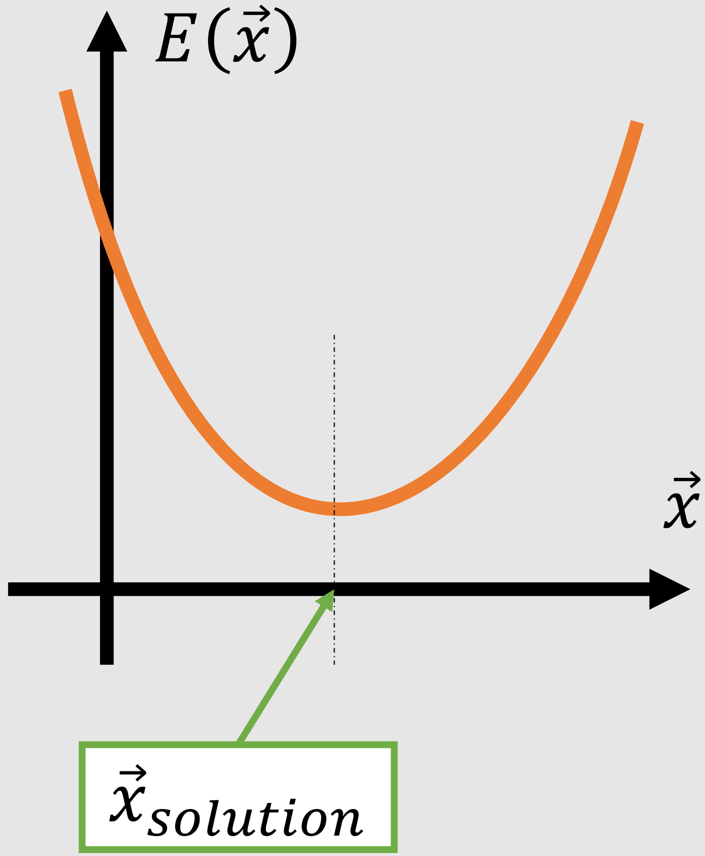
Variational Backward Euler Time Integration

What is Variational Method?

- Solution is expressed by the optimization

$$\vec{X}_{solution} = \underset{\vec{x}}{\operatorname{argmin}} E(\vec{x})$$

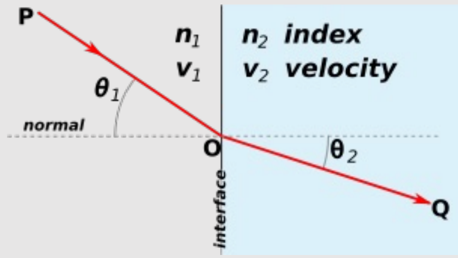


Variational Principles in Physics

- Mechanics

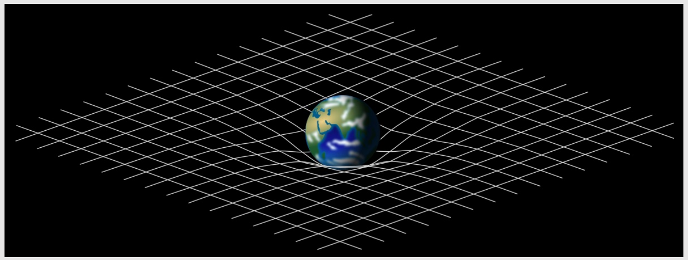
(Wikipedia)

- Optics



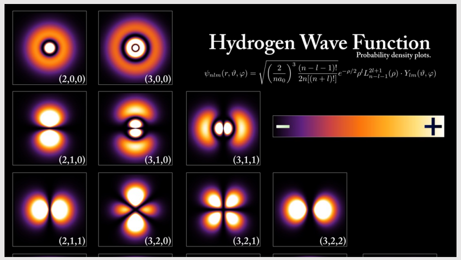
(Wikipedia)

- General relativity



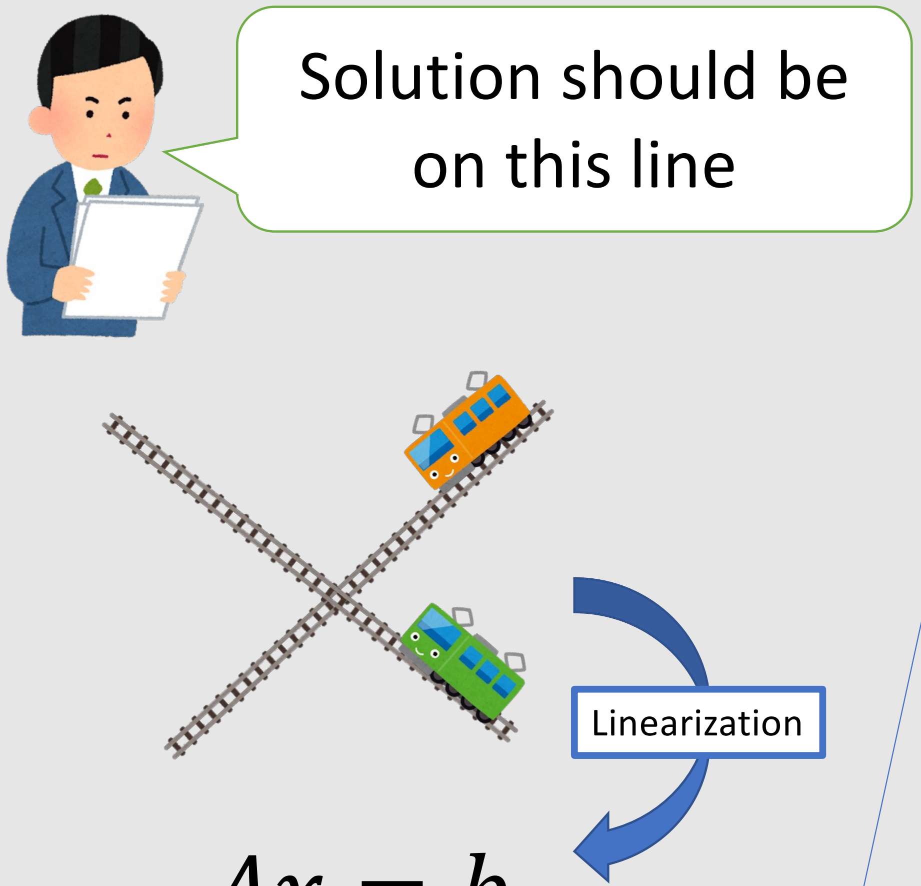
(Wikipedia)

- Quantum physics

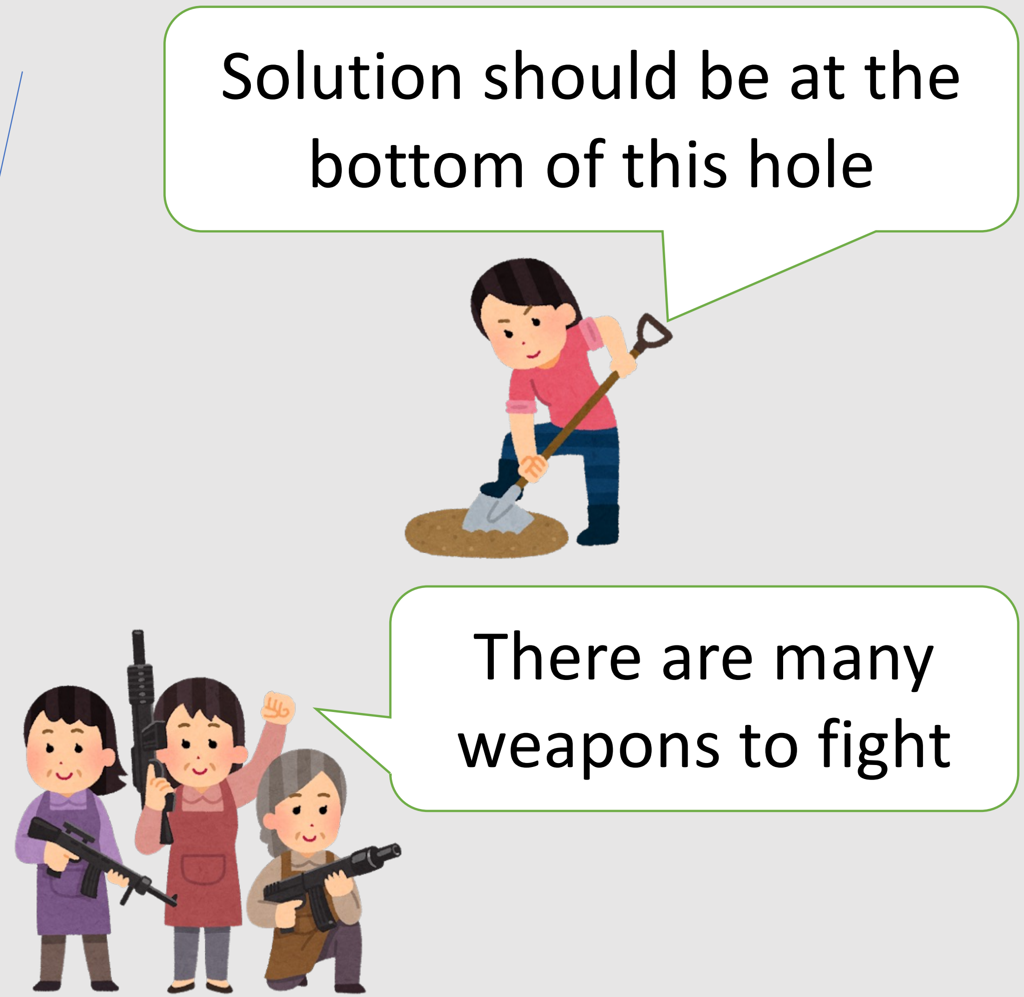


(Wikipedia)

Solving Constraints v.s. Variational Problem

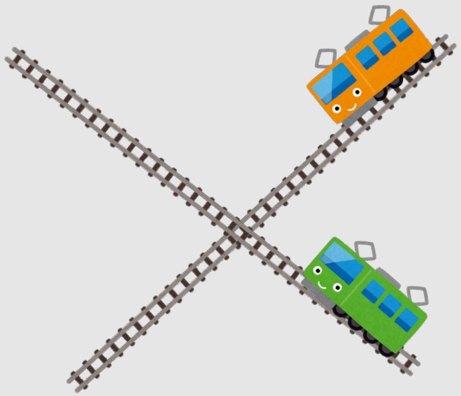


$$Ax = b$$



Solving Constraints v.s. Variational Problem

Solution should be on this line



Solution should be at the bottom of this hole

How can we conver the problem?

Making a Variational Problem

- We only need a single scalar value E to find solution

$$A\vec{x} = \vec{b}$$

L2 norm of residual

$$E(\vec{x}) = \left\|A\vec{x} - \vec{b}\right\|^2$$

integration

$$E(\vec{x}) = \frac{1}{2}\vec{x}^T A\vec{x} - \vec{b}^T\vec{x}$$

Making a Variational Problem

- Integration with $\vec{x}$ will make a variational formula

$$\frac{\partial W(\vec{x})}{\partial \vec{x}} = \vec{b} \xrightarrow{\text{integration}} E(\vec{x}) = W(\vec{x}) - \vec{b}^T \vec{x}$$

$$\frac{\partial W(\vec{x})}{\partial \vec{x}} = -M\vec{x} \xrightarrow{\text{integration}} E(\vec{x}) = W(\vec{x}) + \frac{1}{2} \vec{x}^T M \vec{x}$$

Variational Formulation of Backward Euler

- Review of Backward Euler

$$\frac{ds}{dt} = \frac{s_{i+1} - s_i}{dt} = F(s_{i+1})$$

plug in $s_i = \begin{pmatrix} \vec{v}_\mathrm{i} \\ \vec{x}_\mathrm{i} \end{pmatrix}$, $M\dot{\vec{v}} = \frac{\partial W}{\partial \vec{x}}$

$$\begin{cases} \vec{x}_{i+1} = \vec{x}_i + dt \cdot \vec{v}_i + dt^2 \cdot M^{-1} \dfrac{\partial W}{\partial x_{i+1}} \\ \vec{v}_{i+1} = (\vec{x}_{i+1} - \vec{x}_i)/dt \end{cases}$$

Variational Formulation of Backward Euler

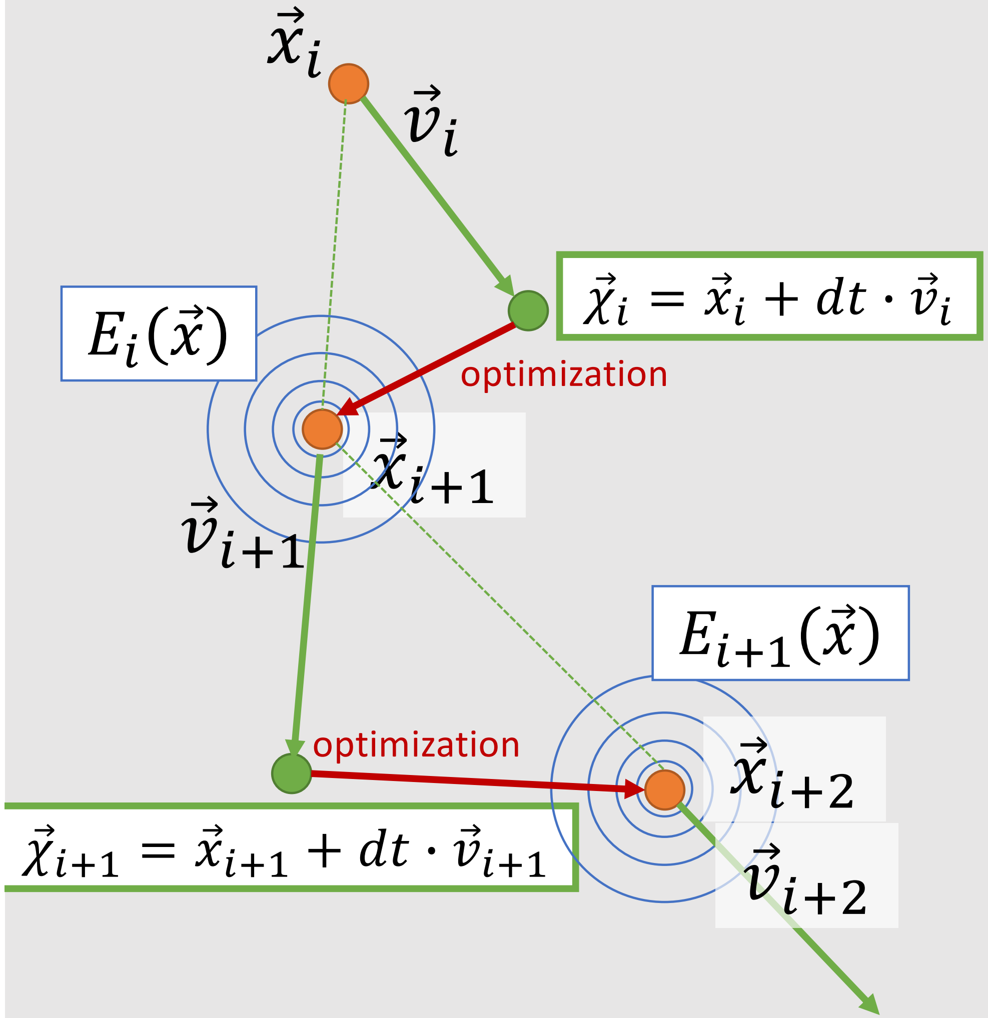
- Getting next time step by minimization

integration

$$
\begin{cases}
\vec{x}_{i+1} = \operatorname*{argmin}_{\vec{x}} E_i(\vec{x}) \\
E_i(\vec{x}) = \mathrm{W}(\vec{x}) + \dfrac{1}{2dt^2}(\vec{x} - \vec{\chi}_i)^T M(\vec{x} - \vec{\chi}_i) \\
\vec{\chi}_i = \vec{x}_i + dt \cdot \vec{v}_i
\end{cases}
$$

$$
\begin{cases}
\vec{x}_{i+1} = \vec{x}_i + dt \cdot \vec{v}_i + dt^2 \cdot M^{-1} \dfrac{\partial W}{\partial x_{i+1}} \\
\vec{v}_{i+1} = (\vec{x}_{i+1} - \vec{x}_i)/dt
\end{cases}
$$

Scheme of Variational Backward Euler



1. compute temporary position

$$\vec{\chi}_i = \vec{x}_i + dt \cdot \vec{v}_i$$

2. optimize $E_i(\vec{x})$ to get $\vec{x}_{i+1}$

3. Set velocity

$$\vec{v}_{i+1} = \frac{(\vec{x}_{i+1} - \vec{x}_i)}{dt}$$

4. Goto 1

Variational Formula Explained

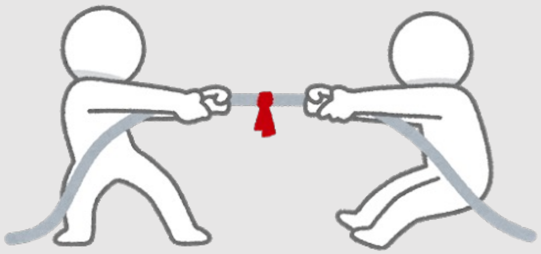
- Solving tradeoff between elasticity & inertia

$$\vec{\chi}_i = \vec{x}_i + dt \cdot \vec{v}_i$$

$$E_i(\vec{x}) = \underbrace{\mathrm{W}(\vec{x})}_{\text{elasticity}} + \underbrace{\frac{1}{2dt^2}(\vec{x} - \vec{\chi}_i)^T M(\vec{x} - \vec{\chi}_i)}_{\text{inertia}}$$

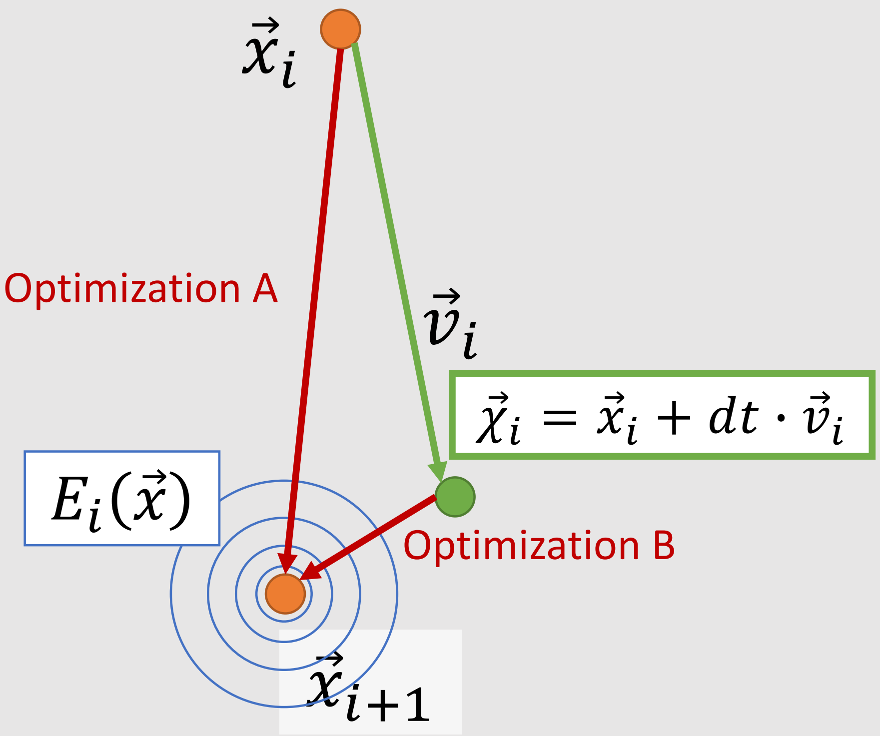
elasticity: Trying to "undeform" shape

inertia: Trying to move shape with velocity $\vec{v}_i$



Optimization with Newton Method

- optimize $E_i(\vec{x}) = \mathrm{W}(\vec{x}) + 1/2dt^2\,(\vec{x} - \vec{\chi}_i)^T M(\vec{x} - \vec{\chi}_i)$ to get $\vec{x}_{i+1}$



Optimization A (bad☹)

$$\vec{x}_{i+1} = \vec{x}_i - \left[\frac{\partial^2 W(\vec{x}_i)}{\partial^2 \vec{x}}\right]^{-1} \left(\frac{\partial W(\vec{x}_i)}{\partial \vec{x}}\right)$$

Optimization B (good ☺)

$$\vec{x}_{i+1} = \vec{\chi}_i - \left[\frac{\partial^2 W(\vec{\chi}_i)}{\partial^2 \vec{x}}\right]^{-1} \left(\frac{\partial W(\vec{\chi}_i)}{\partial \vec{x}}\right)$$