# Lagrangian Mechanics

# Pros & Cons

**Newton's mechanics**

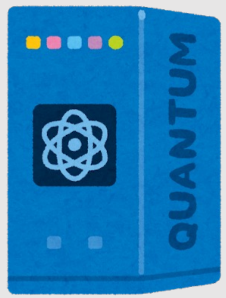
- ☺ Easy to understand

- ☹ Only takes Cartesian coordinate values
- ☹ System of linear equations

**Lagrangian Mechanics**

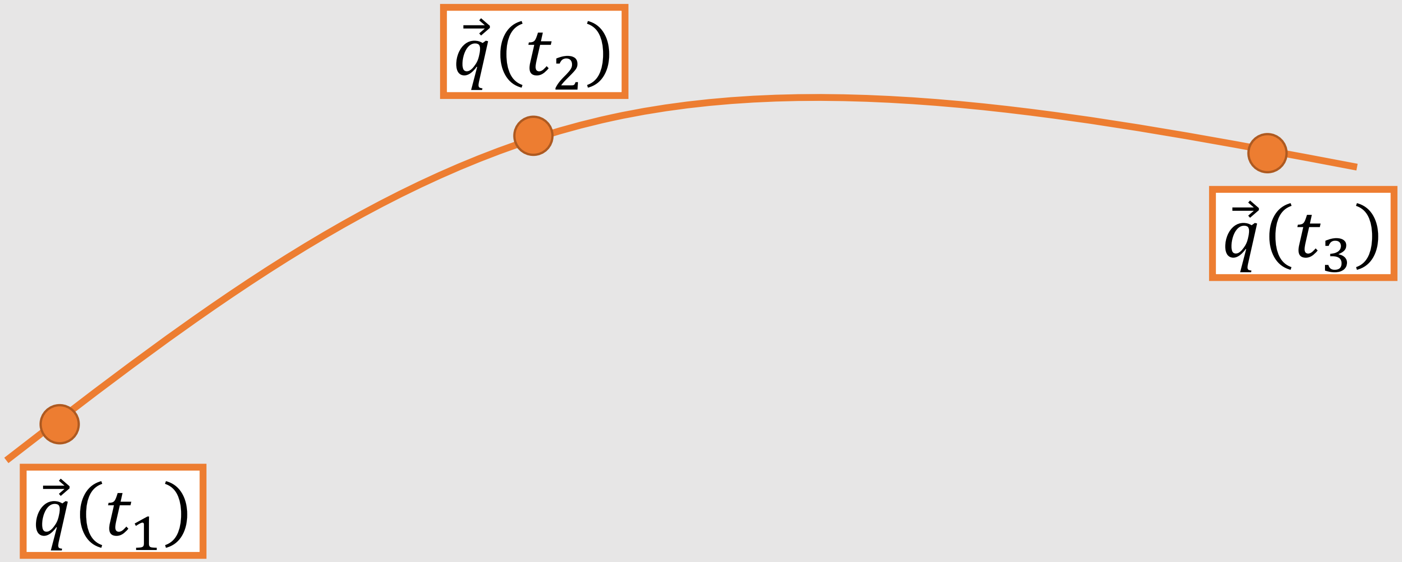
- ☹ Hard to understand

- ☺ Arbitrary variables (e.g., angles)
- ☺ Optimization of scalar value
- ☺ Leading to modern physics
- ☺ Simple & beautiful



# Trajectory of State

- State $\vec{q}$ is the parameters to represent physics phenomena written in any coordinate systems

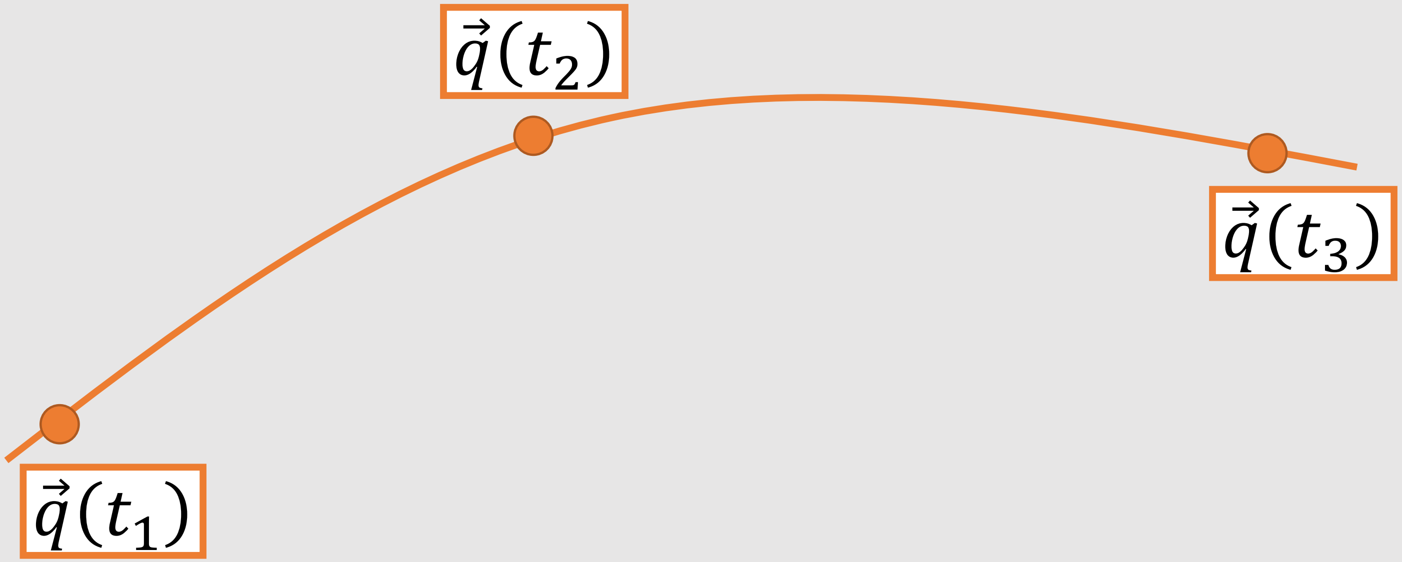


# Lagrangian: Kinetic Enrgy – Potential Enrgy

kinetic energy

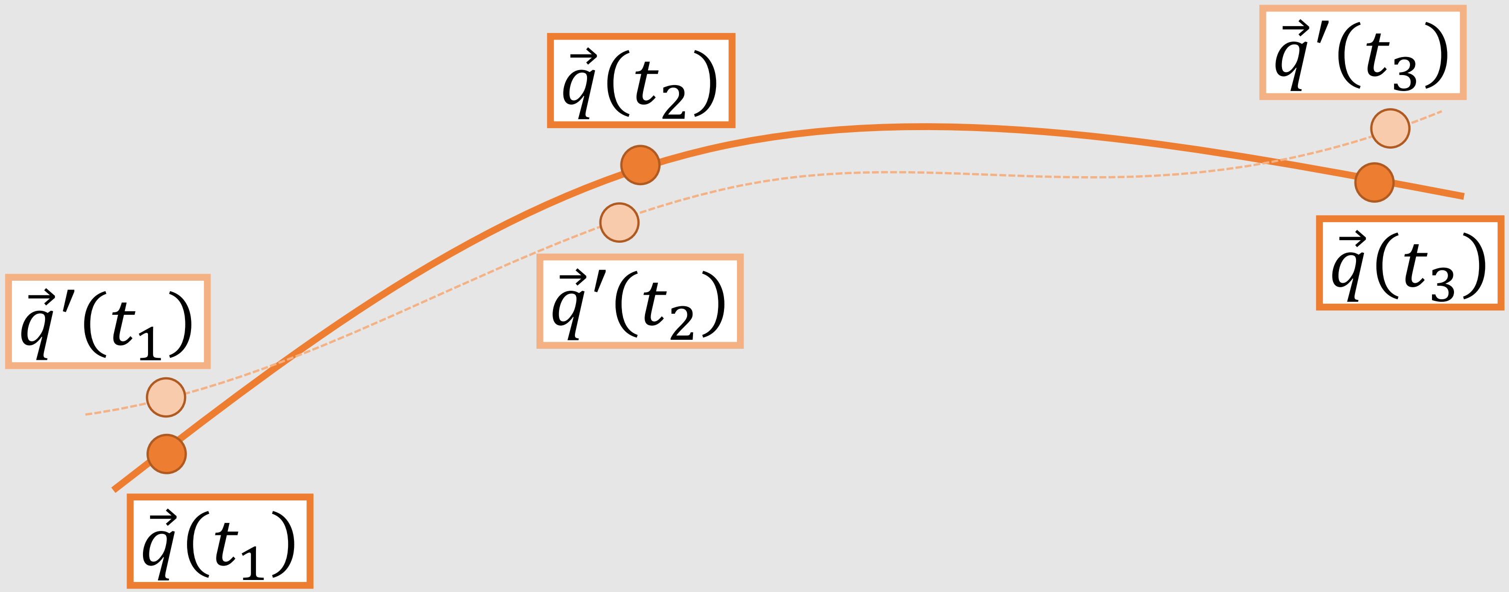
potential energy

$$\mathcal{L}(\vec{q},\dot{\vec{q}}) = \mathcal{K} - \mathcal{P}$$



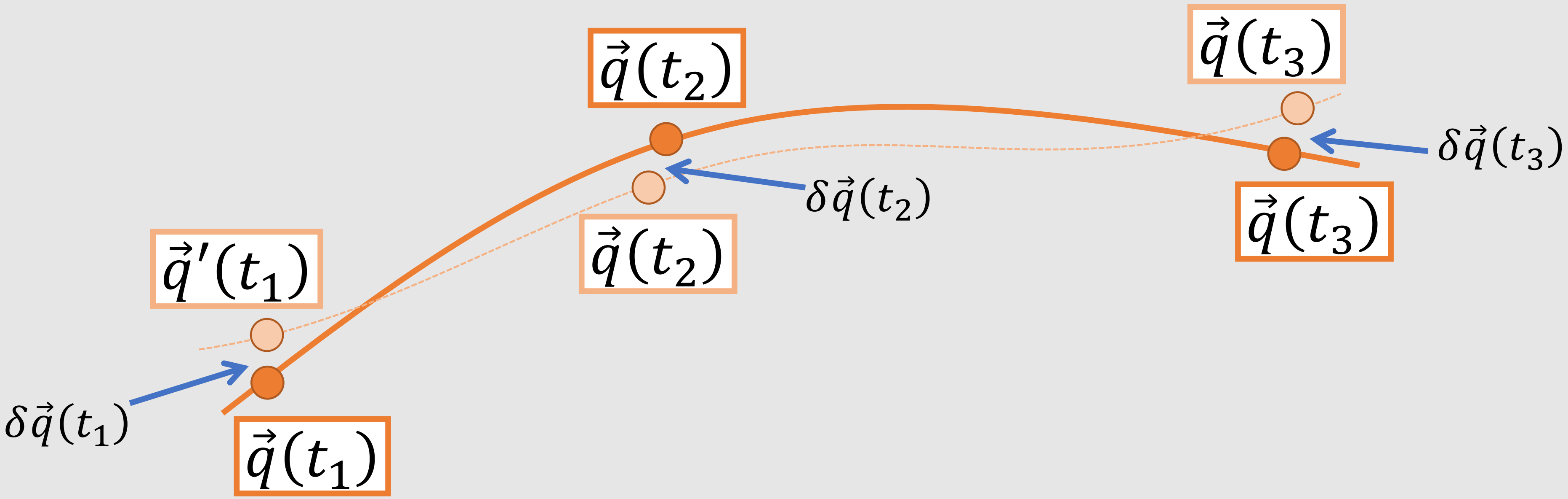
# Perturbation of State

Lagrangian for perturbed state: $\mathcal{L}'(\vec{q}', \dot{\vec{q}}')$



# Perturbation of Lagrangian

$$\delta\mathcal{L}(\vec{q},\dot{\vec{q}},\delta\vec{q},\delta\dot{\vec{q}}) = \mathcal{L}'(\vec{q}',\dot{\vec{q}}') - \mathcal{L}(\vec{q},\dot{\vec{q}})$$



# Euler-Lagrange Equation

- If $\vec{q}(t)$ is the solution, for arbitrary perturbation $\delta\vec{q}(t)$ it holds:

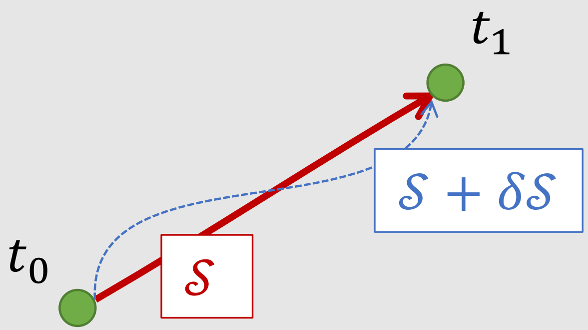
$$\frac{d}{dt}\left(\frac{\partial\delta\mathcal{L}}{\partial\delta\dot{\vec{q}}}\right) - \frac{\partial\delta\mathcal{L}}{\partial\delta\vec{q}} = 0$$

# What Euler-Lagrange Equation Means?

- Action $\mathcal{S}$: time integration of Lagrangian $\mathcal{L}$

$$\mathcal{S}(q, \dot{q}) = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q})\, dt$$

- Hamilton's principle: perturbation of action is zero $\delta\mathcal{S} = 0$

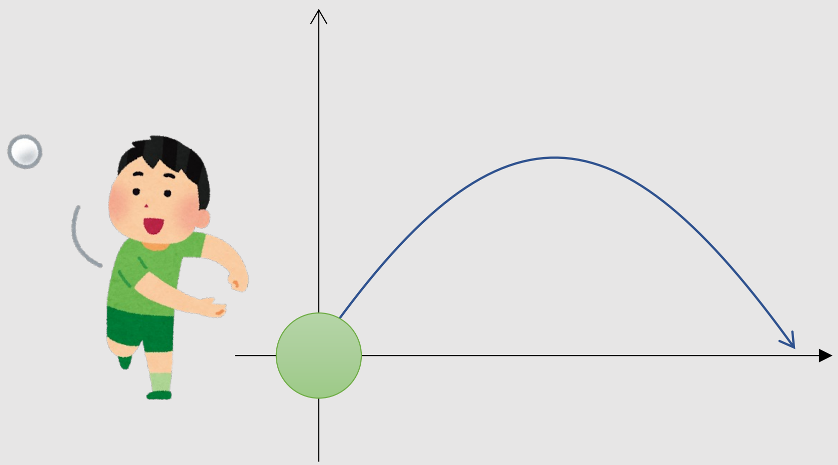


# Hamilton's Principle: Minimal Action

Minimizing time integration of Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{K} - \mathcal{W}$

→ Minimize $\mathcal{K}$ and maximize $\mathcal{W}$

- Minimize $\mathcal{K}$: smooth motion
- Maximize $\mathcal{W}$: movement is slow when potential energy is large

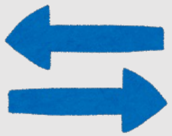


why making potential $\mathcal{W}$ large instead of small?

# Symmetry in Lagrangian = Conservation Law

- Noether's Theorem
- Symmetry = invariance under operation

Emmy Noether


translational symmetry ⟺ conservation of linear momentum


rotational symmetry ⟺ conservation of angular momentum


temporal symmetry ⟺ conservation of energy

# Example of Lagrange's Equation of Motion

Falling ball


$$\mathcal{L}(q, \dot{q}) = \mathcal{K} - \mathcal{W}$$

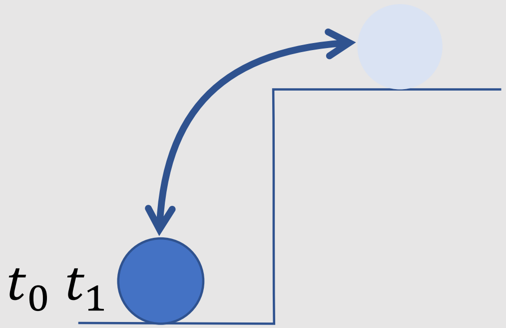
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

(Write Equations Here)

# Hamilton's Principle is *Unintuitive*

Making Lagrangian $\mathcal{L}(q, \dot{q}) = \mathcal{K} - \mathcal{W}$ small?

can a ball climb up a cliff ?




can a ball float in air ?