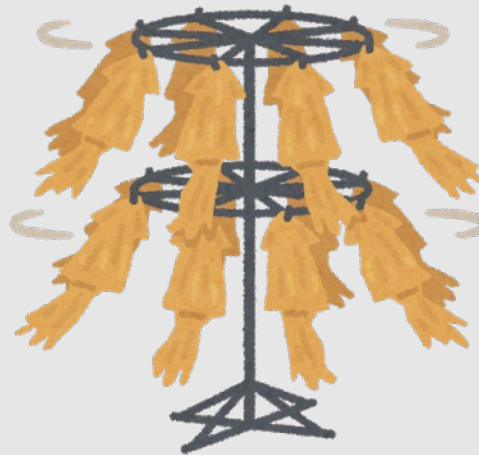


# Rotation

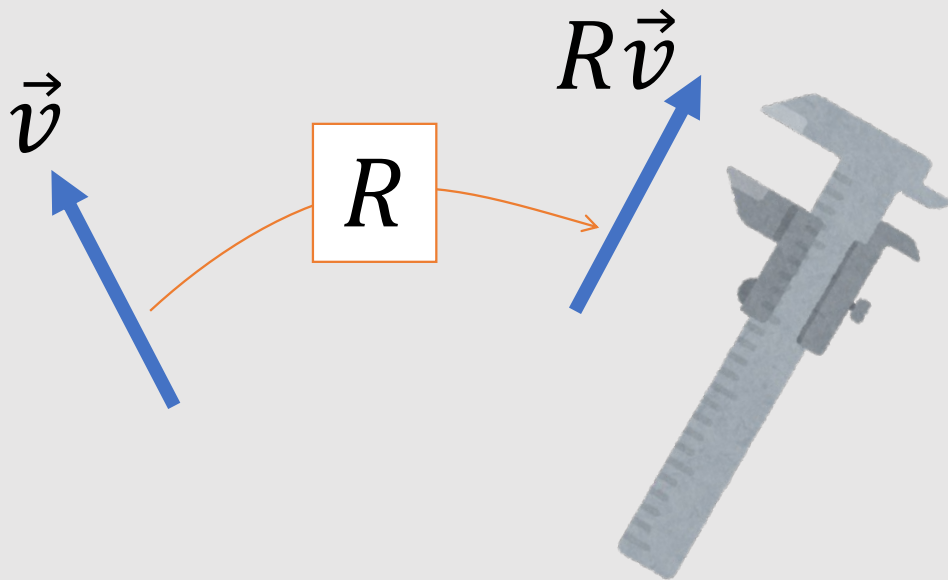
# What is Rotation?

- linear transformation  $R$  is a rotation if **length does not change** and **volume does not flip**



# What is Rotation?

- linear transformation  $R$  is a rotation if **length does not change** and volume does not flip



$$\|\vec{v}\|^2 = \|R\vec{v}\|^2$$

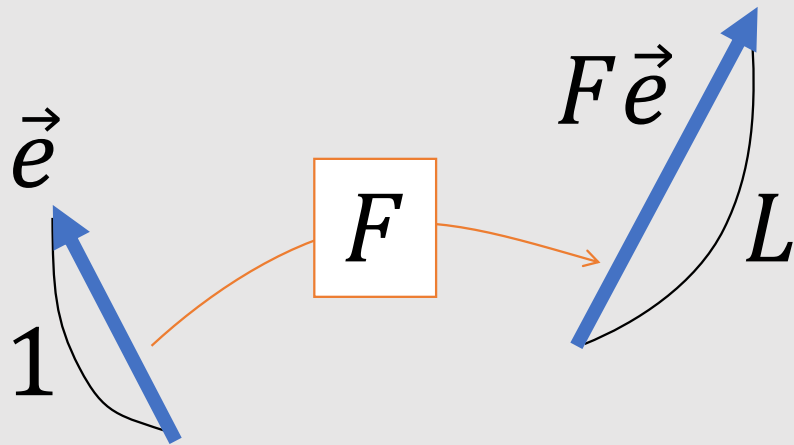
$$\vec{v}^T \vec{v} = \vec{v}^T R^T R \vec{v}$$

$\vec{v}$  can be arbitrary!

$$R^T R = I$$

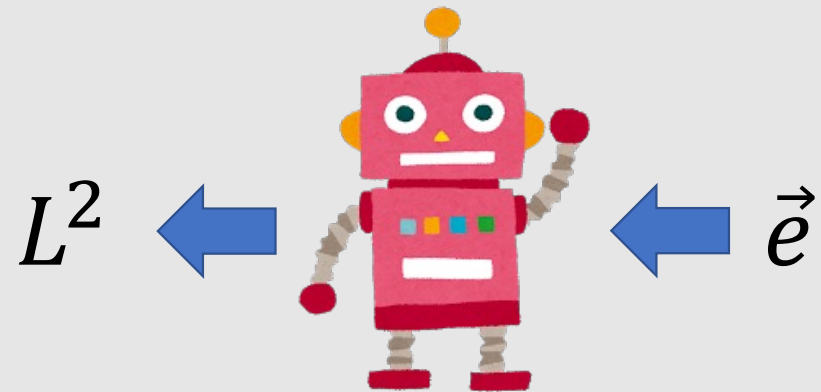


# Gram Matrix $F^T F$ Stands for Deformed Len.



$$L^2(\vec{e}) = \vec{e}^T F^T F \vec{e}$$

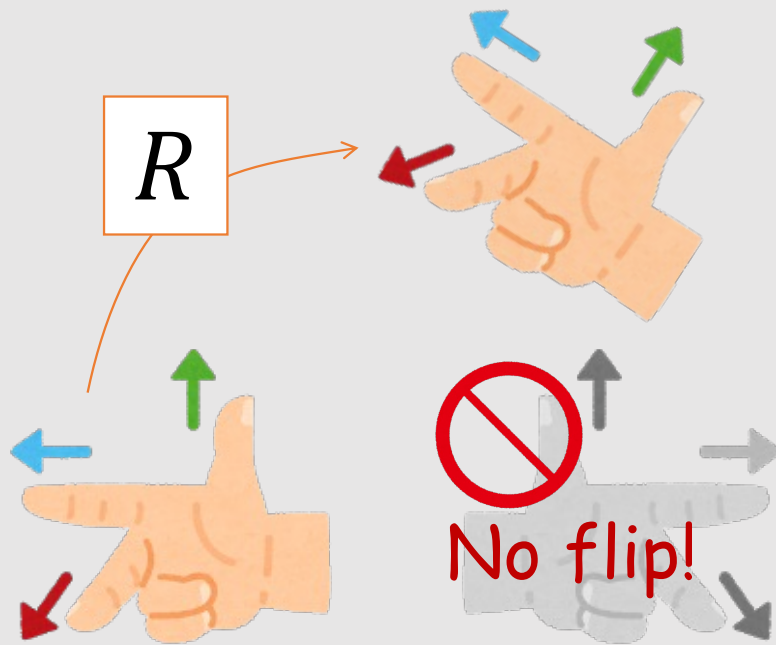
Quadratic form



Positive semi-definite  
matrix

# What is Rotation?

- linear transformation  $R$  is a rotation if length does not change and **volume does not flip**



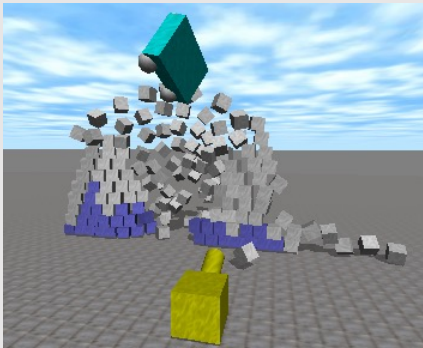
$$\det(R^T R) = \det(I)$$

$$\det(R)^2 = 1$$

$$\det(R) = \pm 1$$

# Example of Applications of Rotation

rigid body  
animation



Credit: Kborer @Wikipedia

character  
animation



Credit: Banlu  
Kemiyatorn@Wikipedia

robotics



Credit: Jo  
Teichmann@Wikipedia

stereo  
reconstruction



Credit: Kbosak@Wikipedia

# Representation of 3D Rotation

*For large rotation:*

- Rotation matrix
- Axis-angle formulation
- Euler angle
- Quaternion



We four are awesome!

*For small rotation:*

- Infinitesimal rotation

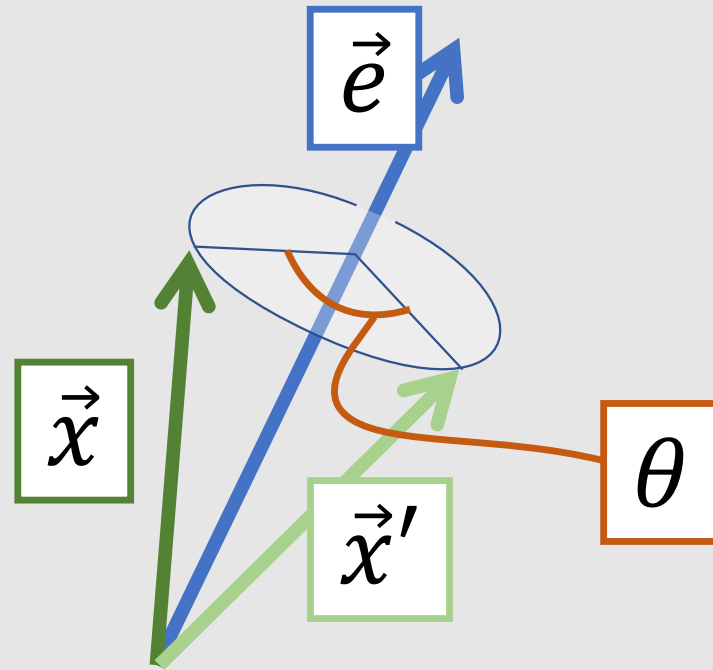


I will be an idol when I grow up!

# Rotation Around Axis

- The rotation is parameterized by axis vector  $\vec{e}$  and angle  $\theta$

$$\vec{x}' = R(\vec{e}, \theta)\vec{x}$$

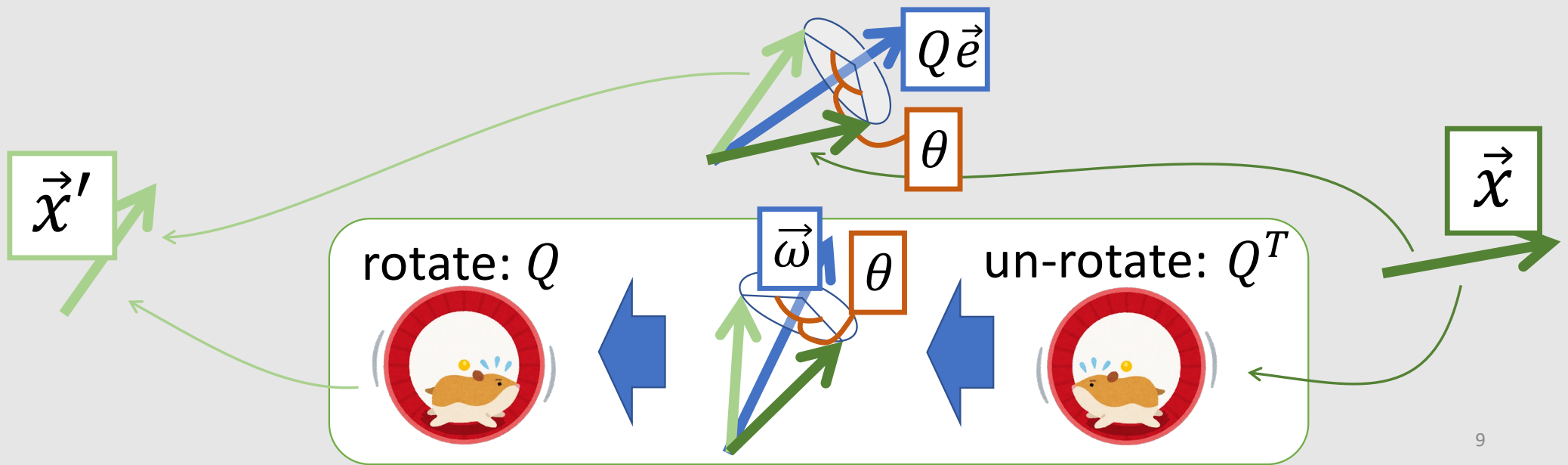




# Rotation around **Rotated Axis**

- Axis is rotated with  $Q \rightarrow$  un-rotate the object with  $Q^T$  then rotate around  $\vec{\omega}$ , then rotate back with  $Q$

$$R(Q\vec{\omega}, \theta) = Q * R(\vec{\omega}, \theta) * Q^T$$





intrinsic rotation:  
axis rotated



extrinsic rotation:  
axis fix

$$R(Q\vec{e}, \theta) = Q * \boxed{R(\vec{e}, \theta)} * Q^T$$



$$R(Q\vec{e}, \theta_2) * Q \neq Q * R(\vec{e}, \theta_2)$$

matrix don't commute!



# Rotation around Rotated Axis: **Robotic Arm**

- Rotation of end-effector in a 4-link articulated body

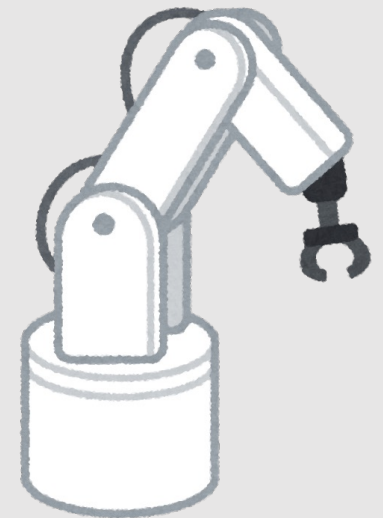
**intrinsic**

$$R_4(R_3R_2R_1\vec{e}_4, \theta_4) * R_3(R_2R_1\vec{e}_3, \theta_3) * R_2(R_1\vec{e}_2, \theta_2) * R_1(\vec{e}_1, \theta_1)$$



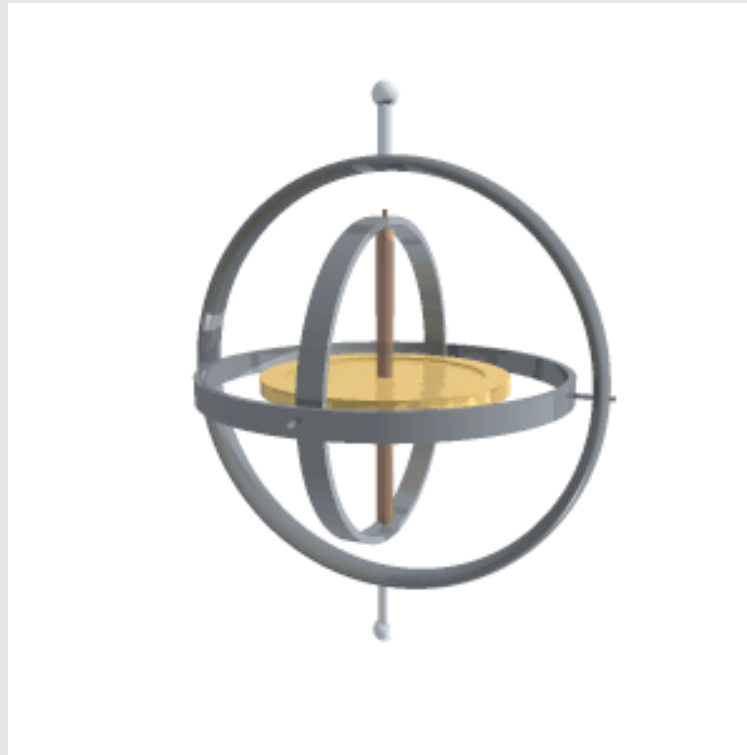
**extrinsic**

$$R_1(\vec{e}_1, \theta_1) * R_2(\vec{e}_2, \theta_2) * R_3(\vec{e}_3, \theta_3) * R_4(\vec{e}_4, \theta_4)$$



# Rotation around Rotated Axis: **Gimbal**

- Gimbal is used to let object freely rotate (e.g, gyroscope)



Credit: Lucas Vieira @ Wikipedia

# Euler Angle for Rotation Parameterization

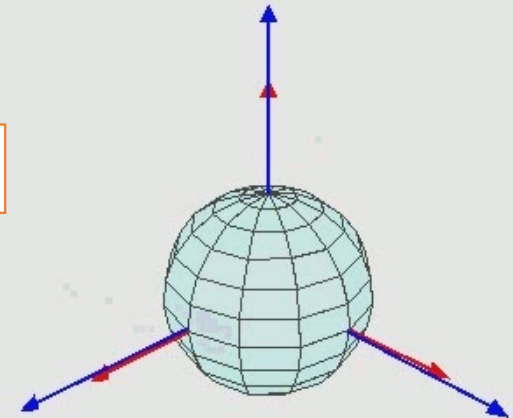
intrinsic

$$\vec{x}' = R_3(R_2 R_1 \vec{e}_Z, \theta_3) * R_2(R_1 \vec{e}_X, \theta_2) * R_1(\vec{e}_Z, \theta_1) \vec{x}$$

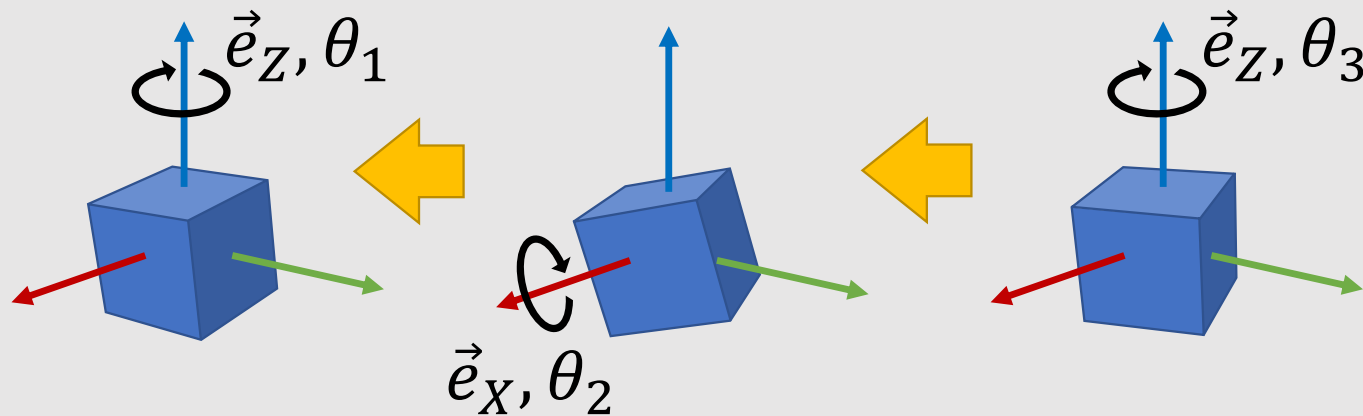


extrinsic

$$\vec{x}' = R_1(\vec{e}_Z, \theta_1) * R_2(\vec{e}_X, \theta_2) * R_3(\vec{e}_Z, \theta_3) \vec{x}$$



credit: Xavax @ Wikipedia



# Vector Rotation Parameterizations

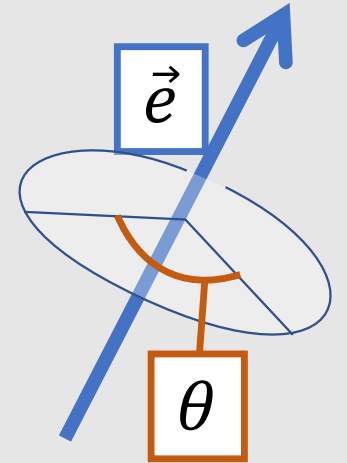
**Euler vector (rotation vector):**  $\vec{\omega} = \vec{e}\theta$

Scaling unit axis vector  $\vec{e}$  with trigonometric func. of  $\theta$

- 360° rotation become zero rotation
- Elegantly composite rotations

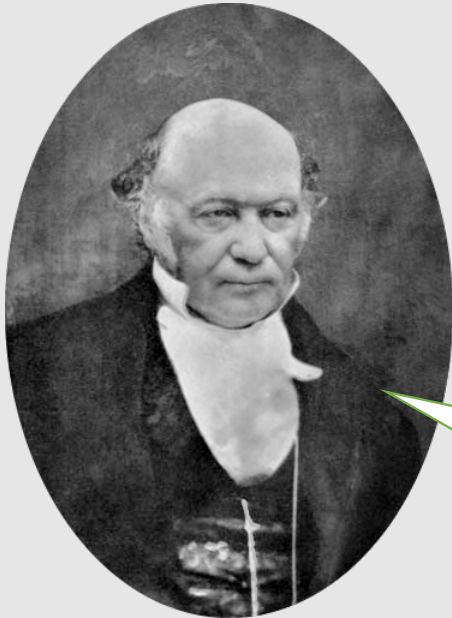
**Cayley-Gibbs-Rodrigues parameterization:**  $\vec{e} \tan \frac{\theta}{2}$

**Euler-Rodrigues parameterization:**  $\left( \vec{e} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right)$



# Discovery of Quaternion

- Sir William Rowan Hamilton (1805–1865) from Ireland



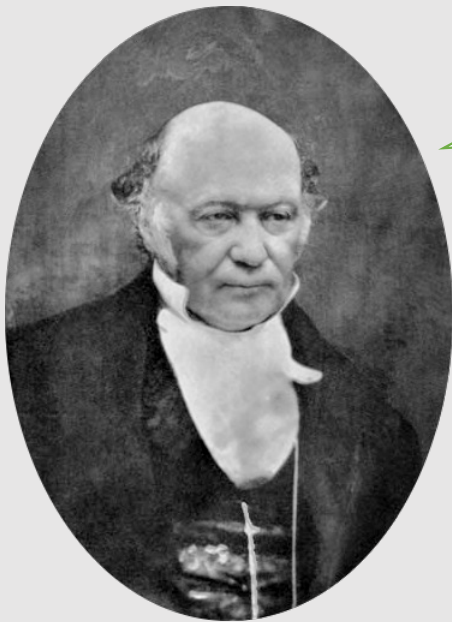
Pa, imaginary number  
for 3D yet?



No clue for 10 years... 😂  
Let me take a walk for a change...

# Discovery of Quaternion

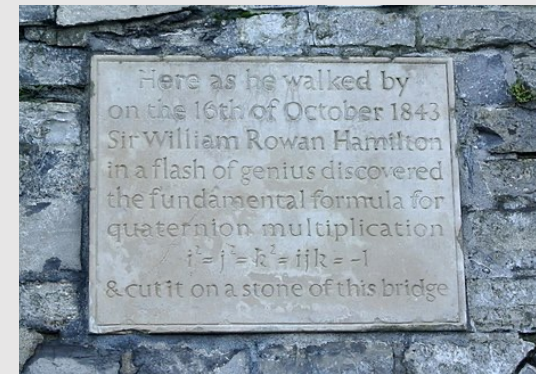
- Sir William Rowan Hamilton (1805–1865) from Ireland



Finally got an idea 🎉  
 $i^2 = j^2 = k^2 = ijk = -1$



Credit: Wisher@wikipedia



Credit: Cone83@wikipedia



# Properties of Quaternion

- One real number and three imaginary numbers

$$q = r + v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

- Conjugate quaternion

$$\bar{q} = r - v_x \mathbf{i} - v_y \mathbf{j} - v_z \mathbf{k}$$

- Norm of quaternion

$$\|q\|^2 = q\bar{q} = r^2 + v_x^2 + v_y^2 + v_z^2$$

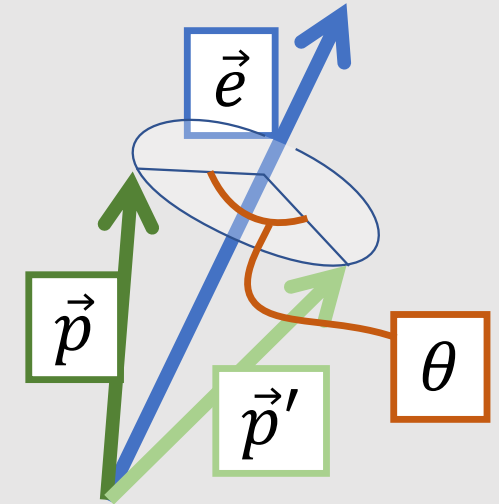
# Rotation with Quaternion

Euler-Rodrigues  
parameterization

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} (e_x \mathbf{i} + e_y \mathbf{j} + e_z \mathbf{k})$$

$$p = p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}$$

$$p' = p'_x \mathbf{i} + p'_y \mathbf{j} + p'_z \mathbf{k}$$



input vector

$$p' = qp\bar{q}$$

rotated vector

# Composite Rotation

- Multiplication of quaternion is associative

rotation with  $q_1$  and then  $q_2$

$$q_{12} = q_2 q_1$$

quaternion don't commute!

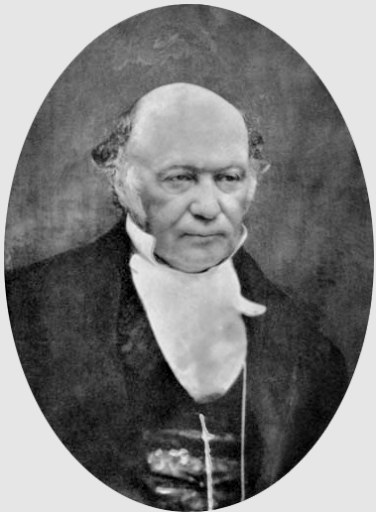


# Cayley–Hamilton Theorem

- Eigen values are the root of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \sum c_i \lambda^i = 0$$

Characteristic polynomial of a matrix  $A$  produces zero ,  
when inputting  $A$



$$p(A) = \sum c_i A^i = 0$$

# Cayley–Hamilton Theorem for a **3x3 Matrix**

- Characteristic polynomial of a matrix  $A$  produces zero , when inputting  $A$

$$p(A) = A^3 + c_1A^2 + c_2A + c_3I = 0$$

$$c_1 = \text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

$$c_2 = 1/2 \{ \text{tr}(A)^2 - \text{tr}(A^2) \} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$c_3 = \det(A) = \lambda_1 \lambda_2 \lambda_3$$

# Infinitesimal Rotation is a Vector

- What if the rotation is very small?

$$R = I + \Omega$$

$$R^T R = (I + \Omega)^T (I + \Omega) = I$$

$$\Omega^T = -\Omega$$

$\Omega$  is a **skew**  
symmetric matrix!



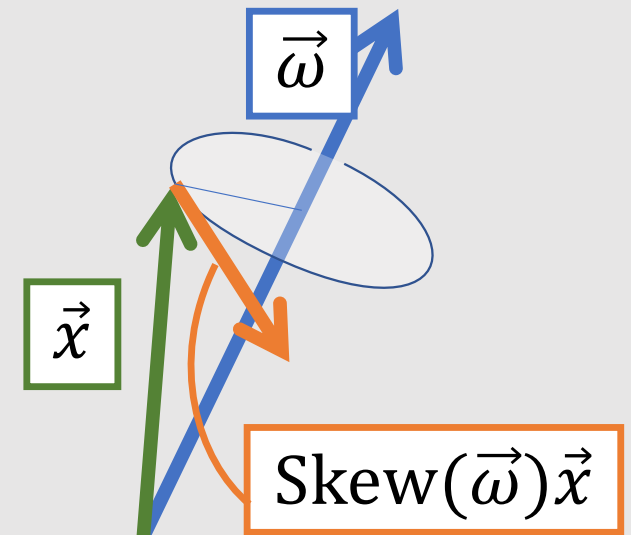
# Properties of 3x3 Skew Symmetric Matrix

- Skew matrix has only three independent elements

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \equiv \text{Skew}(\vec{\omega})$$

- Skew matrix defines cross product

$$\Omega \vec{v} = \vec{\omega} \times \vec{v} \quad \vec{\omega} = (\omega_1, \omega_2, \omega_3)^T$$



# 3x3 Skew Symmetric Matrix Squared

$$\begin{aligned}\Omega^2 = \text{Skew}(\vec{\omega})^2 &= \begin{bmatrix} -\omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_3^2 - \omega_1^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{bmatrix} \\ &= \vec{\omega} \otimes \vec{\omega} - \vec{\omega}^T \vec{\omega} I\end{aligned}$$

$$\begin{aligned}\text{tr}(\Omega^2) &= \omega_1^2 + \omega_2^2 + \omega_3^2 - 3(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ &= -2\|\vec{\omega}\|^2\end{aligned}$$



# 3x3 Skew Symmetric Matrix **Cubed**

Cayley-Hamilton's theorem

$$p(\Omega) = \Omega^3 + (\text{tr}\Omega)\Omega^2 + \frac{(\text{tr}\Omega)^2 - \text{tr}(\Omega^2)}{2} \Omega + (\det \Omega)I = 0$$

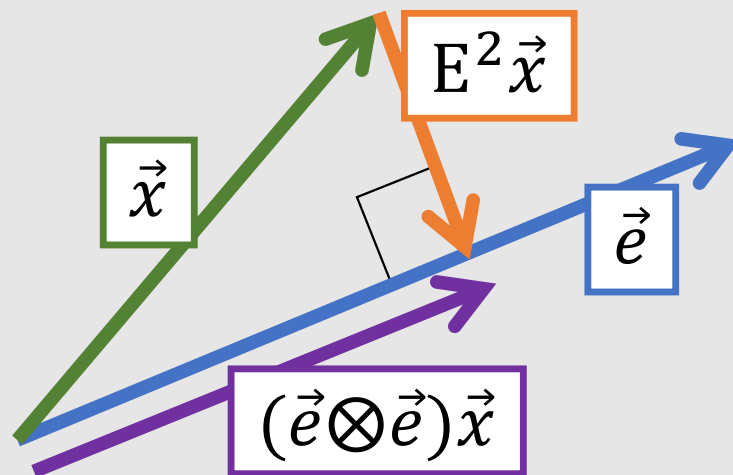

$$\text{tr} \Omega = 0, \det \Omega = 0, \text{tr}(\Omega^2) = -2\|\vec{\omega}\|^2$$

$$\Omega^3 = -\|\vec{\omega}\|^2 \Omega$$

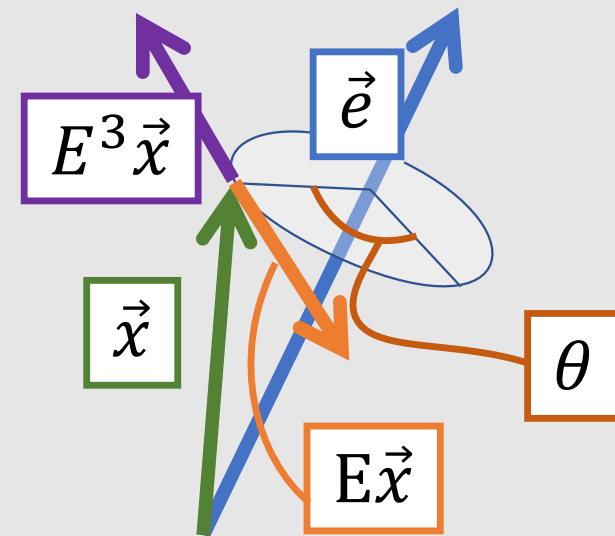
# Skew Symmetric Matrix of **Unit Vector** $\vec{e}$

$$E = \text{Skew}(\vec{e}), \quad \text{where } \|\vec{e}\| = 1$$

$$E^2 = \vec{e} \otimes \vec{e} - I$$



$$E^3 = -E$$



# From Infinitesimal Rotation Vector to Matrix

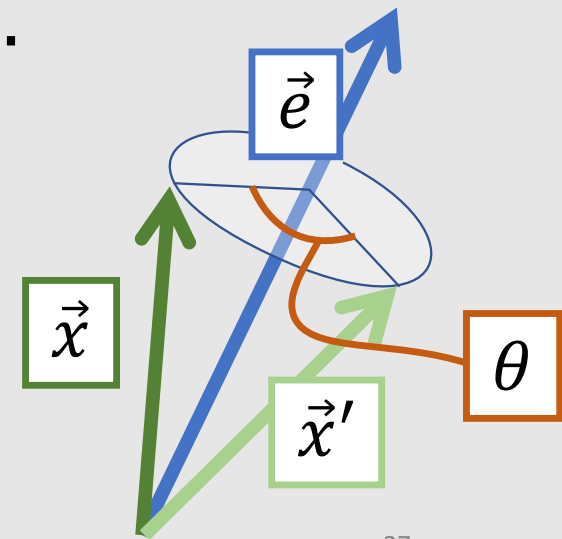
$$R(\vec{e}, \theta) = \exp(\theta E)$$

$$E = \text{Skew}(\vec{e}) = \begin{bmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{bmatrix}$$

$$= I + \frac{\theta}{1!} E + \frac{\theta^2}{2!} E^2 + \frac{\theta^3}{3!} E^3 + \frac{\theta^4}{4!} E^4 \dots$$

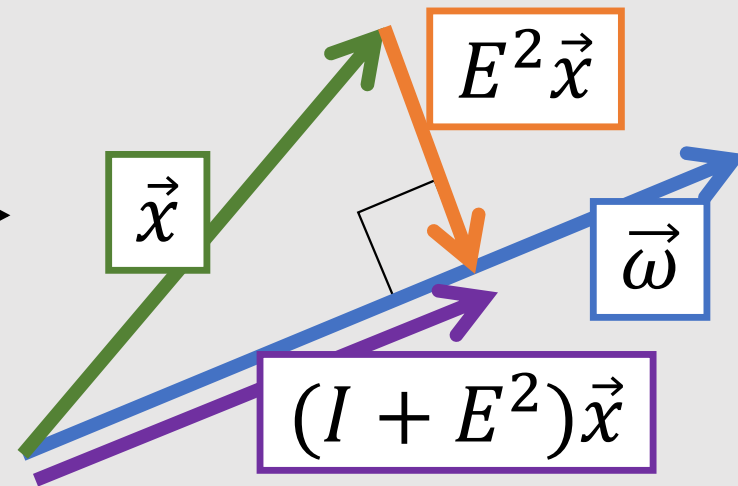
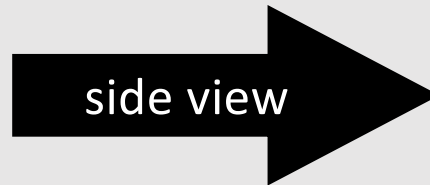
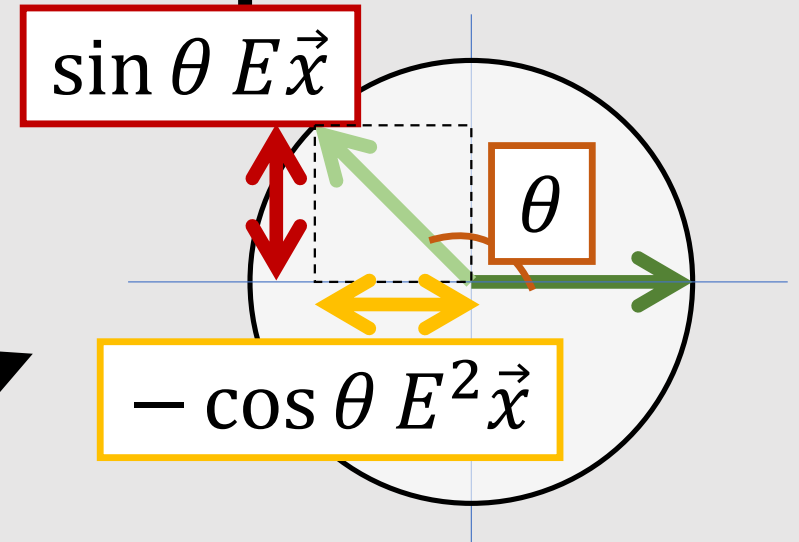
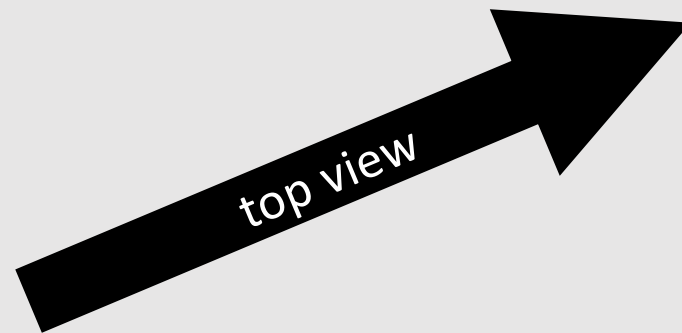
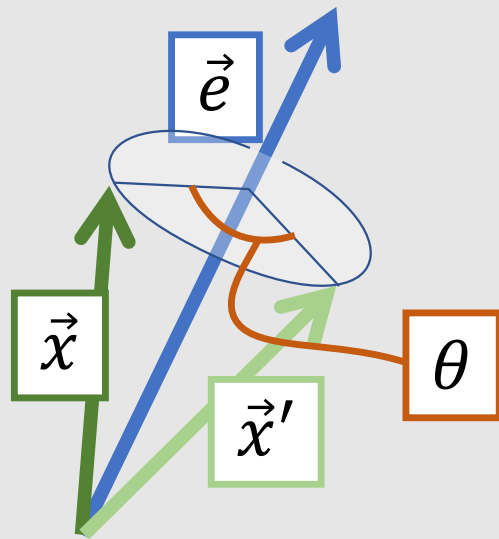
$$= I + \sin\theta E + (1 - \cos\theta) E^2$$

Rodriguez's rotation formula

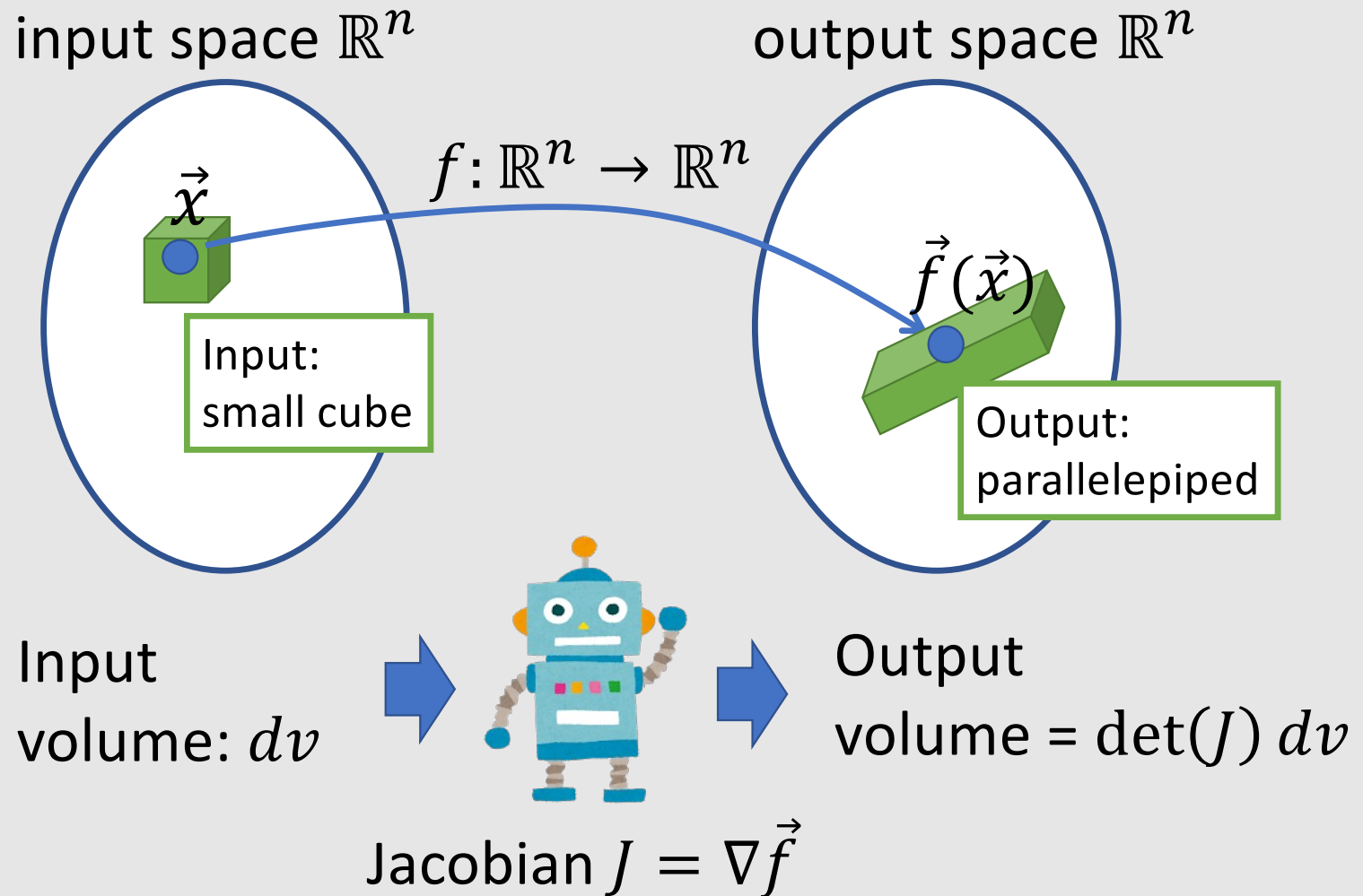


# Rodriguez's Rotation Formula Explained

$$R(\vec{e}, \theta) = I + \sin\theta E + (1 - \cos\theta)E^2$$



# Jacobian Determinant: Volume Change Ratio



# Jacobian Determinant: Volume Change Ratio

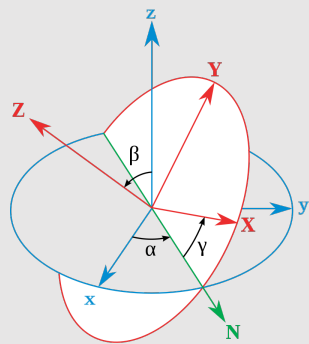
**input:**

small change in Euler angle  
( $d\theta_1, d\theta_2, d\theta_3$ )

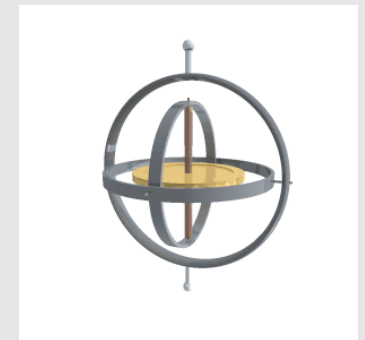
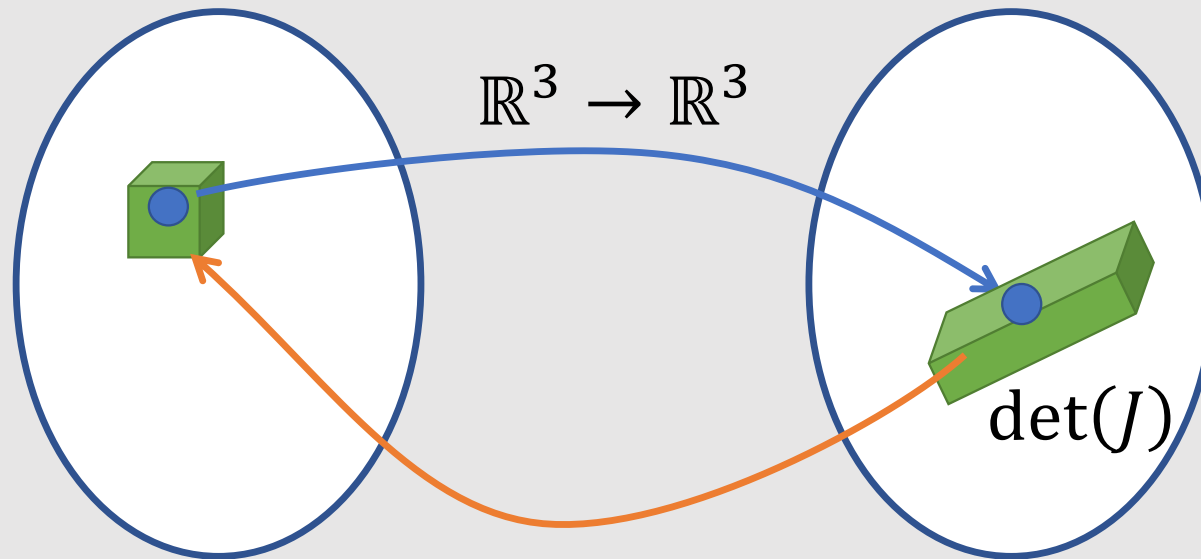
**output:**

infinitesimal rotation

$$d\vec{\omega} = (d\omega_1, d\omega_2, d\omega_3)$$



Credit: Lionel Brits  
@ Wikipedia



Credit: Lucas Vieira  
@Wikipedia

If output volume is not zero  $\det(J) \neq 0$ , there is an inverse map

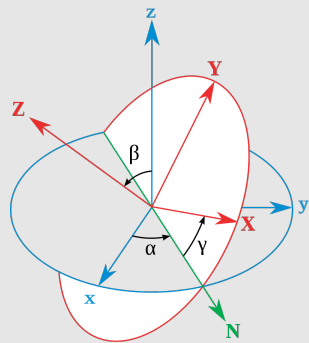
# Gimbal Lock: Zero Jacobian Determinant

**input:**

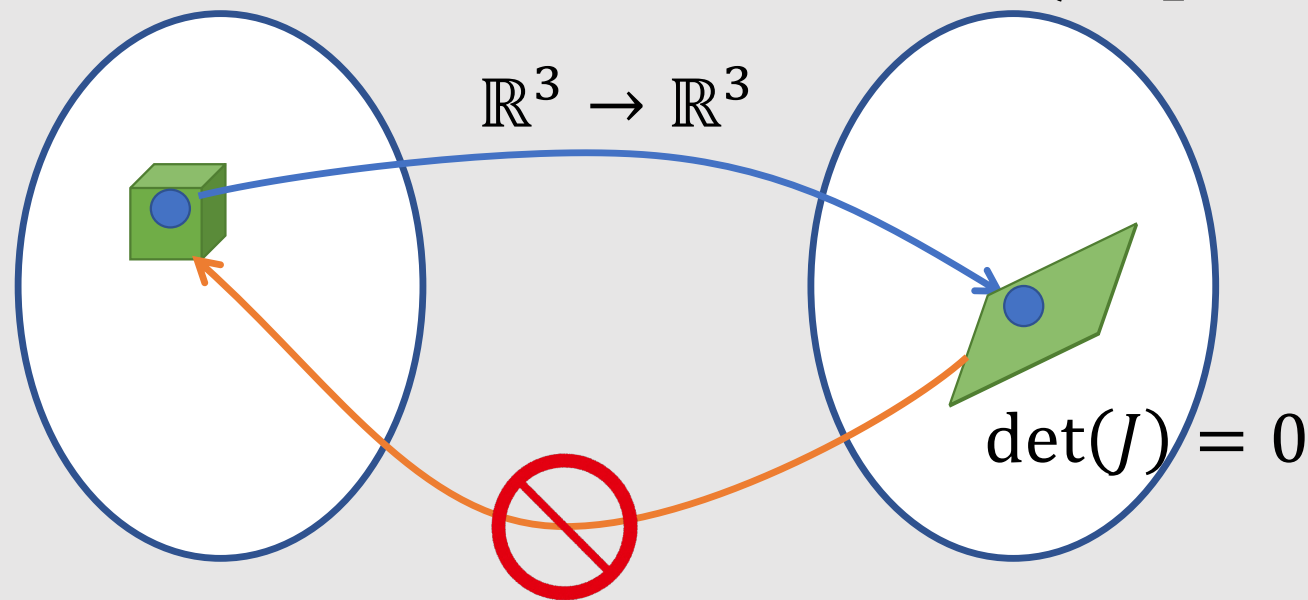
small change in Euler angle  
( $d\theta_1, d\theta_2, d\theta_3$ )

**output:**

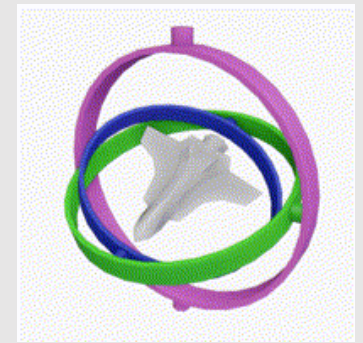
infinitesimal rotation  
 $d\vec{\omega} = (d\omega_1, d\omega_2, d\omega_3)$



Credit: Lionel Brits  
@ Wikipedia



no inverse map when axes aligned!



Credit: Drummyfish  
@ Wikipedia

# Comparison of 3D Rotation Representations

## **Rotation matrix**

- DoF:  $3 \times 3 = 9$
- rotation, scale, sheer, mirror
- 😊 general
- 😞 large

## **Euler angle**

- DoF: 3
- rotation
- 😊 understandable
- 😞 gimbal lock

## **Quaternion**

- DoF:  $1 + 3 = 4$
- rotation, scale
- 😊 compact
- 😞 not understandable

**Axis-angle formulation**

**Infinitesimal rotation**

Rodriguez's formula

exponential

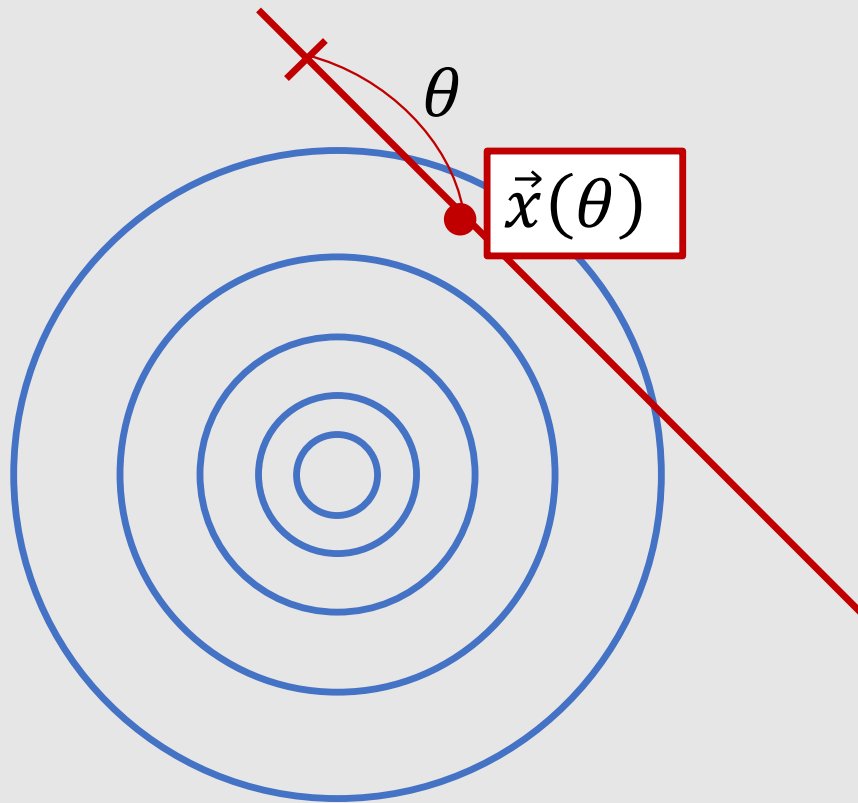
$$\cos \frac{\theta}{2}, \sin \frac{\theta}{2}$$

differentiation



# DoF Elimination using Parameterization

Parameterize solution  $\vec{x}(\theta)$  such that constraints naturally satisfy



$$\operatorname{argmin}_{\vec{x} \in \{\vec{x} | g(\vec{x}) = 0\}} W(\vec{x}) \quad \longrightarrow \quad \operatorname{argmin}_{\theta} W(\vec{x}(\theta))$$

e.g.,  $g(\vec{x}) = x + y + 2 = 0$

$$\longrightarrow \begin{cases} x = +\theta - 1 \\ y = -\theta - 1 \end{cases}$$

# Minimize Parameterized Solution

$$\underset{\theta}{\operatorname{argmin}} W(\vec{x}(\theta))$$

Newton-Raphson method

$$d\theta = - \left[ \frac{\partial^2 W}{\partial \theta^2} \right]^{-1} \left( \frac{\partial W}{\partial \theta} \right)$$

find the **root** of gradient!



# Differentiation of Rotation Matrix

Change under constraints  $R(t)^T R(t) = I \rightarrow$  DoF elimination

walking on the edge!

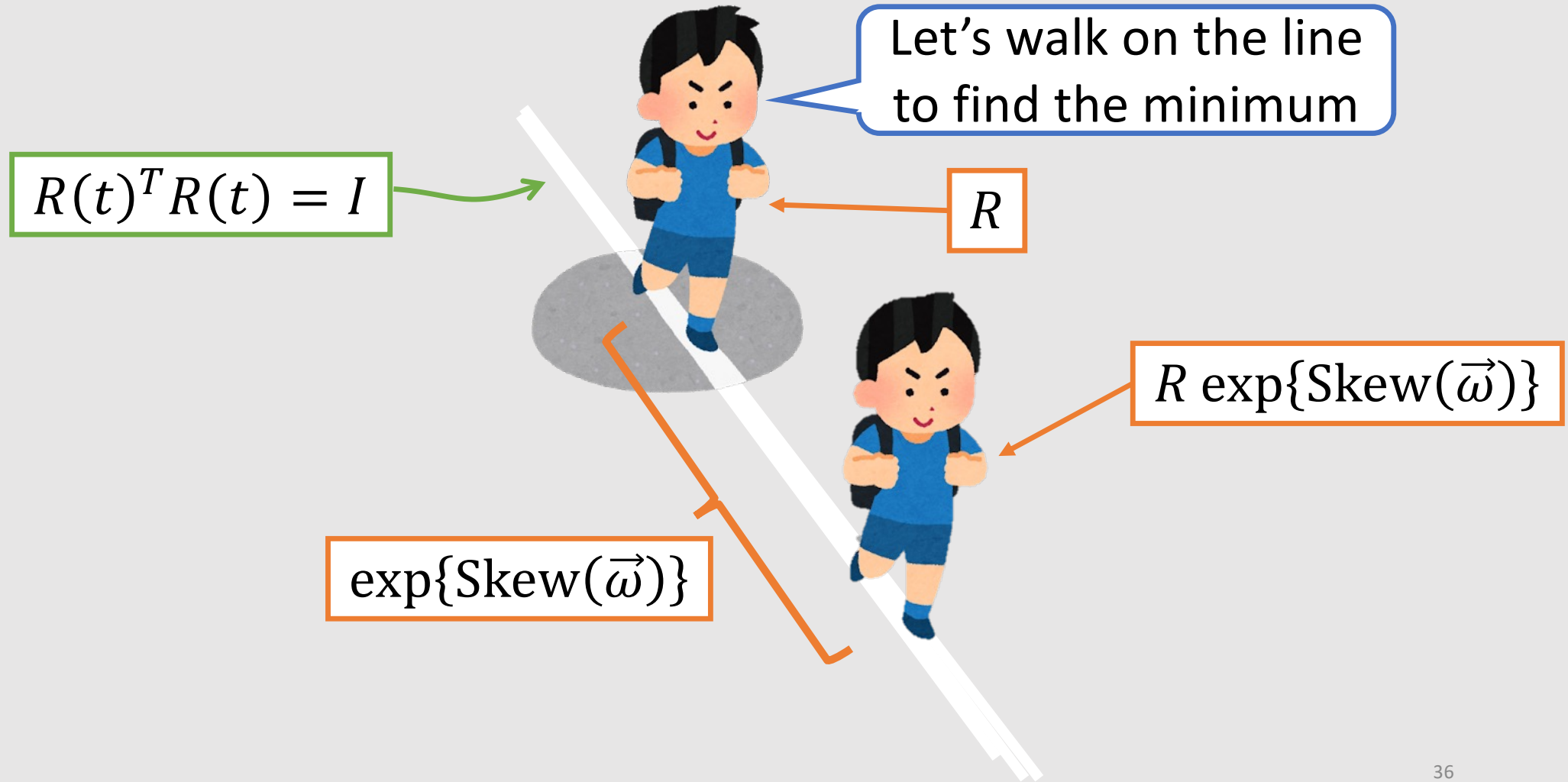


$R(t)$

$R(t)^T R(t) = I$



# Parameterization of Rotation



# Optimization of Rotation: Pattern $R^*dR$

$$R = R_i \exp\{\text{Skew}(\vec{\omega})\}$$

Minimize  $W(R)$

Minimize  $W(R(\vec{\omega}))$

Newton method

$$\vec{\omega} = - \left[ \frac{\partial^2 W}{\partial^2 \vec{\omega}} \right]^{-1} \left( \frac{\partial W}{\partial \vec{\omega}} \right)$$

$$R_{i+1} = R_i \exp\{\text{Skew}(\vec{\omega})\}$$

$$\begin{aligned} \exp\{\text{Skew}(\vec{\omega})\} \\ = I + \text{Skew}(\vec{\omega}) + \frac{1}{2} \text{Skew}(\vec{\omega})^2 \dots \end{aligned}$$

find the **root** of gradient!



# Optimization of Rotation: Pattern $dR^*R$

$$R = \exp\{\text{Skew}(\vec{\omega})\} R_i$$

Minimize  $W(R)$

Minimize  $W(R(\vec{\omega}))$

Newton method

$$\vec{\omega} = - \left[ \frac{\partial^2 W}{\partial^2 \vec{\omega}} \right]^{-1} \left( \frac{\partial W}{\partial \vec{\omega}} \right)$$

$$R_{i+1} = \exp\{\text{Skew}(\vec{\omega})\} R_i$$

$$\begin{aligned} \exp\{\text{Skew}(\vec{\omega})\} \\ = I + \text{Skew}(\vec{\omega}) + \frac{1}{2} \text{Skew}(\vec{\omega})^2 \dots \end{aligned}$$

find the **root** of gradient!



# Gradient and Hessian Including Rotation

$$\begin{aligned}
 W(R) &= \|R\vec{p} - \vec{q}\|^2 \\
 &= (R\vec{p} - \vec{q})^T (R\vec{p} - \vec{q}) = \|\vec{p}\|^2 \underbrace{- 2\vec{q}^T R\vec{p}}_{\text{the only changing term}} + \|\vec{q}\|^2
 \end{aligned}$$

$$\begin{aligned}
 -2\vec{q}^T R(\vec{\omega})\vec{p} &= -2\vec{q}^T \exp\{\text{Skew}(\vec{\omega})\} R\vec{p} \\
 &\cong -2\vec{q}^T \left\{ I + \underbrace{\text{Skew}(\vec{\omega})}_{1^{\text{st}} \text{ order}} + \underbrace{\frac{1}{2} \text{Skew}(\vec{\omega})^2}_{2^{\text{nd}} \text{ order}} \right\} R\vec{p}
 \end{aligned}$$

$$-2\vec{q}^T \text{Skew}(\vec{\omega}) R\vec{p} = -2\vec{\omega}^T \text{Skew}(R\vec{p}) \vec{q} = \vec{\omega}^T (2\vec{q} \times R\vec{p})$$

$$-\vec{q}^T \text{Skew}(\vec{\omega})^2 R\vec{p} = -\vec{q}^T [\vec{\omega} \otimes \vec{\omega} - \vec{\omega}^T \vec{\omega} I] R\vec{p} = \vec{\omega}^T [\vec{q}^T R\vec{p} I - \vec{q} \otimes R\vec{p}] \vec{\omega}$$

# Gradient and Hessian Including Rotation

$$W(R(\vec{\omega})) = \|R(\vec{\omega})\vec{p} - \vec{q}\|^2$$

$$\text{Gradient: } \frac{\partial W}{\partial \vec{\omega}} = \frac{\partial}{\partial \vec{\omega}} \{ \vec{\omega}^T (2\vec{q} \times R\vec{p}) \} = 2\vec{q} \times R\vec{p}$$

$$\begin{aligned} \text{Hessian: } \frac{\partial^2 W}{\partial \vec{\omega}^2} &= \frac{\partial^2}{\partial \vec{\omega}^2} \{ \vec{\omega}^T (\vec{q}^T R\vec{p}I - \vec{q} \otimes R\vec{p}) \vec{\omega} \} \\ &= 2\vec{q}^T R\vec{p}I - \vec{q} \otimes R\vec{p} - R\vec{p} \otimes \vec{q} \end{aligned}$$

Hessian must be symmetric!



# Differentiation w.r.t Vectors

- Transform the equation into a polynomial

$$\begin{aligned}W(\vec{\omega}) &= \dots \\ &= \dots \\ &= a + \vec{b}^T \vec{\omega} + \vec{\omega}^T C \vec{\omega} + \dots\end{aligned}$$

$$\text{Gradient: } \frac{\partial W}{\partial \vec{\omega}} = \vec{b}$$

$$\text{Hessian: } \frac{\partial^2 W}{\partial \vec{\omega}^2} = C^T + C$$