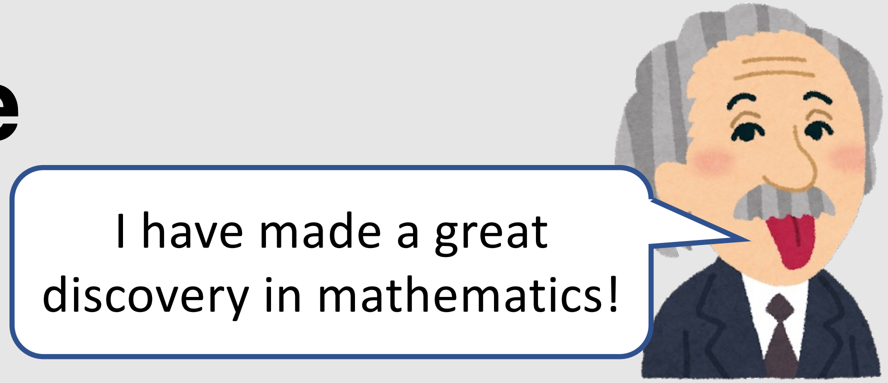
Einstein’s Summation Rule

- Repeated indices are summed over



inner product

$$a \cdot b = a_i b_i = \sum_i a_i b_i$$

Frobenius inner product

$$\langle A, B \rangle_F = A_{ij} B_{ij} = \sum_j \sum_i A_{ij} B_{ij}$$

Let's Practice Einstein's Summation Rule

$tr(A) = ?$

$(A^T B)_{ij} = ?$

$tr(A^T B) = ?$

$\left(\vec{a}^T \vec{b} I\right)_{ij} = ?$

$a_i b_i = ?$

$a_{ii} a_{jj} = ?$

$a_{ii} a_{ii} = ?$

Frobenius Inner Product $\langle \boldsymbol{A}, \boldsymbol{B} \rangle_{\boldsymbol{F}} = \boldsymbol{A}_{\boldsymbol{ij}} \boldsymbol{B}_{\boldsymbol{ij}}$

$$A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$$
$$B = \left[\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\right]$$

$$\langle A, B \rangle_F = \sum_i \vec{a}_i{}^T \vec{b}_i$$

$$\langle RA, B \rangle_F = R_{ik} A_{kj} B_{ij} = \sum_i (R\vec{a}_i)^T \vec{b}_i$$

$$= A_{kj} (R^T B)_{kj} = \langle A, R^T B \rangle_F = \sum_i \vec{a}_i{}^T R \vec{b}_i$$

$$= R_{ik} (B A^T)_{ik} = \langle R, B A^T \rangle_F = \left\langle R, \sum_i \vec{b}_i \otimes \vec{a}_i \right\rangle_F$$

Tensor

What is Tensor?

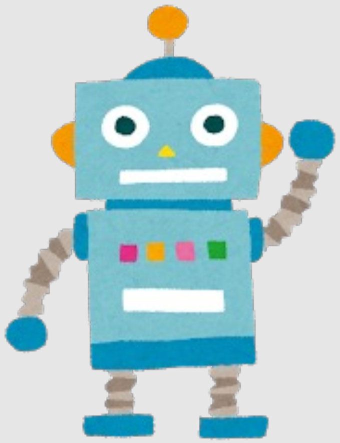
In mathematics, a tensor is an algebraic object that **describes a (multilinear) relationship** between sets of algebraic objects related to a vector space.

https://en.wikipedia.org/wiki/Tensor

scalar a

vector $\vec{u}$

tensor A



tensor

scalar b

vector $\vec{v}$

tensor B

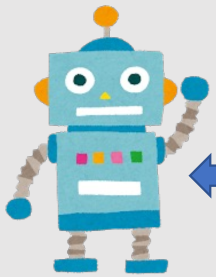
Two ways to Understand 2nd-order Tensor

- Transformation by a tensor is give by the inner product

Linear form

$$\vec{u} = A \cdot \vec{v}$$

vector $\vec{u}$



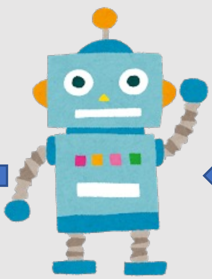
vector $\vec{v}$

tensor

Quadratic form

$$\mathrm{a} = \vec{u} \cdot (A \cdot \vec{v})$$

scalar a



vector $\vec{u}$

vector $\vec{v}$

tensor

Outer Product (Tensor Product)

- Outer product makes a tensor from two vectors

$$\vec{a} \otimes \vec{b} \quad \Rightarrow \quad \left(\vec{a} \otimes \vec{b}\right) \cdot \vec{u} = \vec{a}(\vec{b} \cdot \vec{u})$$

- Tensor product $\vec{e} \otimes \vec{e}$ $(\|e\| = 1)$ defines projection

check it out!

Definition

Projection $\mathcal{P}$

$\mathcal{P}\left(\mathcal{P}(x)\right) = \mathcal{P}$

Outer Product (Tensor Product)

- Transformation for vectors in the outer product

check it out!

$$(A\vec{a})\otimes(B\vec{b})=?$$

Definition

$$\vec{a}\otimes\vec{b}$$

$$\left(\vec{a}\otimes\vec{b}\right)\cdot\vec{u}=\vec{a}(\vec{b}\cdot\vec{u})$$

Tensor + Basis = Matrix

- Inner product with a basis vector gives a coefficient
  - This is true even if the basis is not orthonormal

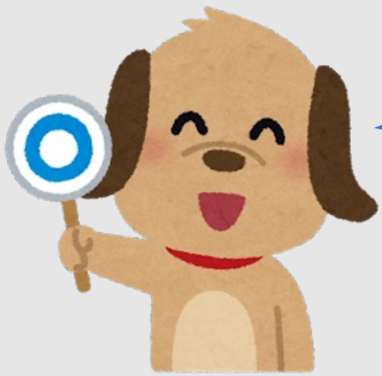
$$v_i = \vec{v} \cdot \vec{e}_i$$

$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

Tensor & Matrix: Common Misunderstanding

Wrong idea! Correct your thought!

Tensor = Basis × Matrix (coefficients)



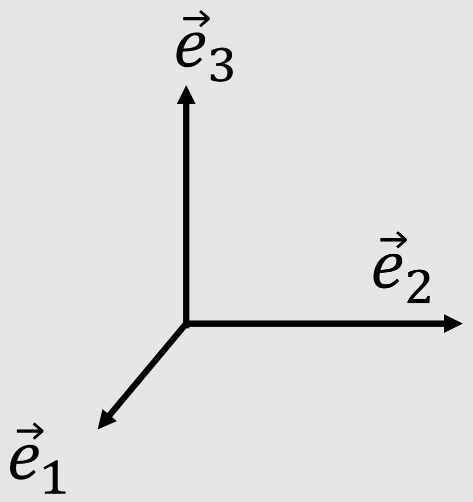
Nice ! Go ahead!

Tensor × Basis = Matrix (coefficients)

Orthonormal Coordinates

- Tensor can be written with bases and coefficients

$$e_i \cdot e_j = \delta_{ij}$$



$$v_i = \vec{v} \cdot \vec{e}_i$$

$$\Rightarrow \vec{v} = v_i \vec{e}_i$$

$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

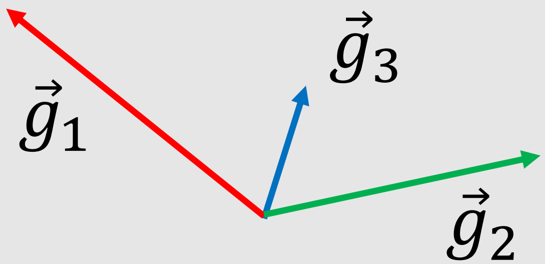
$$\Rightarrow A = a_{ij} \vec{e}_i \otimes \vec{e}_j$$

check it out!

Curvilinear Coordinates

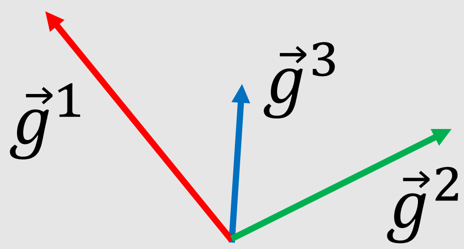
- Non-orthogonal and un-normalized bases
- Dual bases solve the problem

$$\vec{g}_i \cdot \vec{g}_j \neq \delta_{ij}$$



dual basis

$$\vec{g}_i \cdot \vec{g}^j = \delta_i^j$$



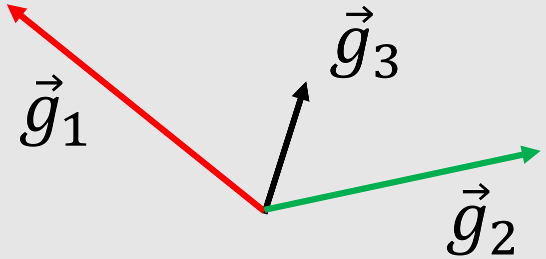
$$\vec{g}^1 = \frac{\vec{g}_2 \times \vec{g}_3}{\vec{g}_2 \cdot (\vec{g}_2 \times \vec{g}_3)}$$

$$\vec{g}^2 = \frac{\vec{g}_3 \times \vec{g}_1}{\vec{g}_3 \cdot (\vec{g}_3 \times \vec{g}_1)}$$

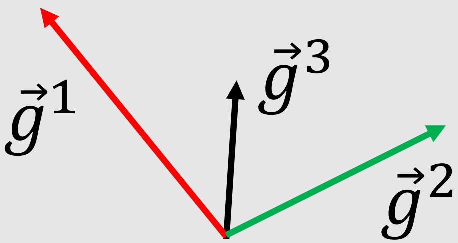
$$\vec{g}^3 = \frac{\vec{g}_1 \times \vec{g}_2}{\vec{g}_1 \cdot (\vec{g}_1 \times \vec{g}_2)}$$

Curvilinear Coordinates

- Expression of a vector in curvilinear coordinates



$\boxed{\text{dual basis}}$ $g_i \cdot g^j = \delta_i^j$

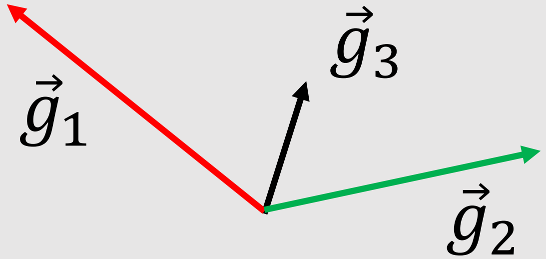


$$v^i = \vec{v} \cdot \vec{g}^i \quad \Rightarrow \quad \vec{v} = v^i \vec{g}_i$$

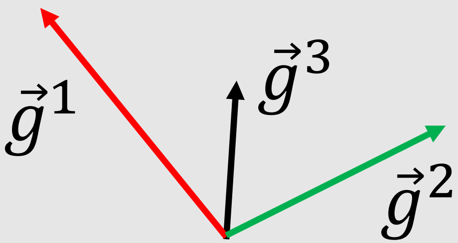
$$v_i = \vec{v} \cdot \vec{g}_i \quad \Rightarrow \quad \vec{v} = v_i \vec{g}^i$$

Curvilinear Coordinates

- Expression of a tensor in curvilinear coordinates



dual basis $g_i \cdot g^j = \delta_i^j$



$$a^{ij} = \vec{g}^i \cdot (A \cdot \vec{g}^j) \Rightarrow A = a^{ij} \vec{g}_i \otimes \vec{g}_j$$

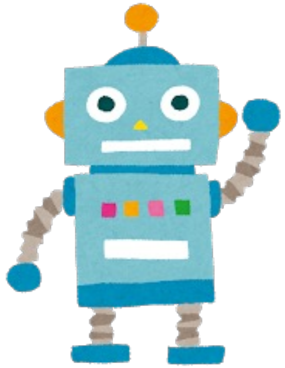
$$a_{ij} = \vec{g}_i \cdot (A \cdot \vec{g}_j) \Rightarrow A = a_{ij} \vec{g}^i \otimes \vec{g}^j$$

$$a^i{}_j = \vec{g}^i \cdot (A \cdot \vec{g}_i) \Rightarrow A = a^i{}_j \vec{g}_i \otimes \vec{g}^j$$

$$a_i^{\ j} = \vec{g}_i \cdot (A \cdot \vec{g}^j) \Rightarrow A = a_i^{\ j} \vec{g}^i \otimes \vec{g}_j$$

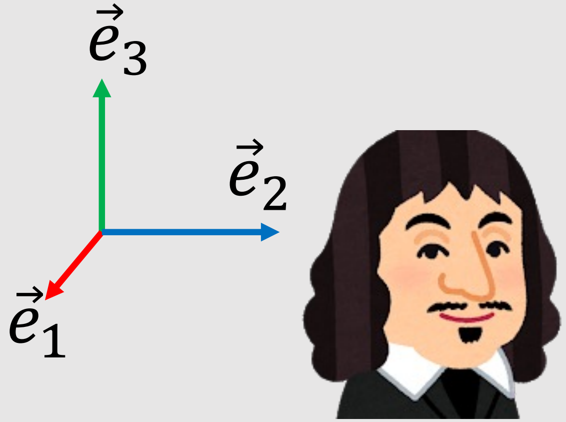
Coordinate Transformation

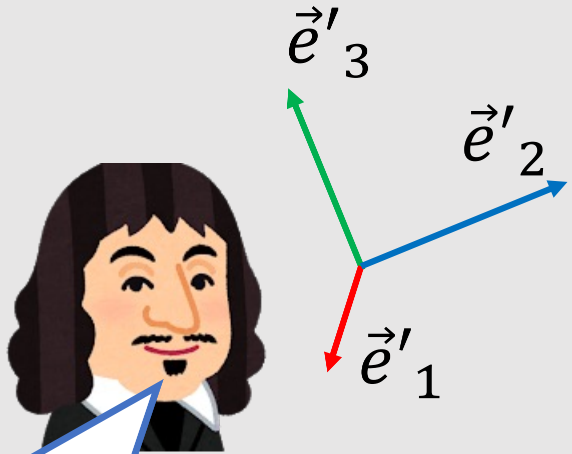
tensor: A



$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

$$a'_{ij} = \vec{e}'_i \cdot (A \cdot \vec{e}'_j)$$

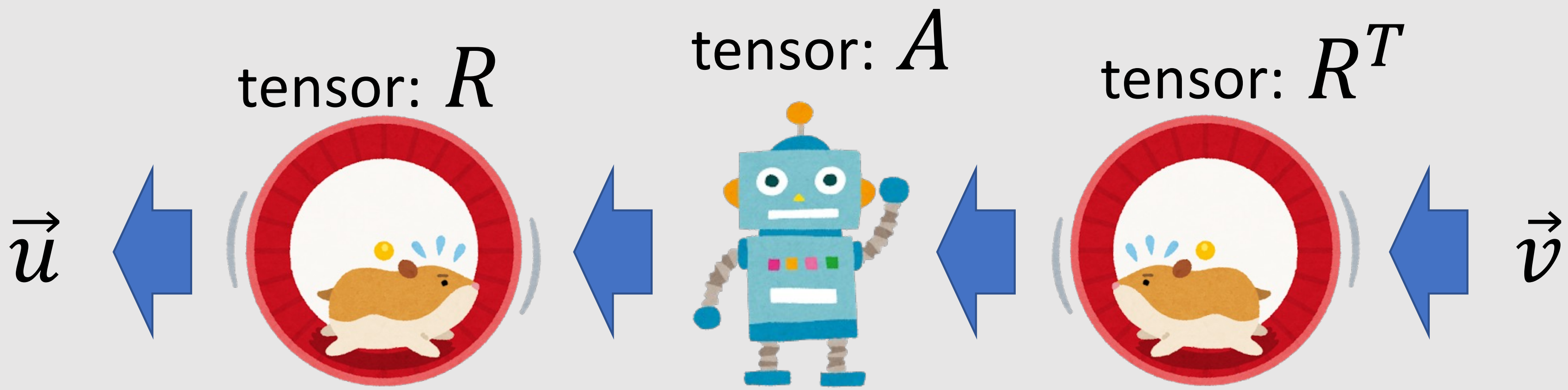




same tensor, different (coefficient) matrix!

Rotation of a Tensor

- Un-rotating input and rotating output



$$\vec{u} = RAR^T\vec{v}$$

Rotation of a Tensor

check it out!

- Rotating bases while using the same coefficients

$$A = a_{ij}\vec{e}_i \otimes \vec{e}_j$$

rotation of basis

$$\begin{aligned} A' &= a_{ij}(R\vec{e}_i) \otimes (R\vec{e}_j) \\ &= a_{ij}R(\vec{e}_i \otimes \vec{e}_j)R^T \\ &= R(a_{ij}\vec{e}_i \otimes \vec{e}_j)R^T \\ &= RAR^T \end{aligned}$$

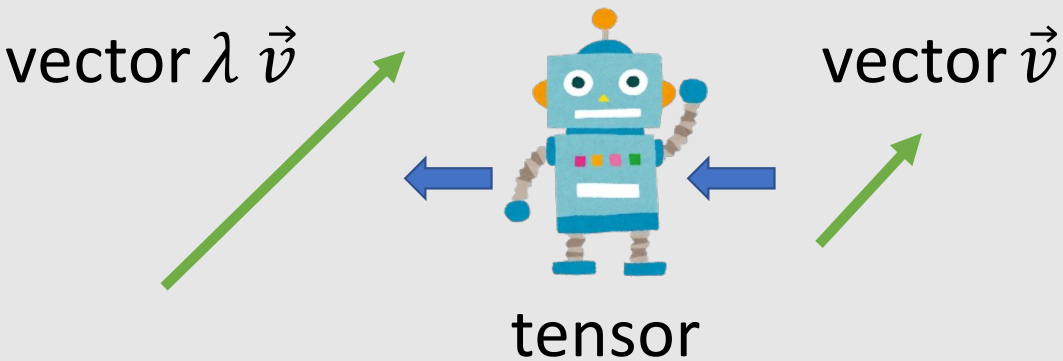
Simple Elastic Potential Energy for Continuum

Eigenvalue of Symmetric Tensor

- Eigenvalue of tensor is defined without matrix & coordinate

Linear form

$$\lambda\vec{v} = A \cdot \vec{v}$$



Eigenvalues and Frobenius Norm

from characteristic equation

$$\boxed{\mathrm{tr(A)} = \lambda_1 + \lambda_2 + \lambda_3}$$

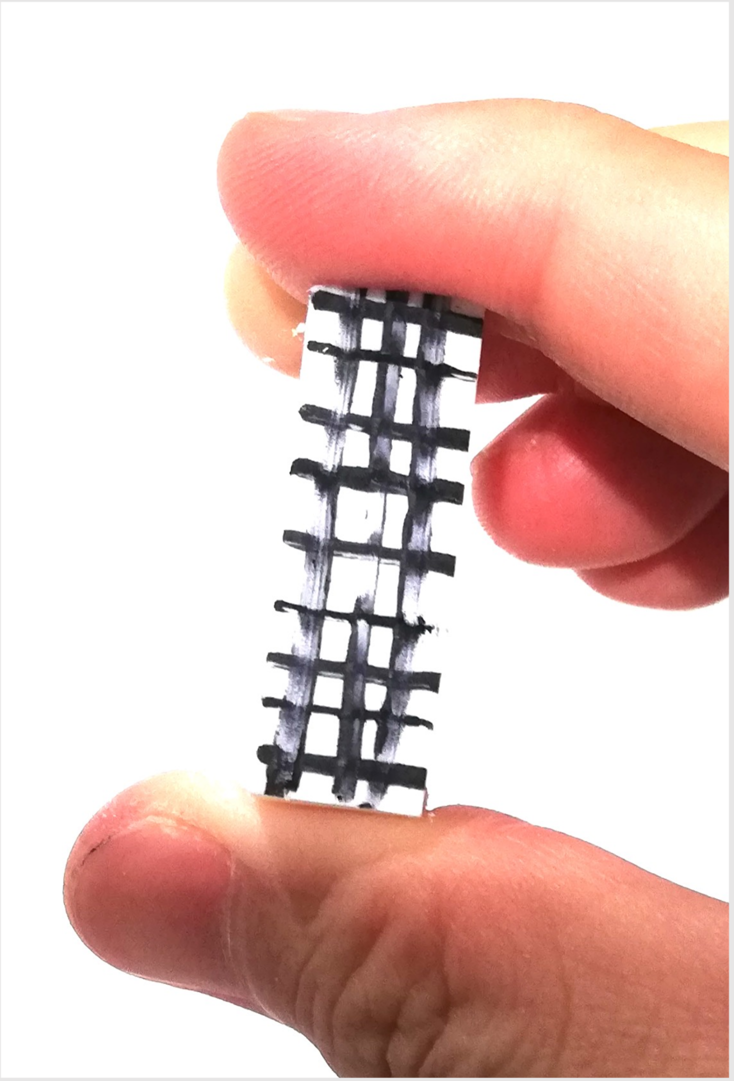
$$A^2\vec{e} = \lambda^2\vec{e}$$

$$\mathrm{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

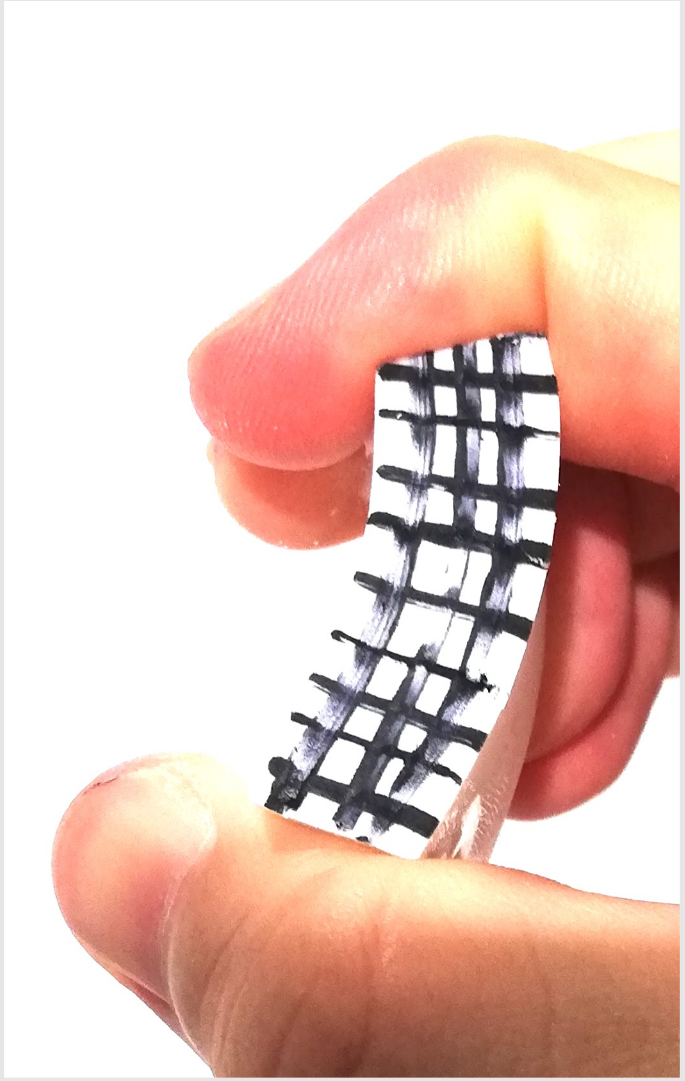
$$= \mathrm{tr}(A^T A) = \sum_{1 \le i,j \le 3} a_{ij}^2 = \|A\|_F^2$$

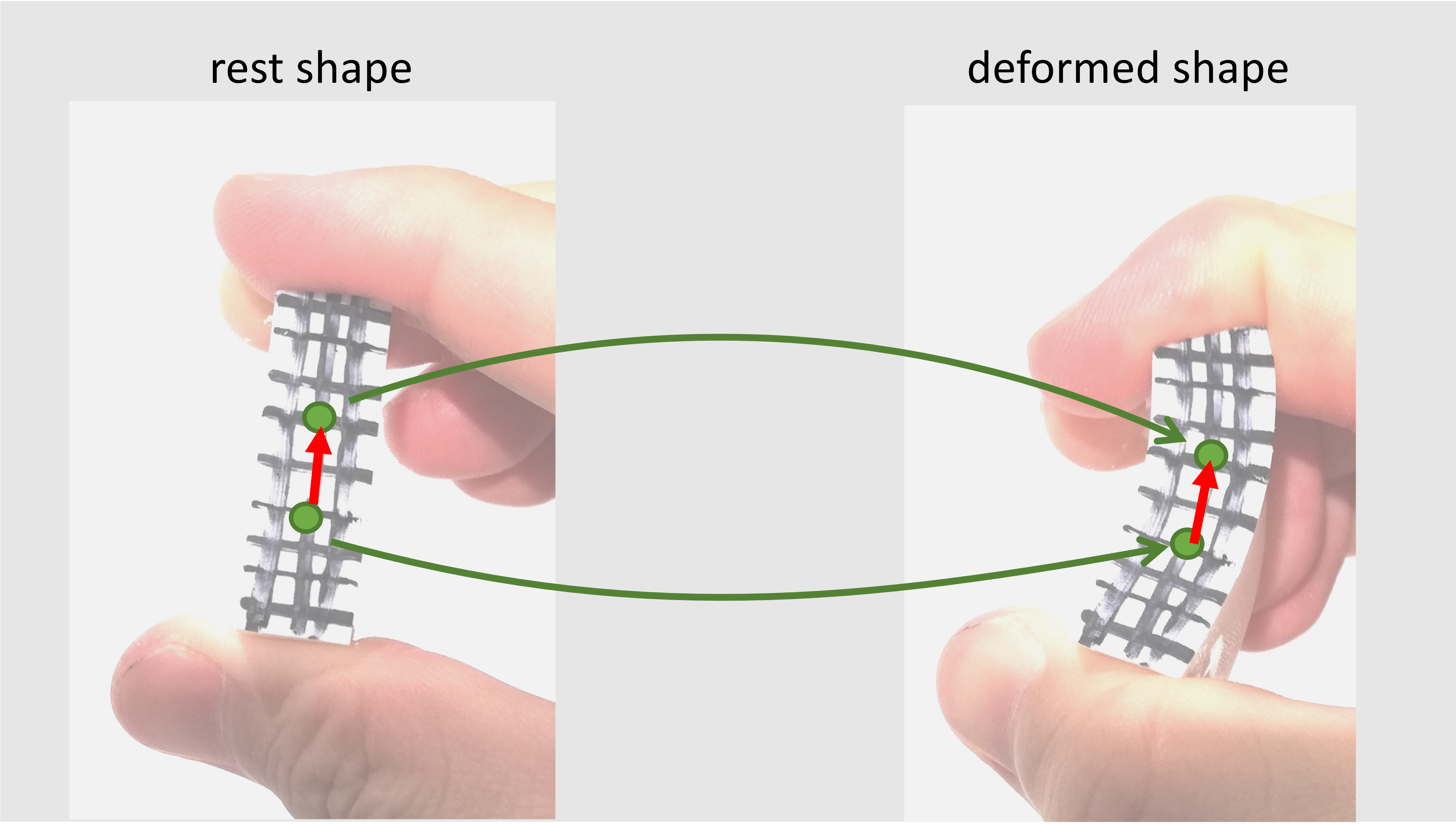
Frobenius norm

rest shape

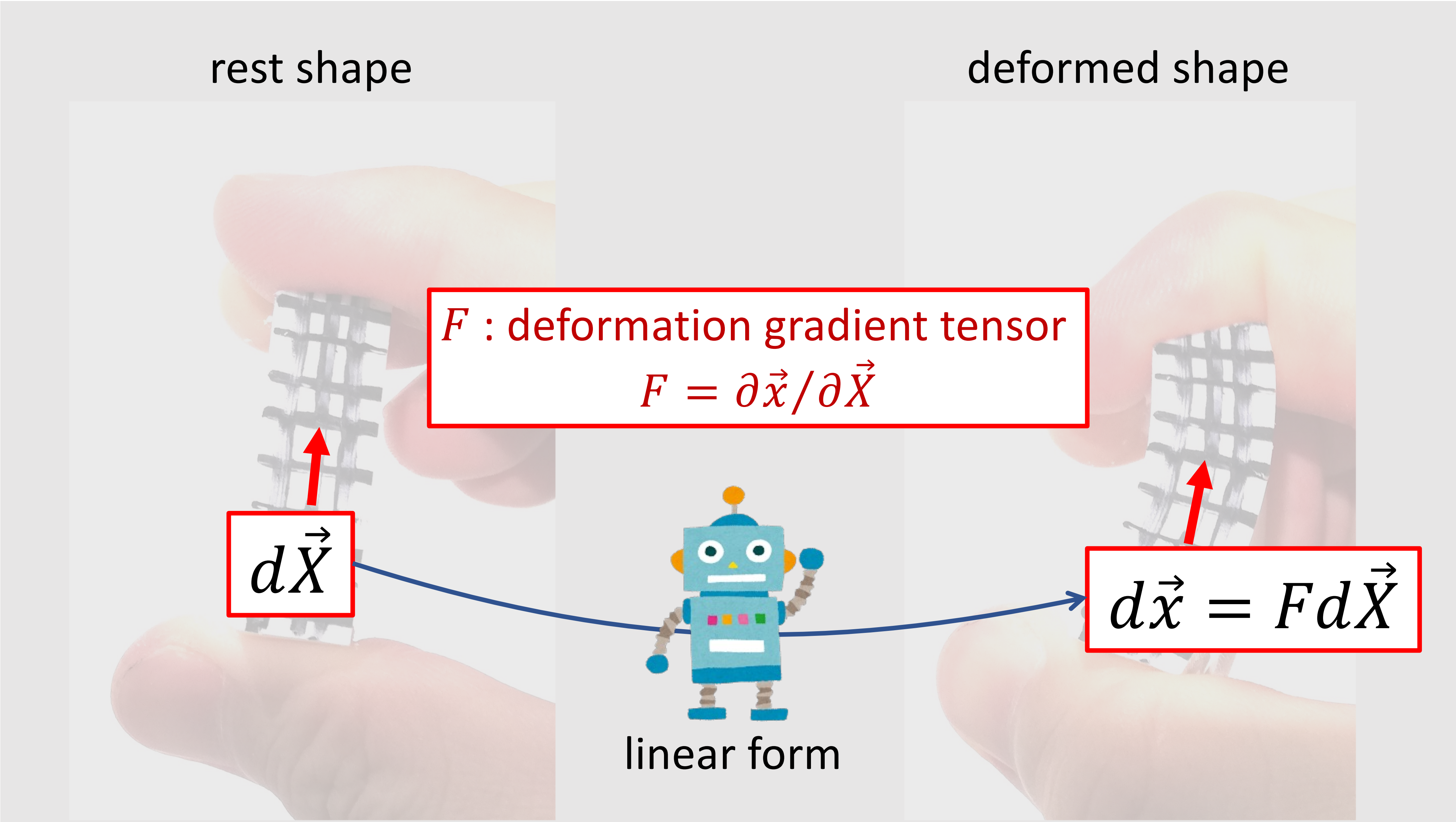


deformed shape



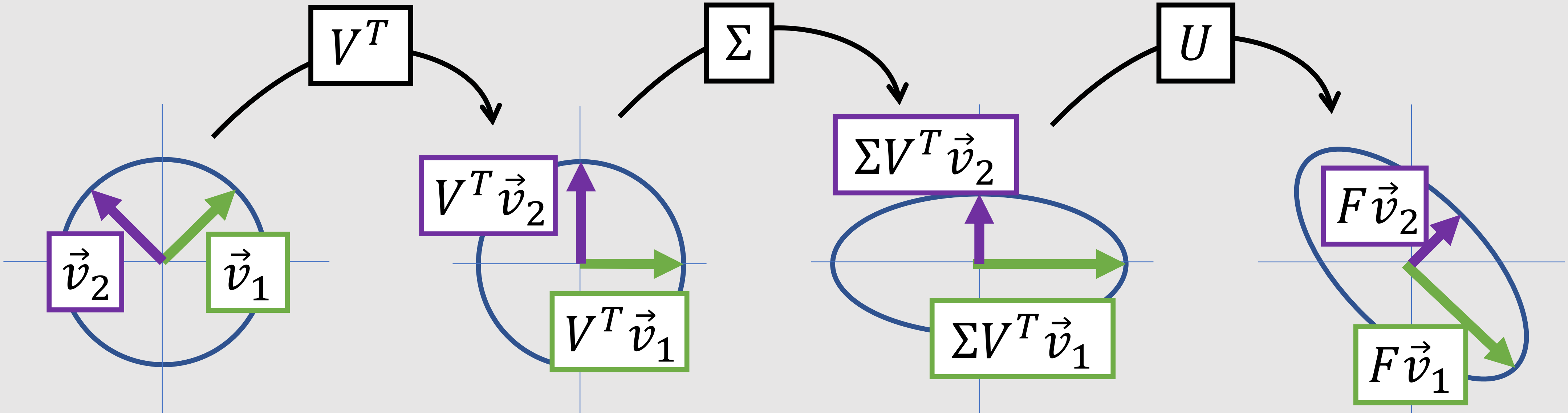


rest shape
deformed shape


rest shape
deformed shape
F : deformation gradient tensor
$F = \partial\vec{x}/\partial\vec{X}$
$d\vec{X}$
$d\vec{x} = Fd\vec{X}$
linear form

SVD of Deformation Gradient Tensor

$$F = U\Sigma V^T \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$



$$V = [\vec{v}_1, \vec{v}_2]$$

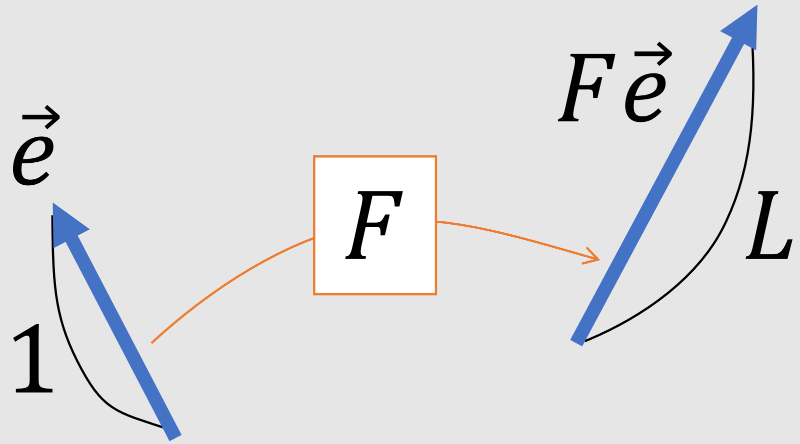
SVD of Deformation Gradient Tensor

$$F = U\Sigma V^T \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$

- Σ is the value we want for energy
- but SVD is costly
- How can we obtain Σ without SVD?

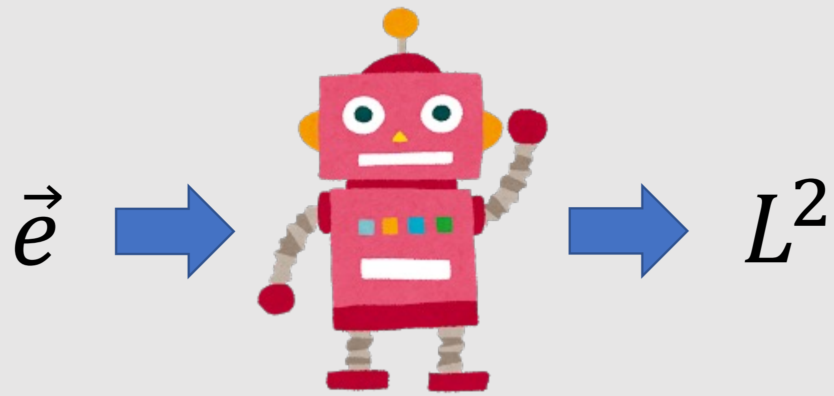
Gram Matrix $F^T F$ Stands for Length Change

- $C = F^T F$: right Cauchy-Green tensor



$$\boxed{L^2(\vec{e}) = \vec{e}^T F^T F \vec{e}}$$

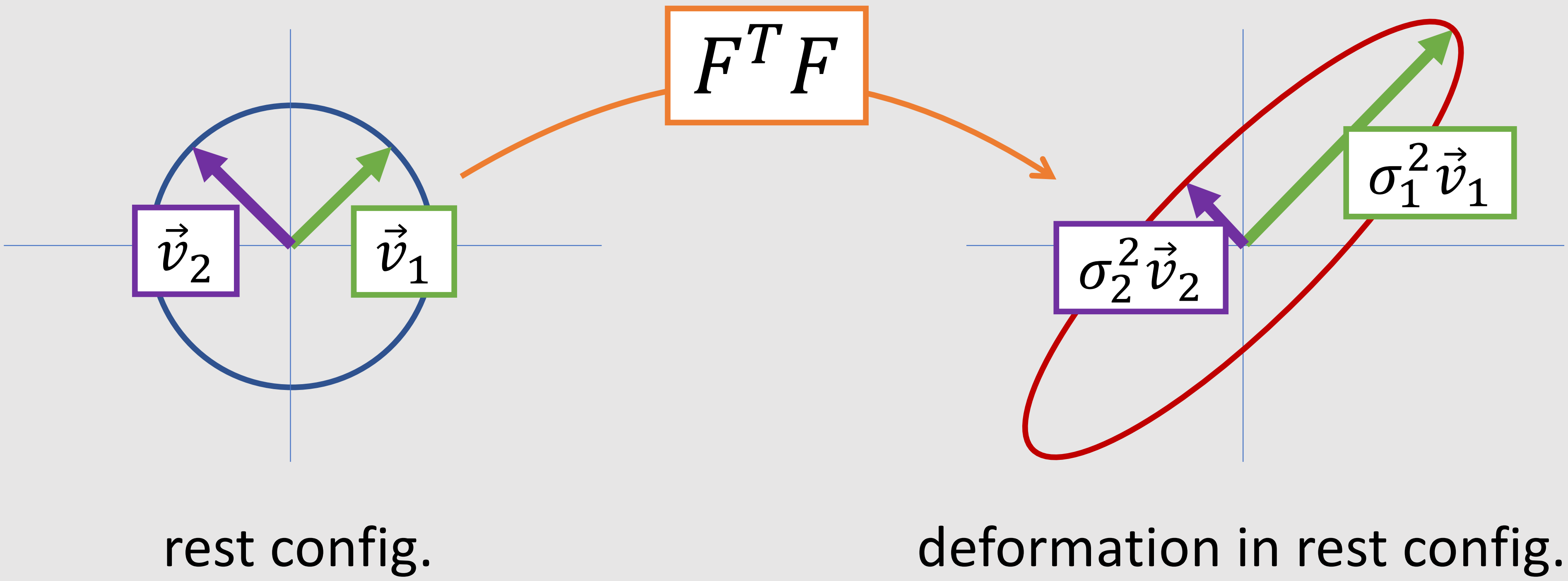
Quadratic form



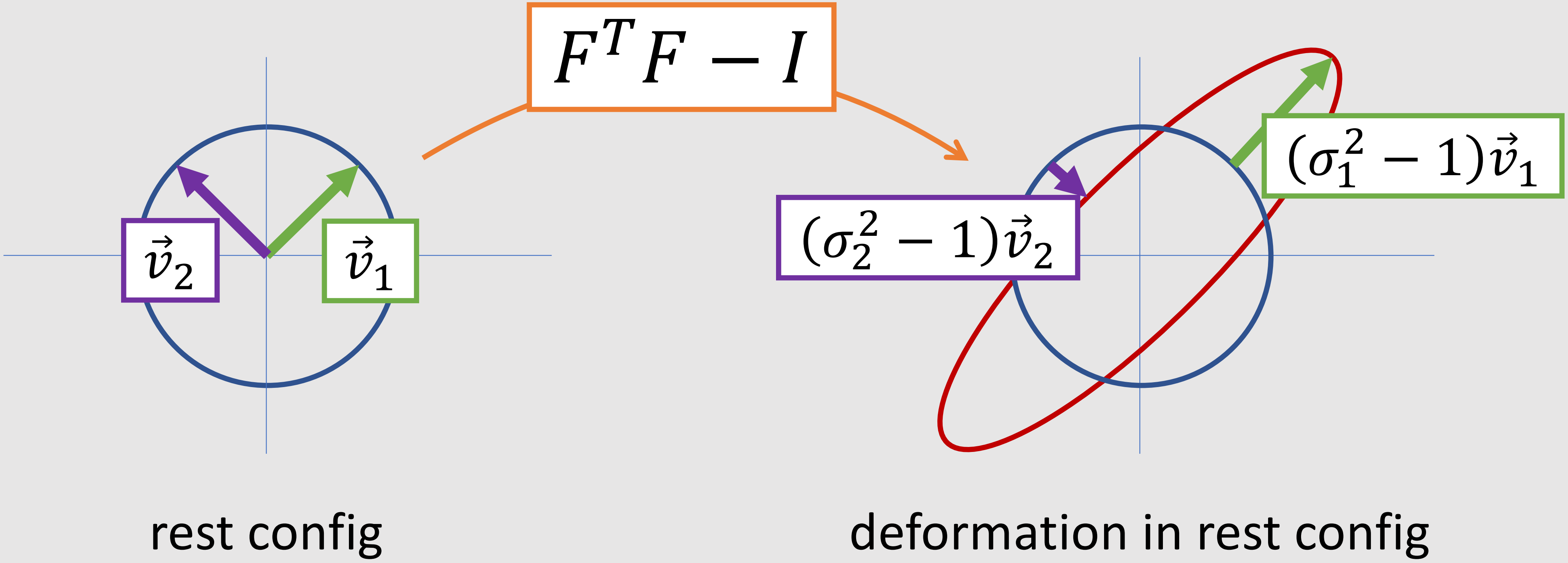
Right Cauchy-Green tensor

Eigenvalue: Right Cauchy Green Tensor $F^T F$

- Right Cauchy Green tensor is symmetric $F^T F = V\Sigma^2 V^T$
- Eigenvalues of $F^T F$ is squared of singular values : $\sigma_1^2, \sigma_2^2, \sigma_3^2$



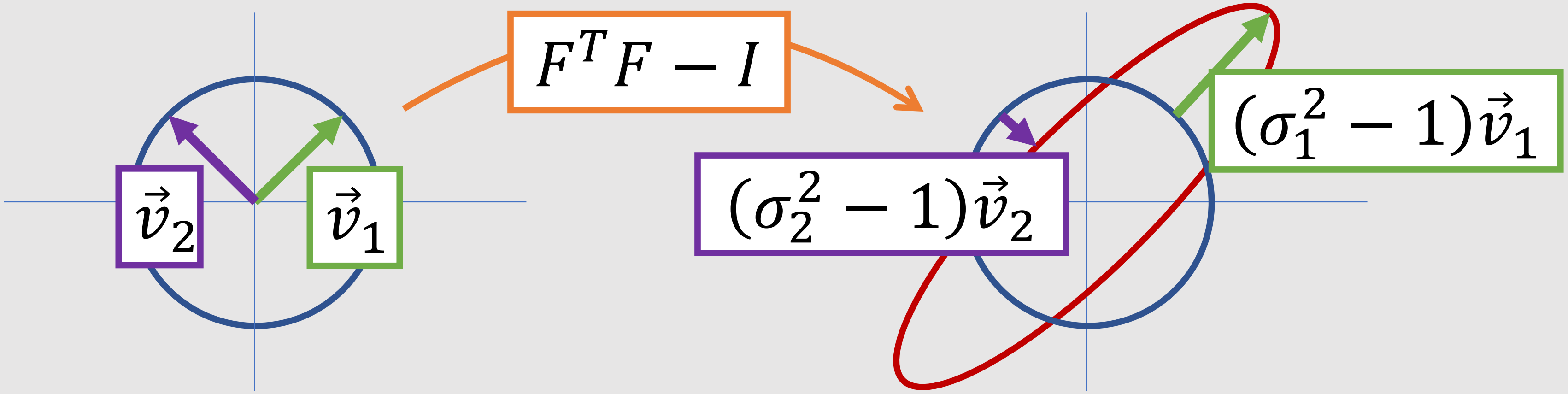
Eigenvalue: Green Lagrange Tensor $F^T F - I$



Making Energy from Eigenvalue

- Energy for isotropic material

$$W(F) = \|F^T F - I\|_F^2 = (\sigma_1^2 - 1)^2 + (\sigma_2^2 - 1)^2 + (\sigma_3^2 - 1)^2$$



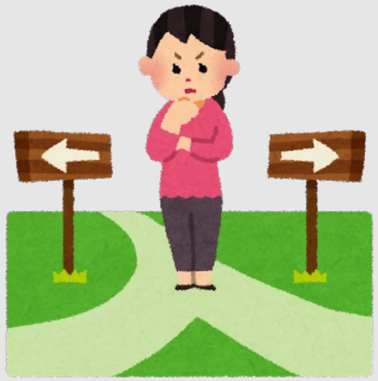
Let's put penalty on the absolute value of
$(\sigma_1^2 - 1)$, $(\sigma_2^2 - 1)$ and $(\sigma_3^2 - 1)$

How can We Formulate Elastic Energy?

Strategy A: Elastic energy W is a function of eigenvalues of Green-Lagrange strain $E = F^T F - I$

$$W = \|F^T F - I\|_F^2, \qquad \text{where } F = \partial\vec{x}/\partial\vec{X}$$

cancel rotation *cancel translation*



Strategy B: Elastic energy W is a sum of square distances after cancelling rotation and translation

$$W = \min_{R,\vec{t}} \sum_i \omega_i \left\| R\vec{X}_i + \vec{t} - \vec{x}_i \right\|^2$$

Making Energy from Eigenvalue

- Energy for isotropic material

$$W(F) = \|F^T F - I\|_F^2 = (\sigma_1^2 - 1)^2 + (\sigma_2^2 - 1)^2 + (\sigma_3^2 - 1)^2$$

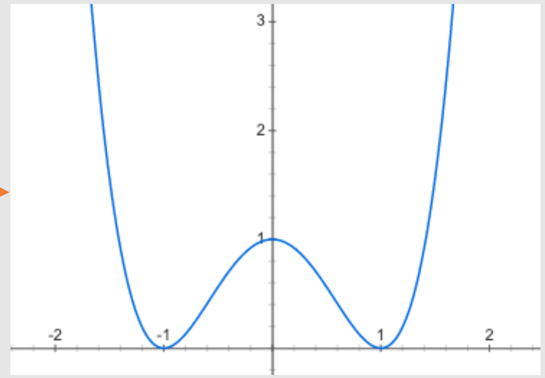
This energy doesn't have costly SVD and eigen decomposition easy to compute gradient & hessian!

Making Energy from Eigenvalue

$$W(F) = \|F^T F - I\|_F^2 = (\sigma_1^2 - 1)^2 + (\sigma_2^2 - 1)^2 + (\sigma_3^2 - 1)^2$$

Wait… $W(F) = 0$ is not always no deformation.
What about mirror reflection $\sigma_i = -1$?

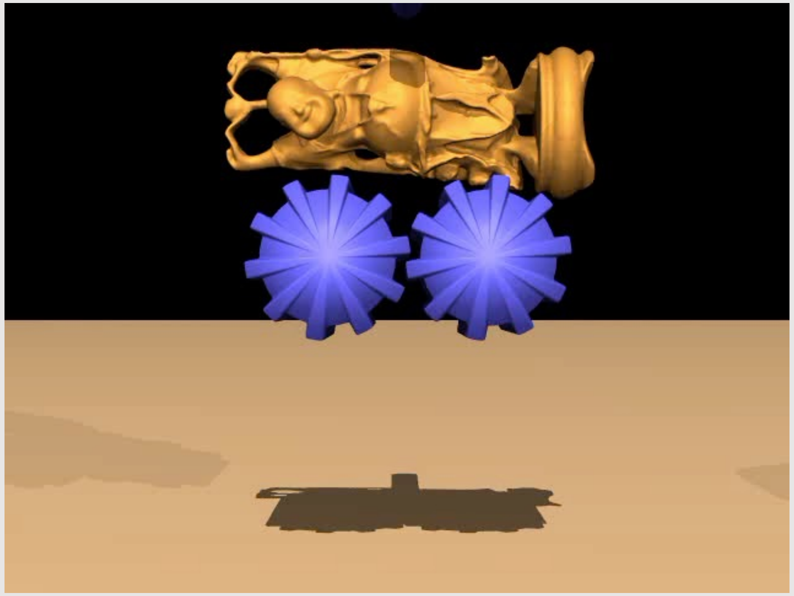
$\boxed{y = (x^2 - 1)^2}$

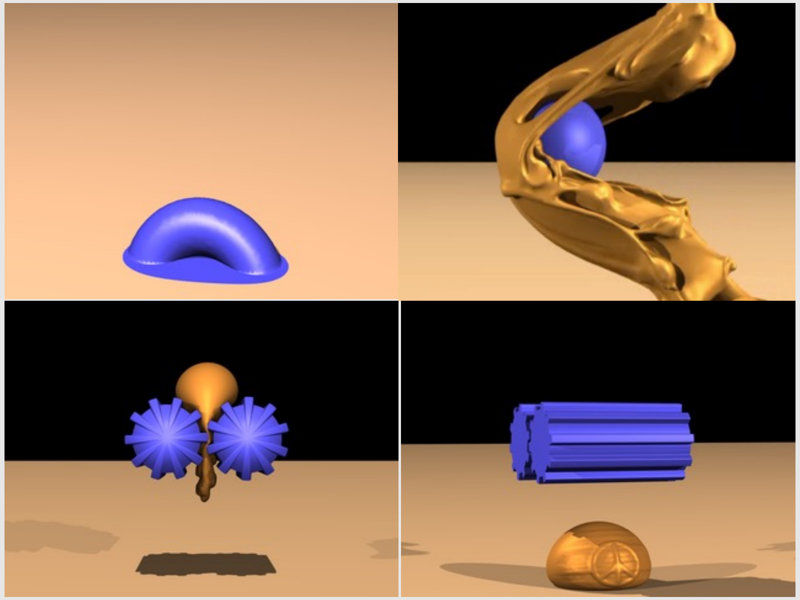


This energy is not
robust to inversion

Invertible FEM [Irving et al. 2004]

- Elastic potential energy based on singular values of F that are σ_i, not on the eigen values of $F^T F$ that are σ_i^2





G. Irving, J. Teran, and R. Fedkiw. 2004. Invertible finite elements for robust simulation of large deformation. In Proceedings of the 2004 ACM SIGGRAPH/Eurographics symposium on Computer animation (SCA '04)