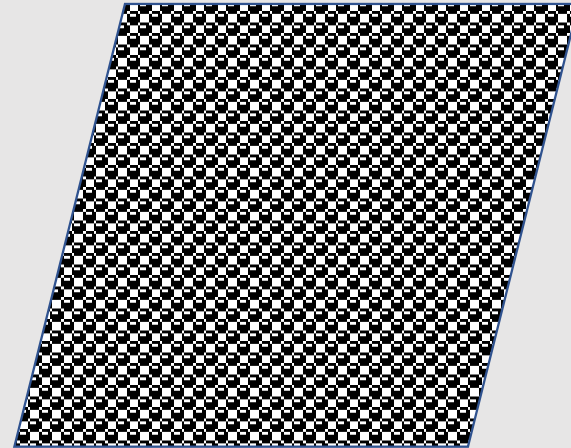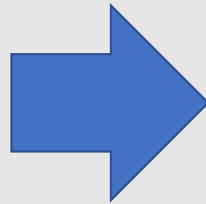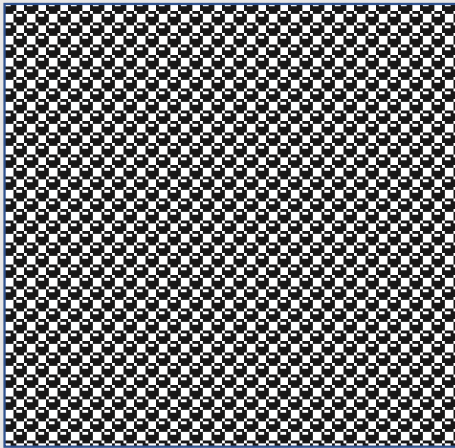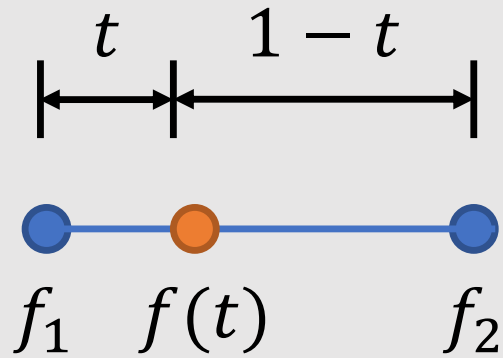# Grid & Mesh Interpolation

Continuum Approximation

- Drastically reducing degrees of freedom (DoFs)

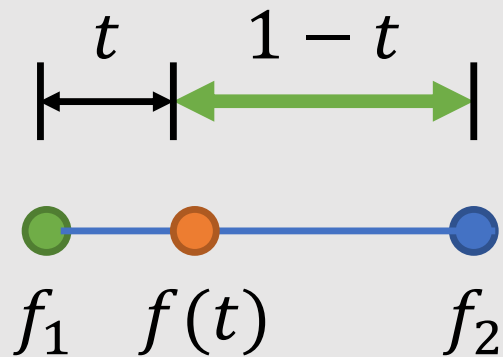# Linear Interpolation (1D)

$$f(0) = f_1$$

$$f(1) = f_2$$

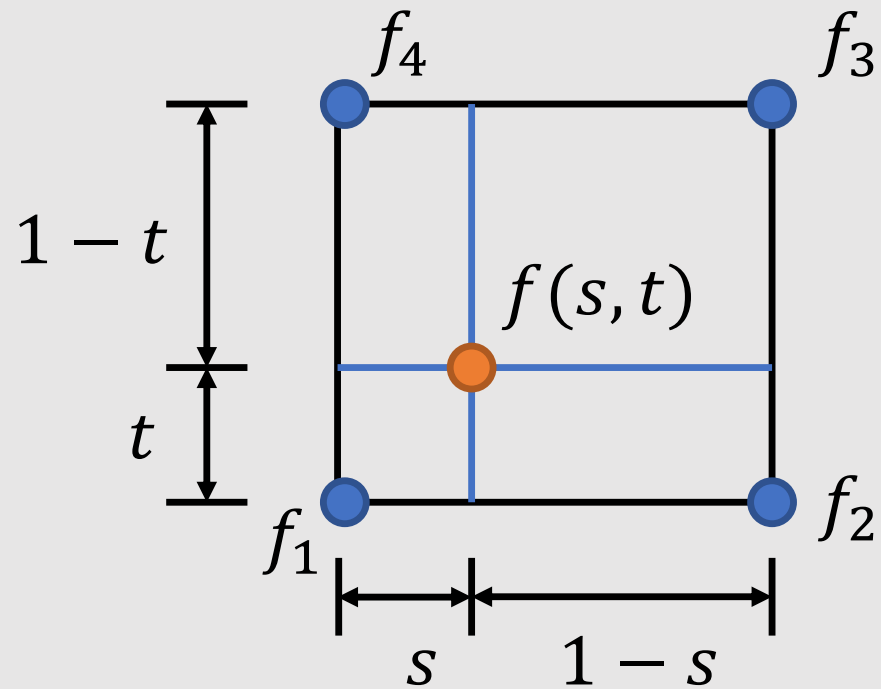# Linear Interpolation (1D)

$$f(0) = f_1$$

$$f(1) = f_2$$

Weights sums to 1

$$f(t) = \boxed{(1 - t)} f_1 + t f_2$$

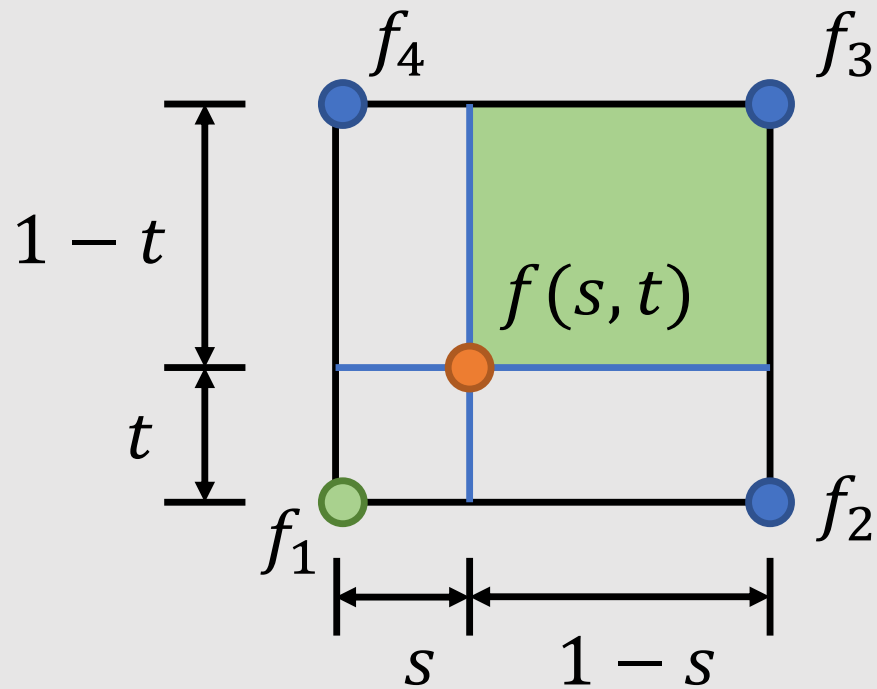# Bilinear Interpolation (2D)

$f(0,0) = f_1$

$f(1,0) = f_2$

$f(1,1) = f_3$

$f(0,1) = f_4$

Bilinear Interpolation (2D)

Weights sums to 1

$$f(s,t) = \boxed{(1-s)(1-t)f_1} + s(1-t)f_2 + stf_3 + (1-s)tf_4$$

Bilinear Interpolation (2D)
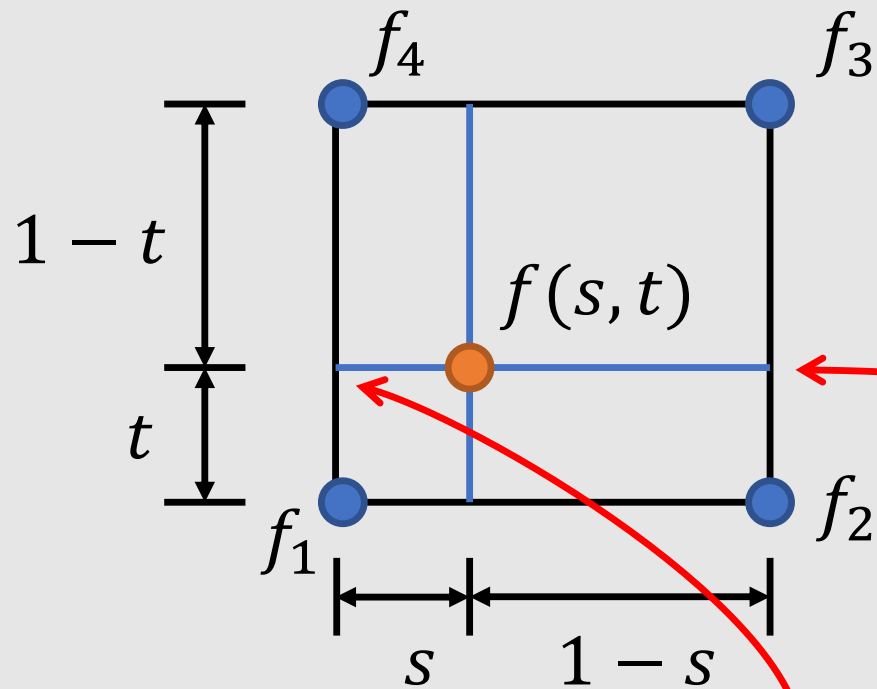
$f(s, t)$ is Linear in horizontal & vertical directions (not in an oblique direction)

$$f(s, t) = (1 - s)[(1 - t)f_1 + tf_4] + s[(1 - t)f_2 + tf_3]$$

Trilinear Interpolation (3D)

$f(s,t,u) =$
$(1-s)(1-t)(1-u)f_1 +$
$s(1-t)(1-u)f_2 +$
$st(1-u)f_3 +$
$(1-s)t(1-u)f_4 +$
$(1-s)(1-t)uf_5 +$
$s(1-t)uf_6 +$
$stuf_7 +$
$(1-s)tuf_8$
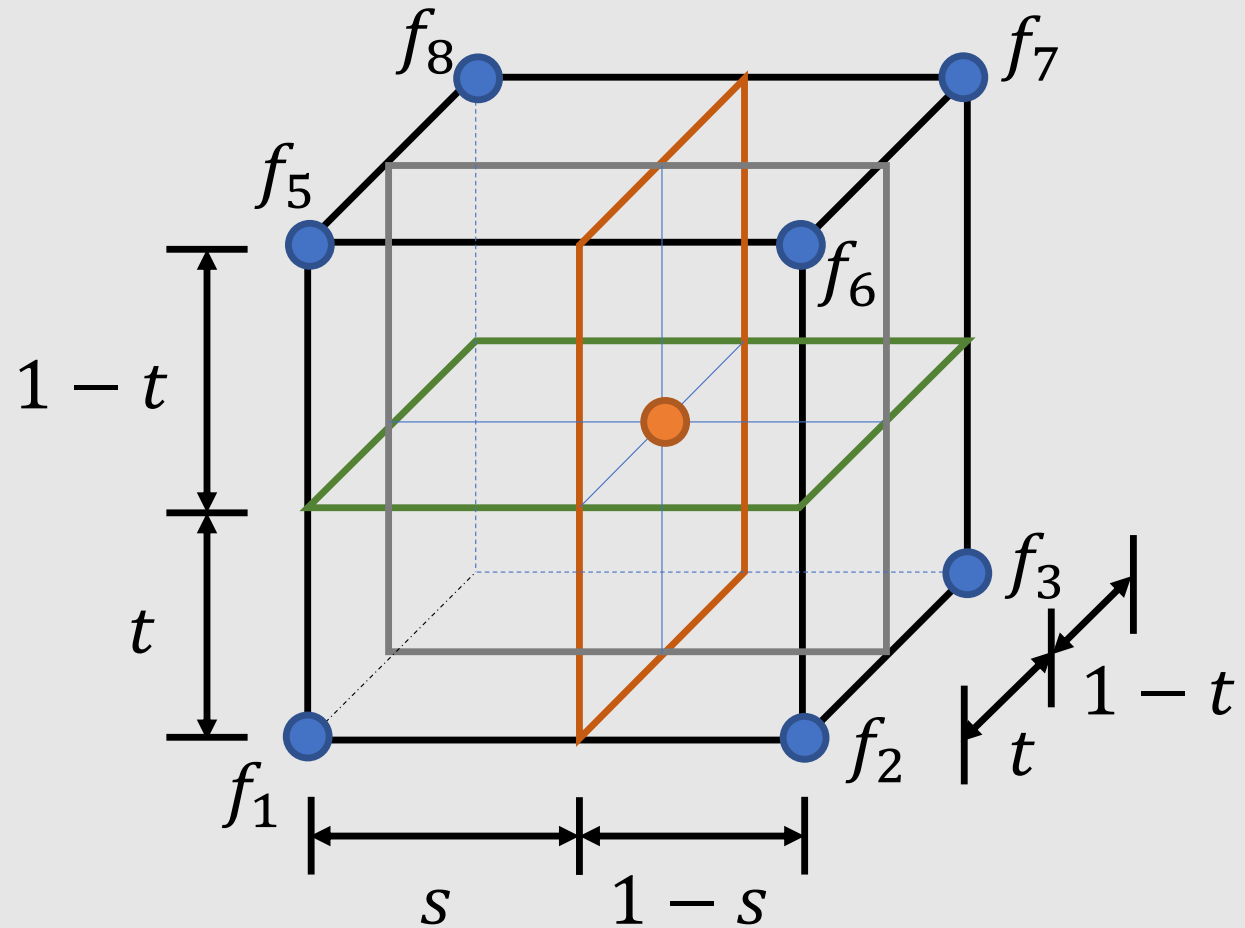
Trilinear Interpolation (3D)

$f(s, t, u) =$
$(1-s)(1-t)(1-u)f_1 +$
$s(1-t)(1-u)f_2 +$
$st(1-u)f_3 +$
$(1-s)t(1-u)f_4 +$
$(1-s)(1-t)uf_5 +$
$s(1-t)uf_6 +$
$stuf_7 +$
$(1-s)tuf_8$
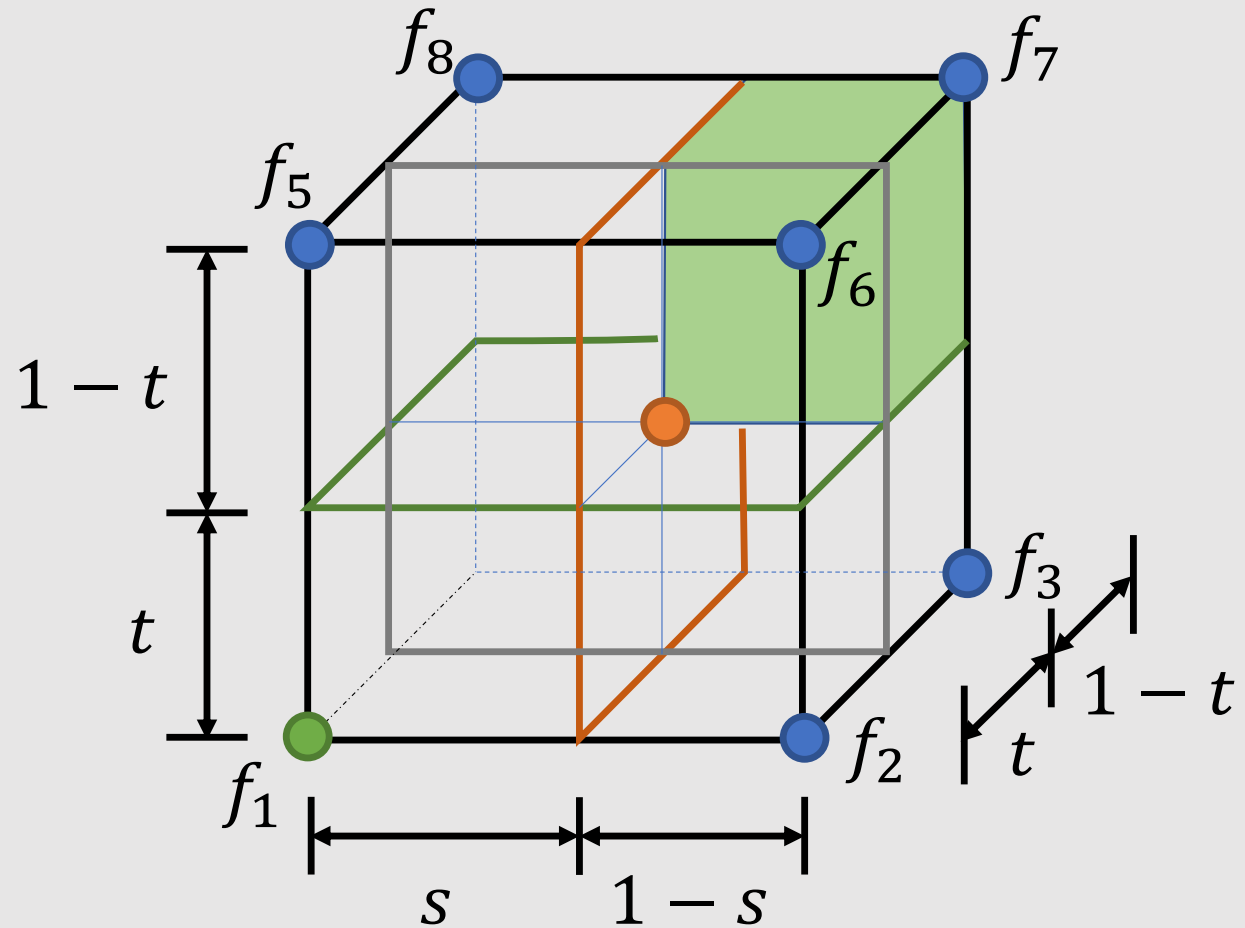
Barycentric Coordinates for a Triangle

$$f_p = f(L_1, L_2, L_3) = L_1 f_1 + L_2 f_2 + L_3 f_3$$

$$L_1 = \frac{Area(p, 2, 3)}{Area(1, 2, 3)}$$

$$L_2 = \frac{Area(1, p, 3)}{Area(1, 2, 3)}$$

$$L_3 = \frac{Area(1, 2, p)}{Area(1, 2, 3)}$$
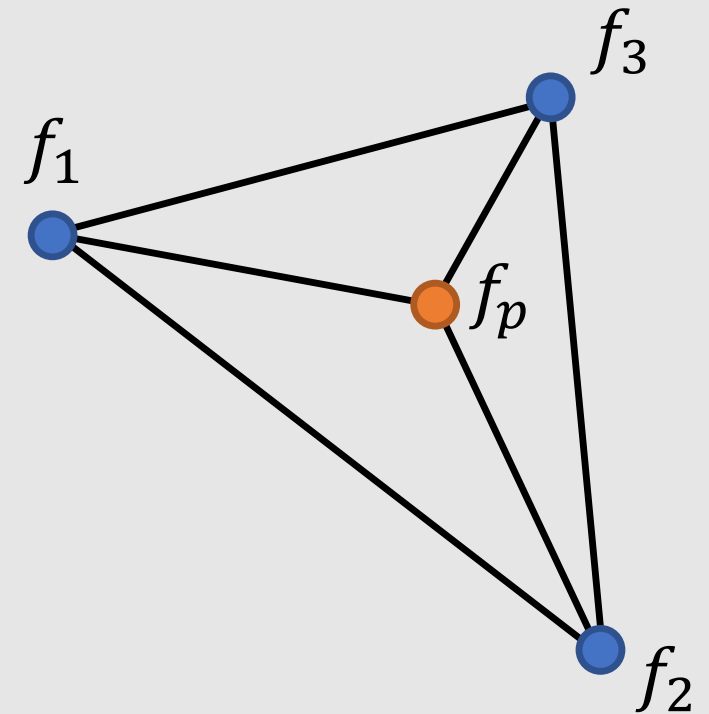
Barycentric Coordinates for a Tetrahedron

$$f_p = f(L_1, L_2, L_3, L_4)$$
$$= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4$$

$$L_1 = \frac{Volume(p, 2, 3, 4)}{Volume(1, 2, 3, 4)}$$

$$L_2 = \frac{Volume(1, p, 3, 4)}{Volume(1, 2, 3, 4)}$$

$$L_3 = \frac{Volume(1, 2, p, 4)}{Volume(1, 2, 3, 4)}$$
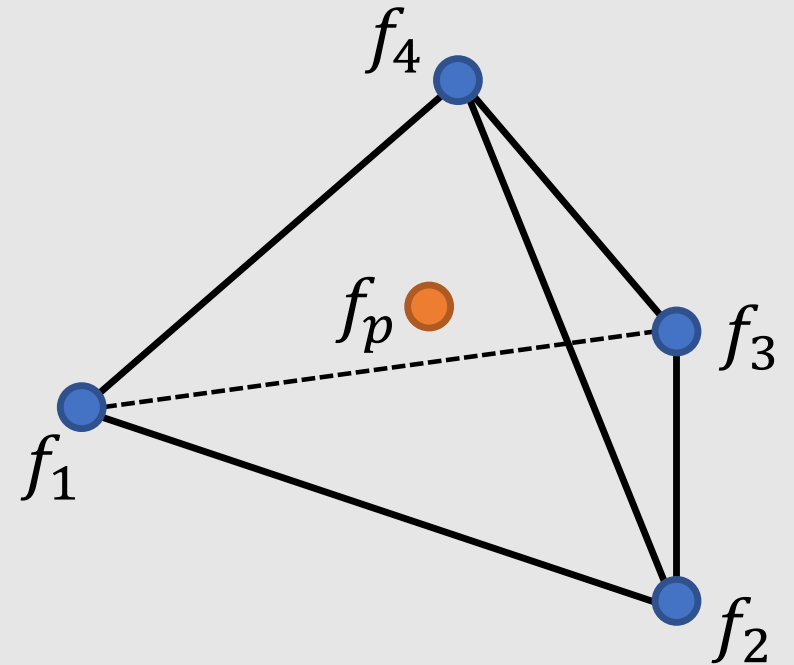
$$L_4 = \frac{Volume(1, 2, 3, p)}{Volume(1, 2, 3, 4)}$$

Barycentric Coordinates for a Tetrahedron

$$f_p = f(L_1, L_2, L_3, L_4)$$
$$= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4$$

$$L_1 = \frac{Volume(p, 2, 3, 4)}{Volume(1, 2, 3, 4)}$$

$$L_2 = \frac{Volume(1, p, 3, 4)}{Volume(1, 2, 3, 4)}$$

$$L_3 = \frac{Volume(1, 2, p, 4)}{Volume(1, 2, 3, 4)}$$

$$L_4 = \frac{Volume(1, 2, 3, p)}{Volume(1, 2, 3, 4)}$$
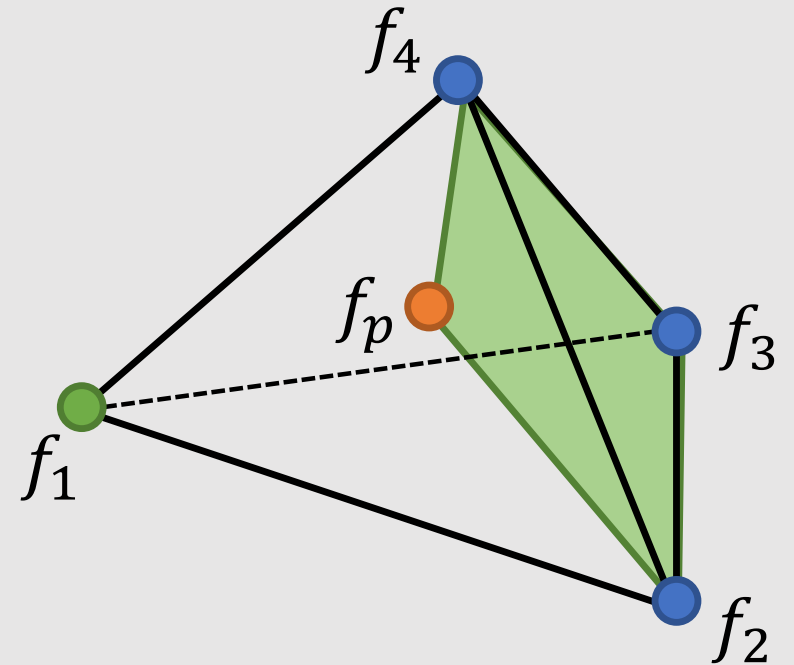
Differentiation of Interpolated Values

Volume of Tetrahedron from Parallelepiped

- Volume of parallelepiped: $V = \vec{a} \cdot (\vec{b} \times \vec{c})$

- Volume of tetrahedron: $V = 1/6 \; \vec{a} \cdot (\vec{b} \times \vec{c})$
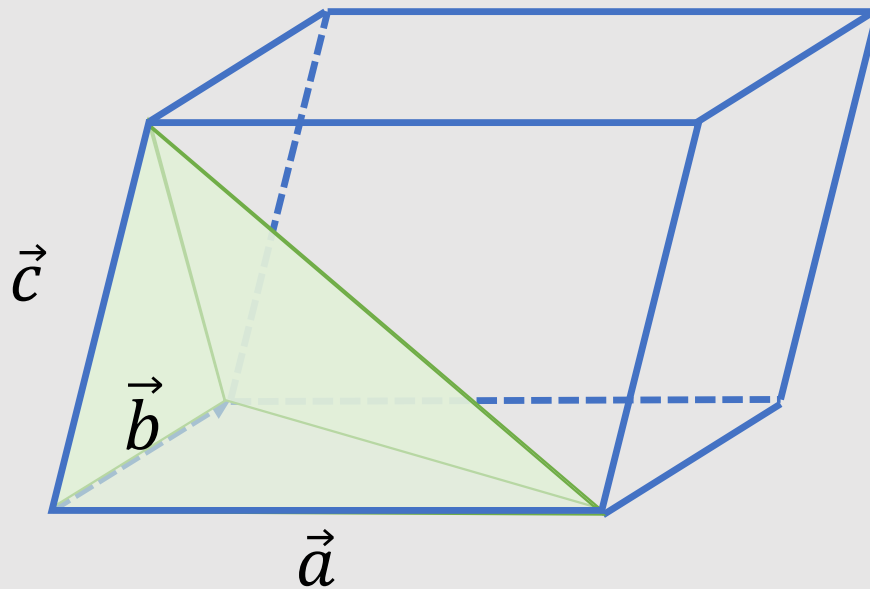
Differential of Scalar Triple Product

$$W(\vec{x}) = \vec{b} \cdot (\vec{a} \times \vec{x})$$

$$= \vec{x} \cdot \left(\vec{b} \times \vec{a} \right)$$

$$\frac{\partial W}{\partial \vec{x}} = \vec{b} \times \vec{a}$$
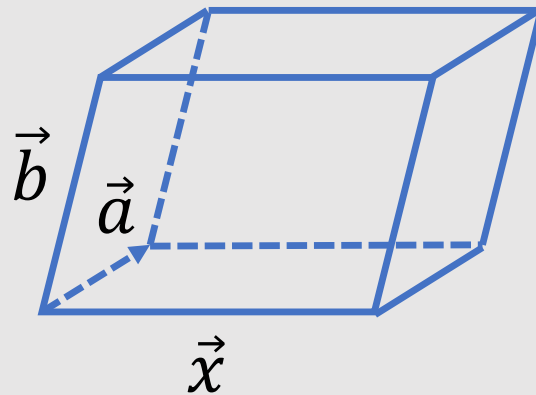
The volume is linear to the position

Volume of Tetrahedron
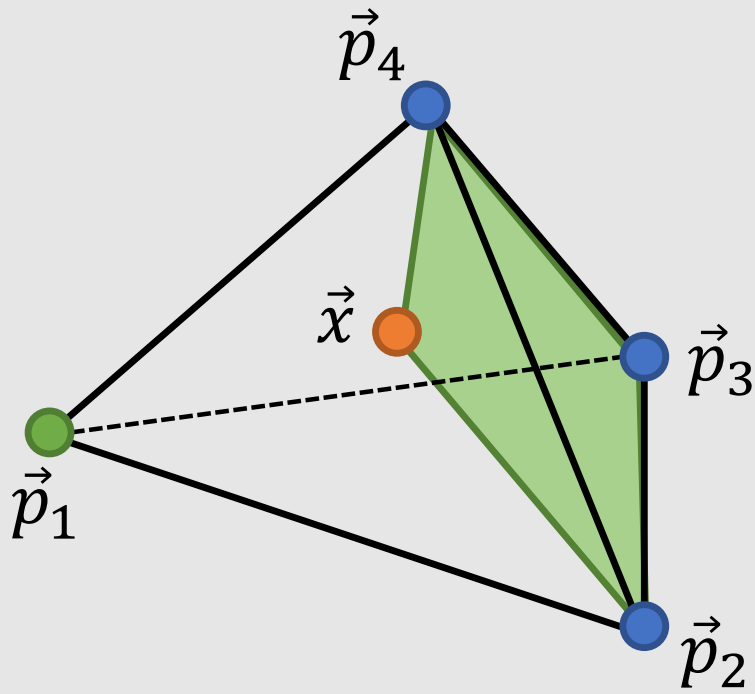
$$L_1 = \frac{Volume(\vec{x}, \vec{p}_2, \vec{p}_3, \vec{p}_4)}{Volume(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}$$

$$Volume(\vec{x}, \vec{p}_2, \vec{p}_3, \vec{p}_4)$$
$$= 1/6 \ (\vec{x} - \vec{p}_2) \cdot \{(\vec{p}_4 - \vec{p}_2) \times (\vec{p}_3 - \vec{p}_2)\}$$

$$\frac{\partial L_1}{\partial \vec{x}} = \frac{1/6 \ (\vec{p}_4 - \vec{p}_2) \times (\vec{p}_3 - \vec{p}_2)}{Volume(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4)}$$

Gradient of Interpolated Value

$$f_{\vec{x}} = f(L_1, L_2, L_3, L_4)$$
$$= L_1 f_1 + L_2 f_2 + L_3 f_3 + L_4 f_4$$

differentiation

$$\frac{\partial f_{\vec{x}}}{\partial \vec{x}} = \frac{\partial L_1}{\partial \vec{x}} f_1 + \frac{\partial L_2}{\partial \vec{x}} f_2 + \frac{\partial L_3}{\partial \vec{x}} f_3 + \frac{\partial L_4}{\partial \vec{x}} f_4$$
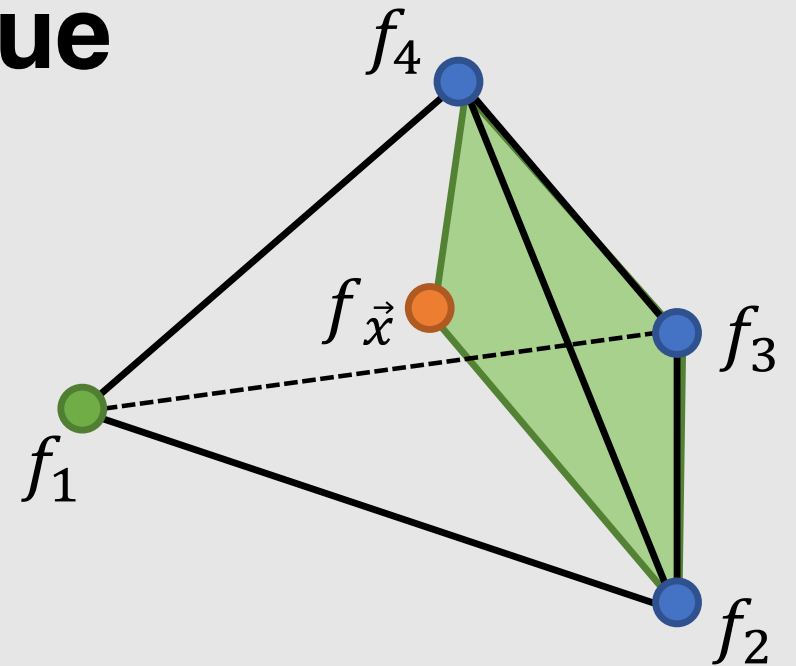
Integration of Polynomial over Simplex

Integration Rule for Triangle

$$\int_{\vec{x} \in Tri} L_1^i L_2^j L_3^k \, d\vec{x} = \frac{2! \, i! \, j! \, k! \, Area}{(2 + i + j + k)!}$$

$$\vec{x}$$

$$f(\vec{x}) = L_1^i L_2^j L_3^k$$

Baldoni, Velleda, Nicole Berline, Jesus De Loera, Matthias Köppe, and Michèle Vergne. "How to integrate a polynomial over a simplex." *Mathematics of Computation* 80, no. 273 (2011): 297–325. https://arxiv.org/abs/0809.2083
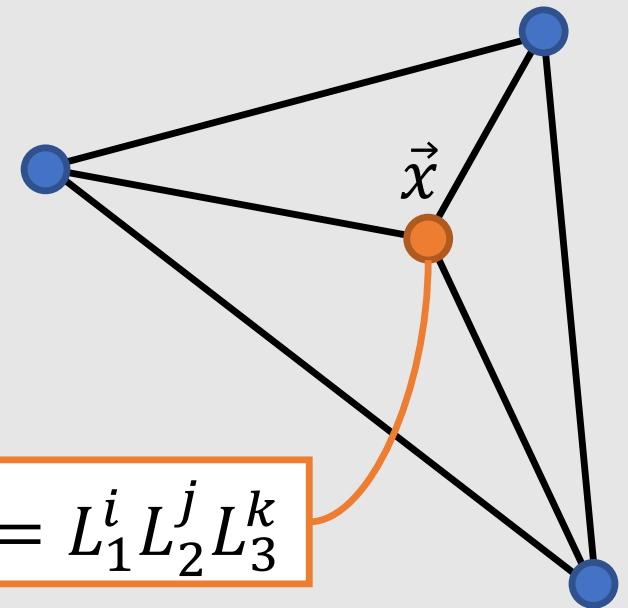
Example of Integration over Triangle

$$\int_{\vec{x}\in Tri} L_1 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^0 L_3^0 d\vec{x} = \frac{2!\,1!\,0!\,0!\,Area}{(2+1+0+0)!} = \frac{1}{3} Area$$

$$\int_{\vec{x}\in Tri} L_1^2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^2 L_2^0 L_3^0 d\vec{x} = \frac{2!\,2!\,0!\,0!\,Area}{(2+2+0+0)!} = \frac{1}{6} Area$$
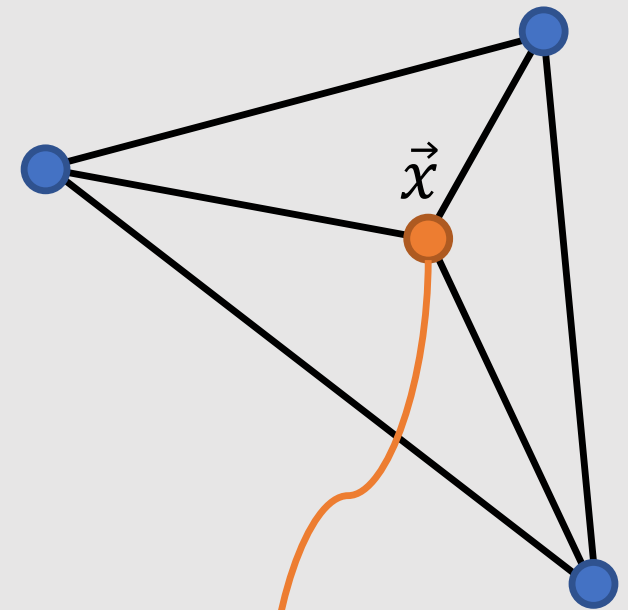
$$\int_{\vec{x}\in Tri} L_1 L_2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^1 L_3^0 d\vec{x} = \frac{2!\,1!\,1!\,0!\,Area}{(2+1+1+0)!} = \frac{1}{12} Area$$

$$\int_{\vec{x}\in Tri} L_a L_b d\vec{x} = \frac{1}{12} Area(1 + \delta_{ab})$$
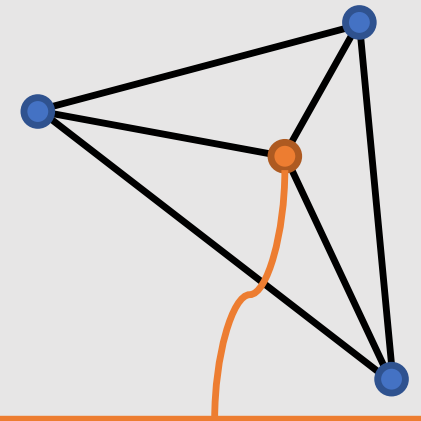
Center of The Gravity of a Triangle in 3D

$$\vec{p}_{cg} = \frac{1}{Area} \int_{\vec{x} \in \text{Tri}} \vec{x} d\vec{x}$$

$$= \frac{1}{Area} \int_{\vec{x} \in \text{Tri}} L_1 \vec{p}_1 + L_2 \vec{p}_2 + L_3 \vec{p}_3 d\vec{x}$$

$$= \frac{1}{3} (\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$

$\vec{x}$

$$\vec{x} = L_1 \vec{p}_1 + L_2 \vec{p}_2 + L_3 \vec{p}_3$$
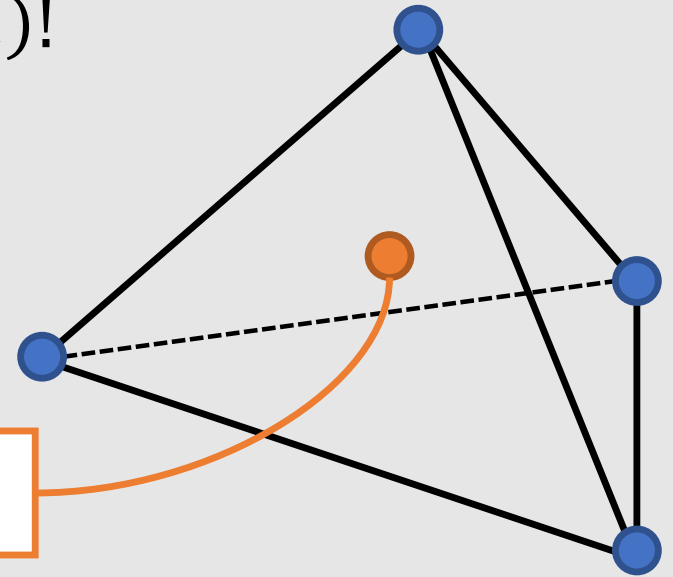
Inertia Tensor of a Triangle in 3D

$$I_{in} = \int_{\vec{x}\in\text{Tri}} -\text{Skew}(\vec{x})\text{Skew}(\vec{x})d\vec{x}$$

$$= \int_{\vec{x}\in\text{Tri}} -\text{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3)\text{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3)d\vec{x}$$

$$= \int_{\vec{x}\in\text{Tri}} \sum_{1\leq a\leq 3}\sum_{1\leq b\leq 3} -\text{Skew}(L_a\vec{p}_b)\text{Skew}(L_a\vec{p}_b)d\vec{x}$$

$$= \sum_{1\leq a\leq 3}\sum_{1\leq b\leq 3} -\text{Skew}(\vec{p}_a)\text{Skew}(\vec{p}_b)\int_{\vec{x}\in\text{Tri}} L_a L_b d\vec{x}$$

$$= \sum_{1\leq a\leq 3}\sum_{1\leq b\leq 3} (\vec{p}_a^T\vec{p}_b I - \vec{p}_a\otimes\vec{p}_b)\frac{Area}{12}(1 + \delta_{ab})$$

$$\vec{x} = L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3$$

Integration Rule for Tetrahedron

$$\int_{\vec{x} \in \text{Tet}} L_1^i L_2^j L_3^k L_4^l d\vec{x} = \frac{3!\, i!\, j!\, k!\, l!\, Volume}{(3 + i + j + k + l)!}$$

$$\vec{f}(\vec{x}) = L_1^i L_2^j L_3^k L_4^l$$

Baldoni, Velleda, Nicole Berline, Jesus De Loera, Matthias Köppe, and Michèle Vergne. "How to integrate a polynomial over a simplex." *Mathematics of Computation* 80, no. 273 (2011): 297–325. https://arxiv.org/abs/0809.2083

Example of Integration over Tetrahedra

$$\int_{\vec{x}\in Tet} L_1 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^0 L_3^0 L_4^0 d\vec{x} = \frac{3!\,1!\,0!\,0!\,0!\,Volume}{(3+1+0+0+0)!} = \frac{1}{4}Volume$$

$$\int_{\vec{x}\in Tet} L_1^2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^2 L_2^0 L_3^0 L_4^0 d\vec{x} = \frac{3!\,2!\,0!\,0!\,0!\,Volume}{(3+2+0+0+0)!} = \frac{1}{10}Volume$$

$$\int_{\vec{x}\in Tet} L_1 L_2 d\vec{x} = \int_{\vec{x}\in Tri} L_1^1 L_2^1 L_3^0 L_4^0 d\vec{x} = \frac{2!\,1!\,1!\,0!\,0!\,Volume}{(3+1+1+0+0)!} = \frac{1}{20}Volume$$

$$\int_{\vec{x}\in Tet} L_a L_b d\vec{x} = \frac{1}{20}Volume(1+\delta_{ab})$$

Inertia Tensor of a Tetrahedron

$$I_{in} = \int_{\vec{x} \in \text{Tet}} -\text{Skew}(\vec{x})\text{Skew}(\vec{x})d\vec{x}$$

$$= \int_{\vec{x} \in \text{Tet}} -\text{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3 + L_4\vec{p}_4)\text{Skew}(L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3 + L_4\vec{p}_4)d\vec{x}$$

$$= \int_{\vec{x} \in \text{Tet}} \sum_{1 \le a \le 4} \sum_{1 \le b \le 4} -\text{Skew}(L_a\vec{p}_b)\text{Skew}(L_a\vec{p}_b)d\vec{x}$$

$$= \sum_{1 \le a \le 4} \sum_{1 \le b \le 4} -\text{Skew}(\vec{p}_a)\text{Skew}(\vec{p}_b) \int_{\vec{x} \in \text{Tet}} L_a L_b d\vec{x}$$

$$= \sum_{1 \le a \le 4} \sum_{1 \le b \le 4} (\vec{p}_a^T\vec{p}_b I - \vec{p}_a \otimes \vec{p}_b)\frac{Volume}{20}(1 + \delta_{ab})$$
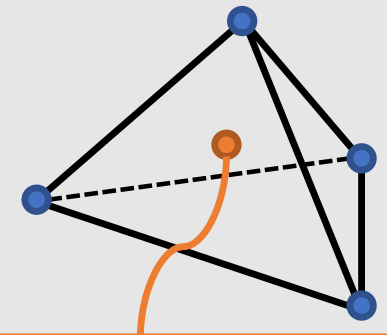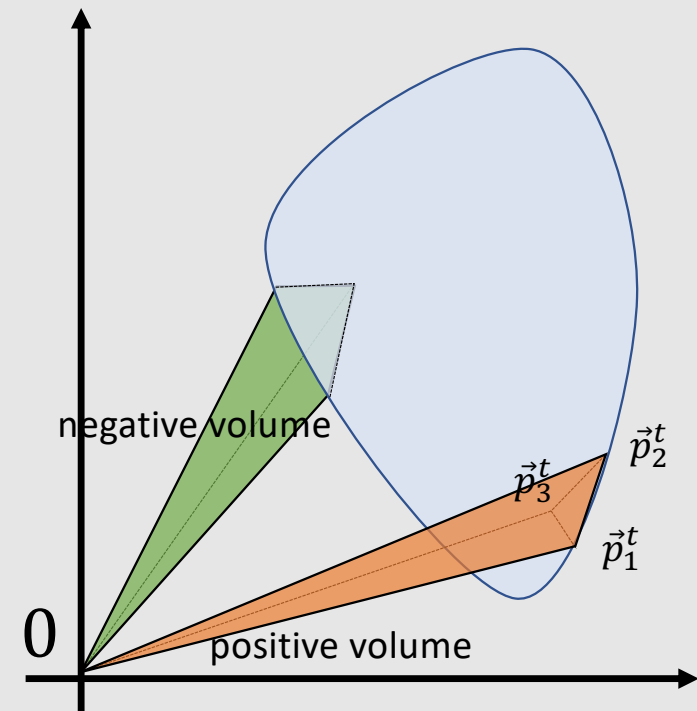

$$\vec{x} = L_1\vec{p}_1 + L_2\vec{p}_2 + L_3\vec{p}_3 + L_4\vec{p}_4$$

Inertia Tensor of a Solid 3D **Triangle** Mesh

- Summing over inertia tensors of tetrahedra connecting the origin of the coordinate and the three corner points of the triangle

$$I_{inertia} = \sum_{t \in Tri} I_{inertia}(0, \vec{p}_1^{\,t}, \vec{p}_2^{\,t}, \vec{p}_3^{\,t})$$
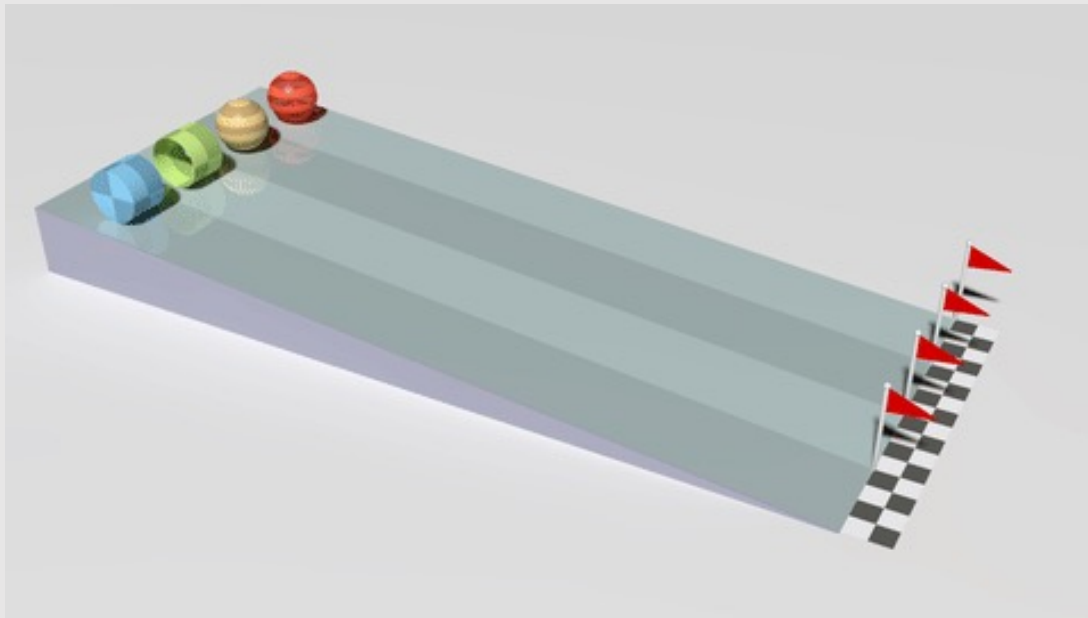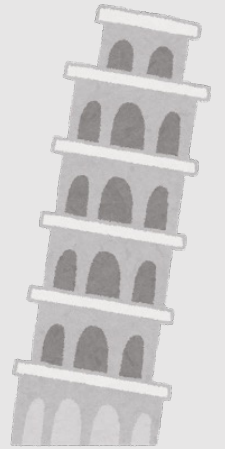
Inertia Tensor = How Hard to Rotate

Same radius, same mass, different speed!?

Galileo Galilei

$$\mathcal{K} = \frac{1}{2} M \left\| \vec{v}_{cg} \right\|^2 + \frac{1}{2} \vec{\Omega}^T I_{in} \vec{\Omega}$$

Credit: Lucas Vieira @ Wikipedia