random element $g \in \mathbb{Z}_N^*$.

Shor's order finding algorithm [Shor94] – *factoring* $N \in \mathbb{Z}$

1. Compute the superposition

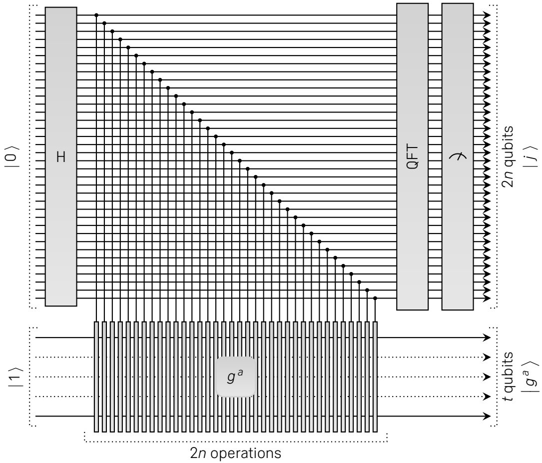
$$\frac{1}{2^n} \sum_{a=0}^{2^{2n}-1} \left| a,\, g^{a} \right\rangle$$

   where $g \in \mathbb{Z}_N^*$ and $n \sim \log_2 N$.

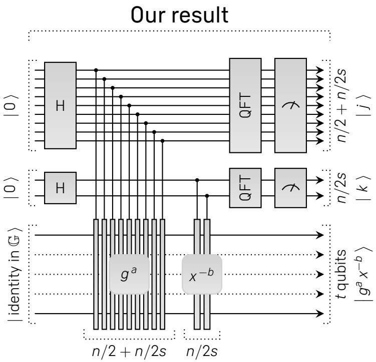
2. Compute a QFT of size 2^{2n}.
3. Observe frequency j.
4. Expect

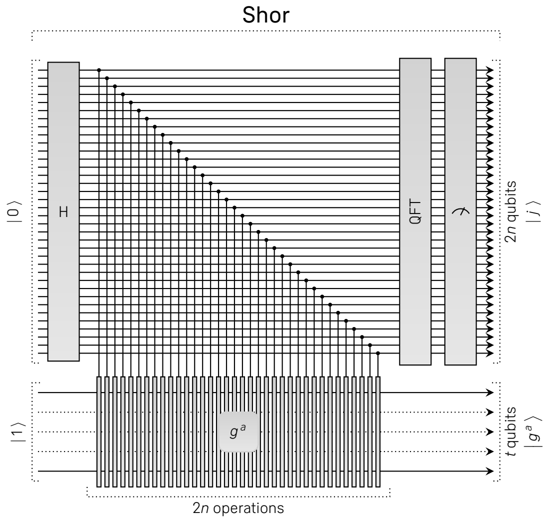
$$\frac{z}{r} \approx \frac{j}{2^{2n}} \quad \text{for some} \quad z \in \mathbb{Z}.$$

   Solve via continued fractions expansion.



Our advantage when solving an n bit RSA IFP





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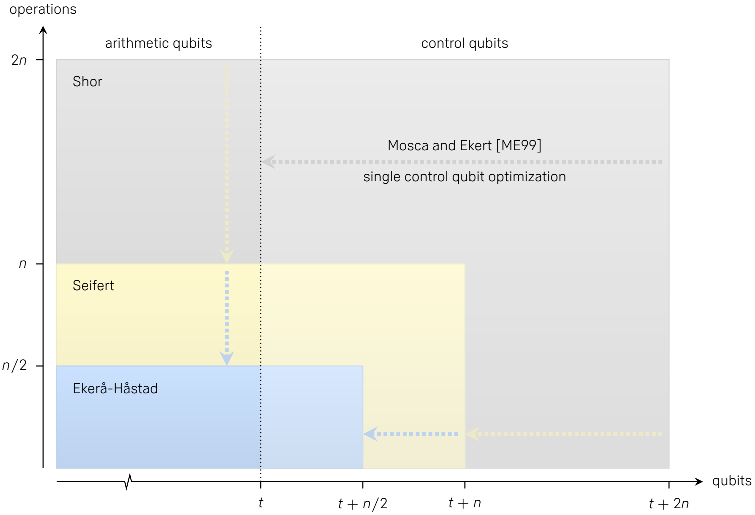

Summary and conclusion

Solving short m bit DLP

- Exponent reduced to $m + 2m/s$ bits for small $s \geq 1$.

- The group order may be unknown.

Factoring n bit RSA integers

- Exponent reduced from $2n$ bits to $n/2 + n/s$ bits for small $s \geq 2$.

- Reduced number of group operations, circuit depth, execution and coherence times.
- May result in a reduced number of control qubits.

Summary and conclusion

Implications for parameterization

- Safe-prime groups with short $d \sim 2^m$ yield $m + 2m/s$ bit exponents.
- Schnorr groups of order $r \sim 2^m$ yield $2m$ bit exponents.
  - Expect reduction to $m + 2m/s$ using tradeoffs.

- Not a reason to prefer safe-prime groups with short d over Schnorr groups.

Additional contributions

- We provide a full analysis of the algorithm and rigorous proofs.

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