Finite Element Method

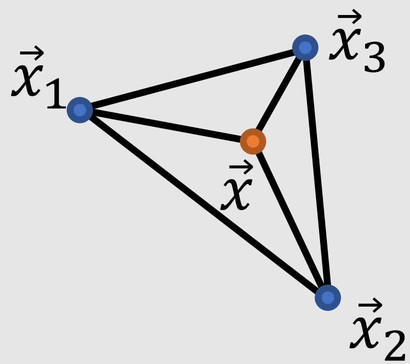
What is Finite Element Method?

- Solution by energy minimization

$$\vec{x}_{solution} = \underset{\vec{x}}{\operatorname{argmin}} W(\vec{x})$$

- Value inside element is interpolated

$$\vec{x} = \sum_{i \in Nodes} w_i \vec{x}_i$$



- Energy is sum of the element-wise energy

$$W(\vec{x}) = \sum_{e \in Elements} W_e(\vec{x})$$

FEM of Laplace Equation on Triangle

Discrete Laplacian → the energy is sum of the squared differences between neighbors

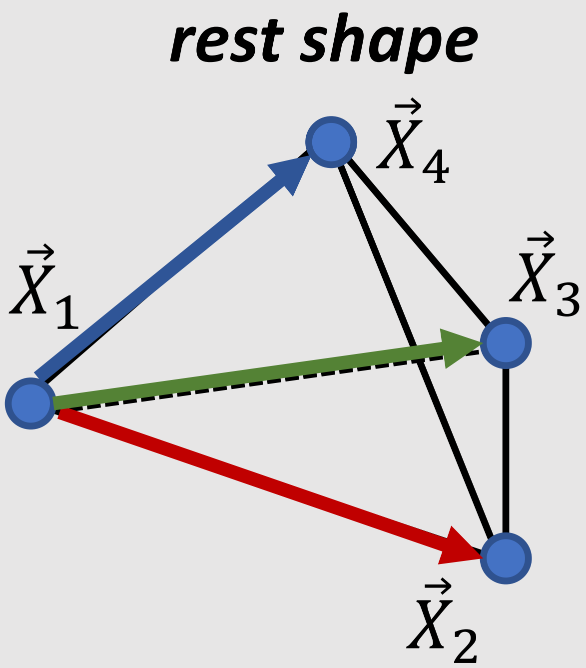
Continuous Laplacian → the energy is integration of the squared gradient

Making the solution as smooth as possible!

$$W(\phi) = \int_{\Omega} \nabla\phi \cdot \nabla\phi \; d\Omega$$

$$W_e(\phi) = \int_{\vec{x}\in Tri} \frac{\partial L_a \phi_a}{\partial \vec{x}} \cdot \frac{\partial L_b \phi_b}{\partial \vec{x}} \, d\vec{x} = \phi_a \phi_b \int_{\vec{x}\in Tri} \frac{\partial L_a}{\partial \vec{x}} \cdot \frac{\partial L_b}{\partial \vec{x}} \, d\vec{x}$$

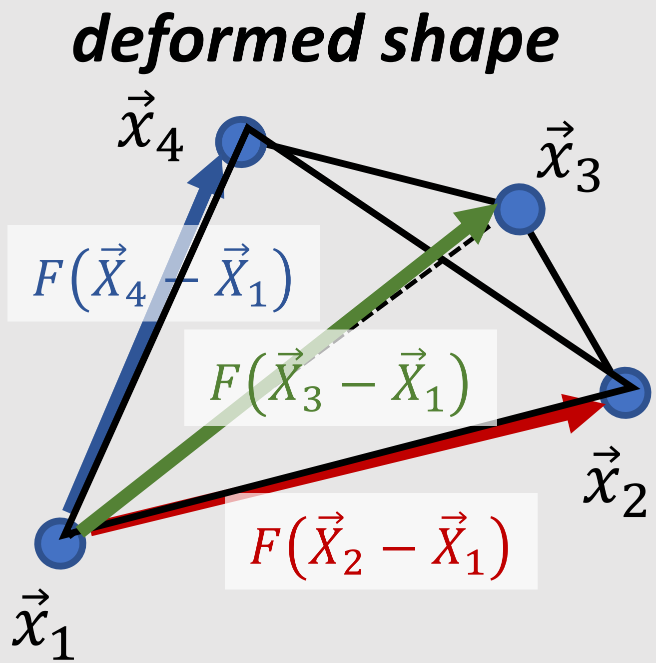
Deformation Gradient Tensor F for Tet.



$$(\vec{x}_2 - \vec{x}_1) = F(\vec{X}_2 - \vec{X}_1)$$

$$(\vec{x}_3 - \vec{x}_1) = F(\vec{X}_3 - \vec{X}_1)$$

$$(\vec{x}_4 - \vec{x}_1) = F(\vec{X}_4 - \vec{X}_1)$$

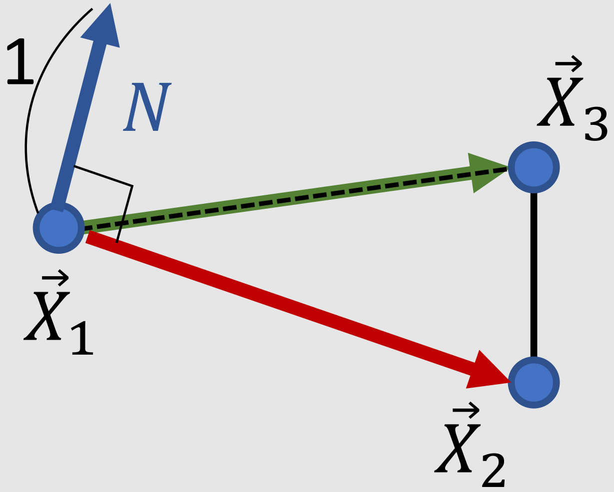


$$[\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, \vec{x}_4 - \vec{x}_1] = F[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, \vec{X}_4 - \vec{X}_1]$$

$$F = [\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, \vec{x}_4 - \vec{x}_1][\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, \vec{X}_4 - \vec{X}_1]^{-1}$$

Deformation Gradient Tensor F for 3D Tri.

rest shape

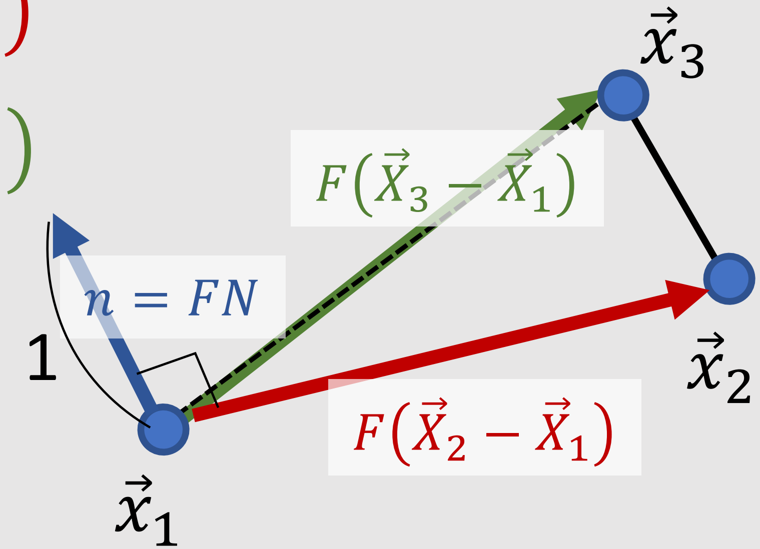


$$(\vec{x}_2 - \vec{x}_1) = F\left(\vec{X}_2 - \vec{X}_1\right)$$

$$(\vec{x}_3 - \vec{x}_1) = F\left(\vec{X}_3 - \vec{X}_1\right)$$

$$n = FN$$

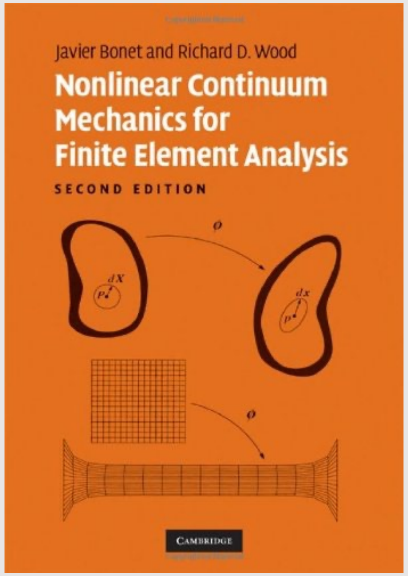
deformed shape



$$[\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, n] = F\left[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, N\right]$$

$$F = [\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, n]\left[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, N\right]^{-1}$$

Reference



•Bonet, Javier, and Richard D. Wood. 1997. *Nonlinear continuum mechanics for finite element analysis*. Cambridge: Cambridge University Press.