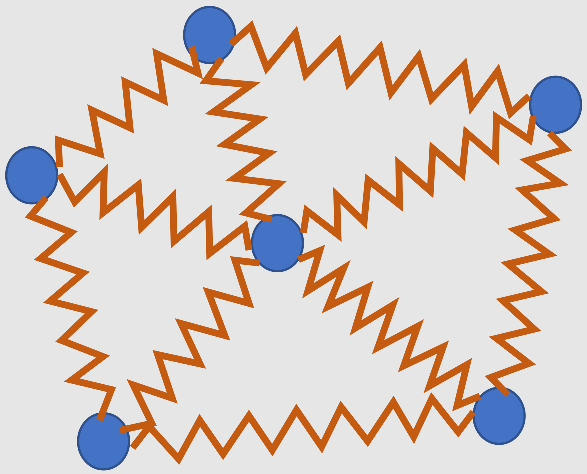
Rigid Body Approximation

剛体近似

Rigid Body Approximation

- In mass-spring system, 5 points in 3D has 15 degrees of freedom

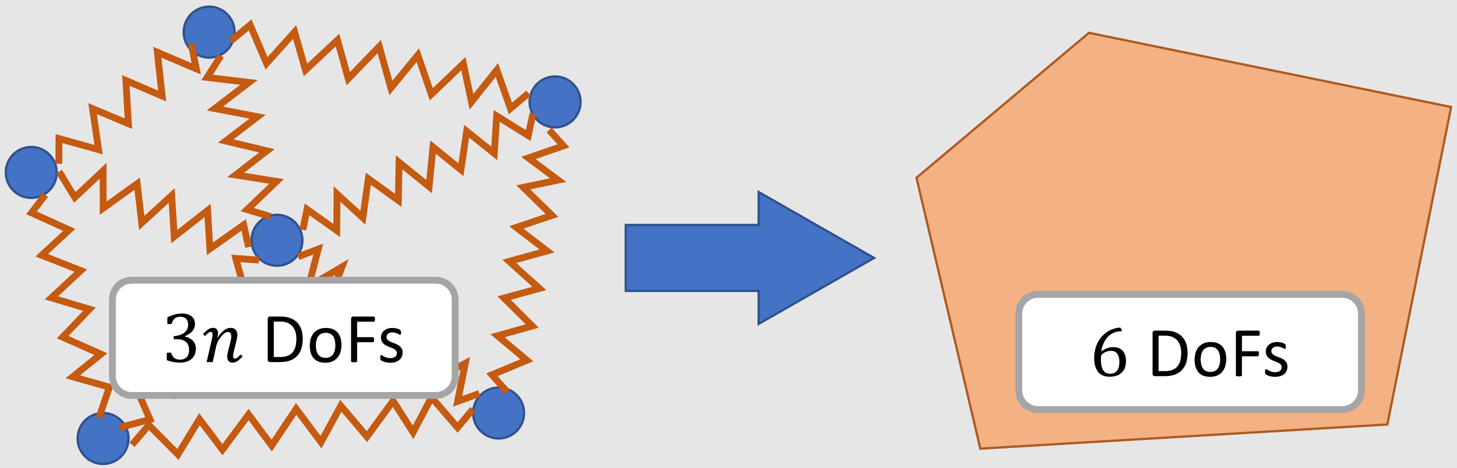


Rigid Body Approximation

- If deformation is negligible, rigid body approximation makes sense

$\vec{x}_{cg}(t)$: the center of gravity's position

$R(t)$: rotation

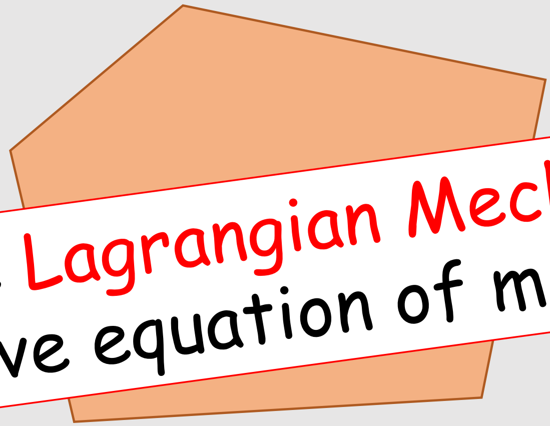


Rigid Body Approximation

- Equation of motion for rigid body?

$p(t)$: the center of gravity's position

$R(t)$: rotation



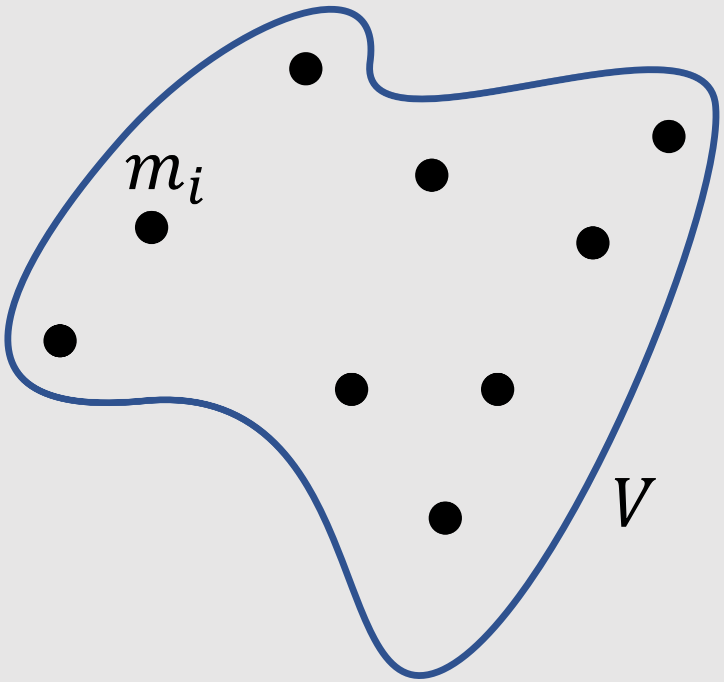
We use Lagrangian Mechanics to derive equation of motion!

Rigid Body is just an Approximation

- Everything deforms as a reaction to force

Mass(質量)

- Total weight of the object

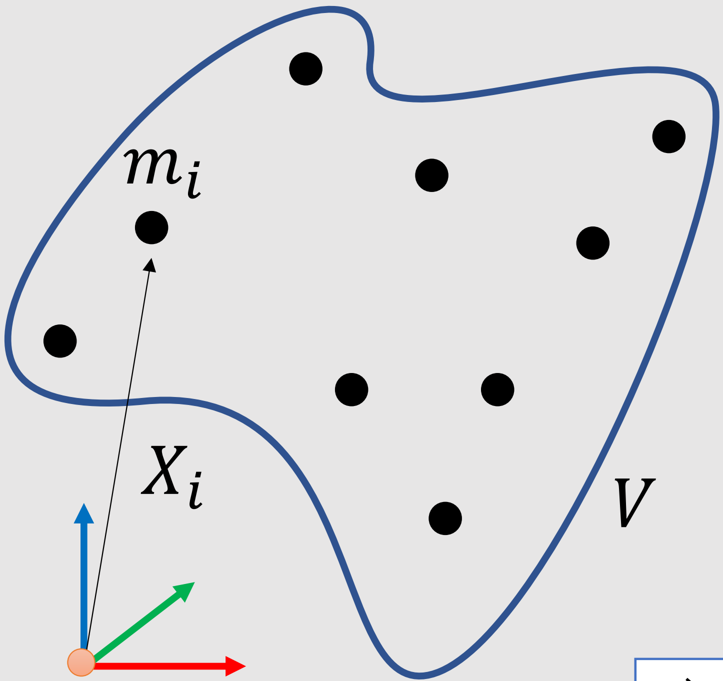


$$M = m_1 + \cdots + m_i$$
$$= \sum_i m_i$$

$$M = \int_V \rho \, dV$$

The Center of the Gravity(重心)

- Average of the positions weighted by mass density



$$\vec{X}_{cg} = \frac{\vec{X}_1 m_1 + \cdots + \vec{X}_i m_i}{m_1 + \cdots + m_i}$$

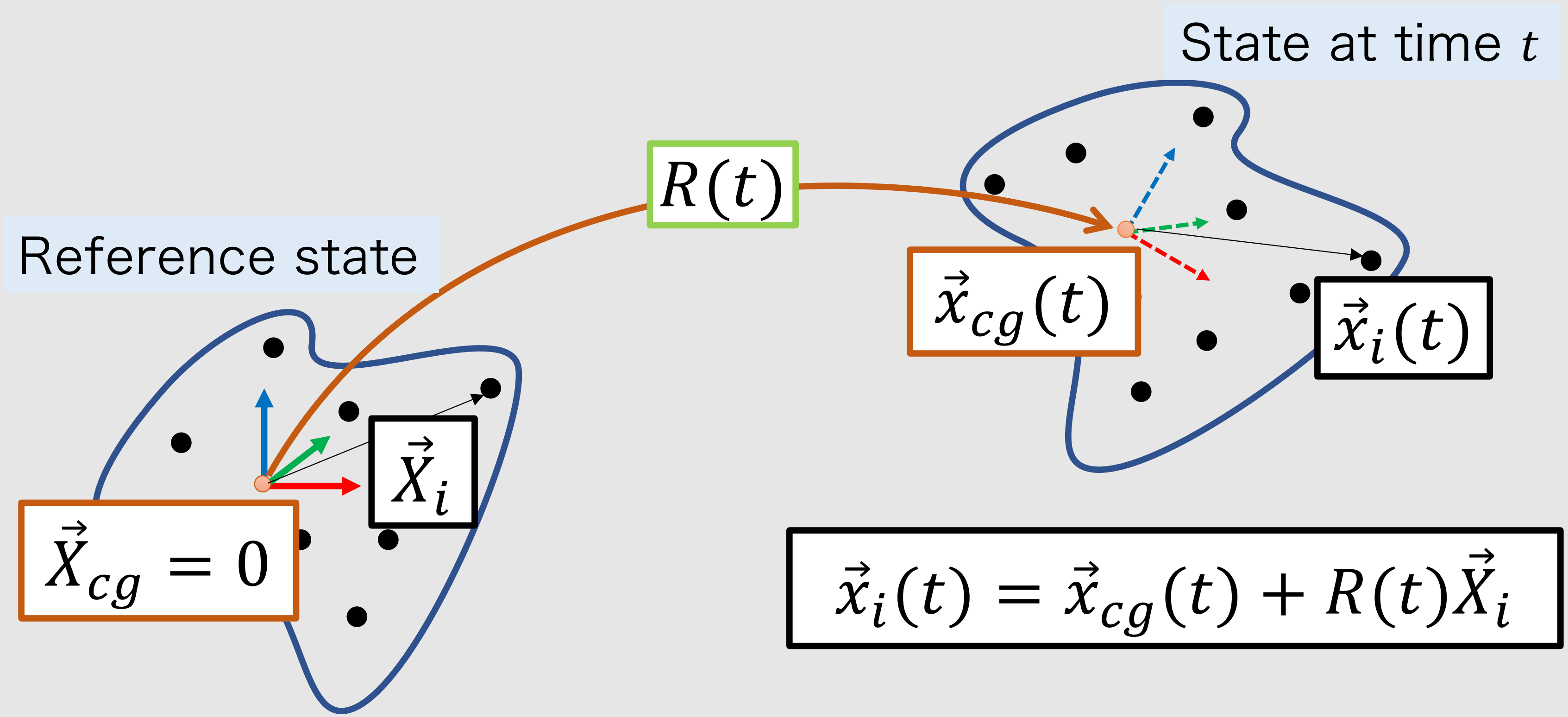
$$= \sum_i \vec{X}_i m_i \Big/ \sum_i m_i$$

$$\vec{X}_{cg} = \int_V \rho \vec{X}\, dV \Big/ \int_V \vec{X}\, dV$$

$\vec{X}_{cg}$ is the centroid(図心) if is constant

Transformation of a Point on a Rigid Body

- For simplicity, let's put $\vec{X}_{cg}$ at the origin of coordinate: $\vec{X}_{cg} = 0$



$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t)\vec{X}_i$$

Linear Momentum (運動量)

$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t)\vec{X}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t)\vec{X}_i$$

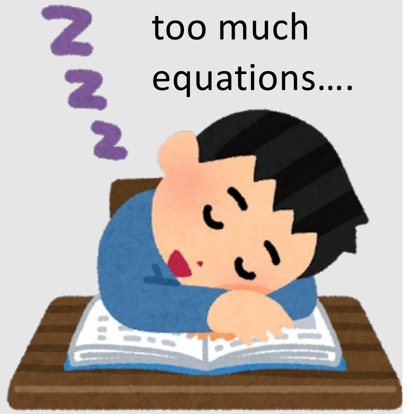
$$\dot{R}(t) = R\ \mathrm{Skew}(\vec{\Omega})$$

$$\vec{P} = \sum_i m_i \vec{v}_i$$

$$\vec{P} = \sum_i m_i \vec{v}_{cg}(t) + \sum_i m_i R\ \mathrm{Skew}(\vec{\Omega})\vec{X}_i$$

$$\sum_i m_i R\ \mathrm{Skew}(\vec{\Omega})\vec{X}_i = R\ \mathrm{Skew}(\vec{\Omega}) \sum_i m_i \vec{X}_i = R\ \mathrm{Skew}(\vec{\Omega})\ \vec{X}_{cg} = R\ \mathrm{Skew}(\vec{\Omega})\ 0 = 0$$

$$\vec{P} = M\vec{v}_{cg}$$



Kinetic Energy (運動エネルギー)

$$\mathcal{K} = \frac{1}{2}\sum_i m_i \vec{v}_i^{\,T} \vec{v}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t)\vec{X}_i$$

$$= \mathrm{R\ Skew}(\vec{\Omega})\vec{X}_i$$

$$= -\mathrm{R\ Skew}(\vec{X}_i)\vec{\Omega}$$

$$\mathcal{K} = \frac{1}{2}\sum_i m_i \vec{v}_{cg}^{\,T} \vec{v}_{cg} + \frac{1}{2}\sum_i m_i \{R\ \mathrm{Skew}(\vec{X}_i)\vec{\Omega}\}^T \{R\ \mathrm{Skew}(\vec{X}_i)\vec{\Omega}\}$$

inertia tensor

$$\mathcal{K} = \frac{1}{2} M \|\vec{v}_{cg}\|^2 + \frac{1}{2}\vec{\Omega}(t)^T \left\{-\sum_i m_i \mathrm{Skew}(\vec{X}_i)\ \mathrm{Skew}(\vec{X}_i)\right\}\vec{\Omega}(t)$$

Momentum (運動量)

$$\vec{L} = \sum_i \vec{x}_i \times (m_i \vec{v}_i)$$

$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t)\vec{X}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t)\vec{X}_i$$

$$= R\ \mathrm{Skew}(\vec{\Omega})$$

$$\vec{L} = \vec{x}_i \times \sum_i m_i \vec{v}_{cg} + \sum_i (R\vec{X}_i) \times (m_i R\ \mathrm{Skew}(\vec{\Omega})\vec{X}_i)$$

$$= \mathrm{Skew}(R\vec{X}_i) = R\mathrm{Skew}(\vec{X}_i)R^T$$

$$= -\mathrm{Skew}(\vec{X}_i)\vec{\Omega}$$

inertia tensor

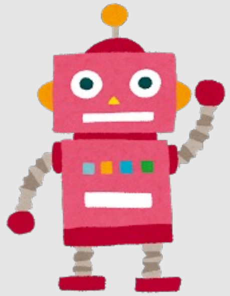
$$\vec{L} = M\vec{x}_{cg}(t) \times \vec{v}_{cg}(t) + R(t)\left\{-\sum_i m_i \mathrm{Skew}(\vec{X}_i)\ \mathrm{Skew}(\vec{X}_i)\right\}\vec{\Omega}(t)$$

Inertia Tensor (慣性テンソル)

$$I_{in} \equiv -\sum_i m_i \mathrm{Skew}(\vec{X}_i)\,\mathrm{Skew}(\vec{X}_i)$$

$$\Rightarrow\ \mathcal{K} = \frac{1}{2} M \left\| \vec{v}_{cg} \right\|^2 + \frac{1}{2} \vec{\Omega}^T I_{in} \vec{\Omega}$$

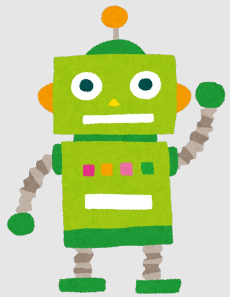
Quadratic form

$\vec{\Omega}$ →  → $\mathcal{K}$

inertia tensor I_{in}
positive semi definite

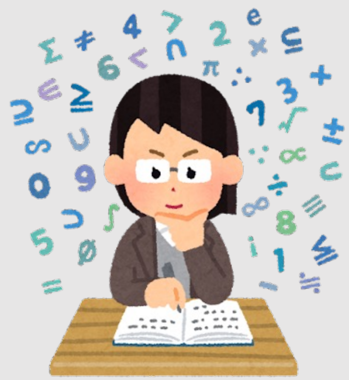
$$\widetilde{I_{in}} \equiv R I_{in} R^T$$

$$\Rightarrow\ \mathcal{K} = \frac{1}{2} M \left\| \vec{v}_{cg} \right\|^2 + \frac{1}{2} \vec{\omega}^T \widetilde{I_{in}} \vec{\omega}$$

$\vec{\omega}$ →  → $\mathcal{K}$

inertia tensor $\widetilde{I_{in}}$
positive semi definite

Equation for Rigid Body



$$\vec{L} = R(t) I_{in} \vec{\Omega}(t)$$

$$\dot{\vec{L}} = \dot{R}(t) I_{in} \vec{\Omega}(t) + R(t) I_{in} \dot{\vec{\Omega}}(t)$$

$$R(t)\mathrm{Skew}(\vec{\Omega}(t))$$

$$R^T(t)\dot{\vec{L}} = \mathrm{Skew}(\vec{\Omega}(t)) I_{in} \vec{\Omega}(t) + I_{in}\dot{\vec{\Omega}}(t)$$

Torque