Equation of Rigid Body

Euler's Equation of Motion of Rigid Body

kinetic energy of a point

$$\mathcal{K} = \frac{1}{2}\vec{v}^T m\vec{v}$$

equation of motion

$$m\dot{\vec{v}} = \vec{F}$$

kinetic energy of rigid body

$$\mathcal{K} = \frac{1}{2}\overrightarrow{\Omega}^T I_{in}\overrightarrow{\Omega}$$

wrong!!

$$\frac{1}{2} I_{in}\dot{\overrightarrow{\Omega}} = \vec{F}$$

equation of motion

$$I_{in}\dot{\overrightarrow{\Omega}} + \mathrm{Skew}\left(\overrightarrow{\Omega}\right) I_{in}\overrightarrow{\Omega} = \vec{F}$$

Euler-Lagrange Equation

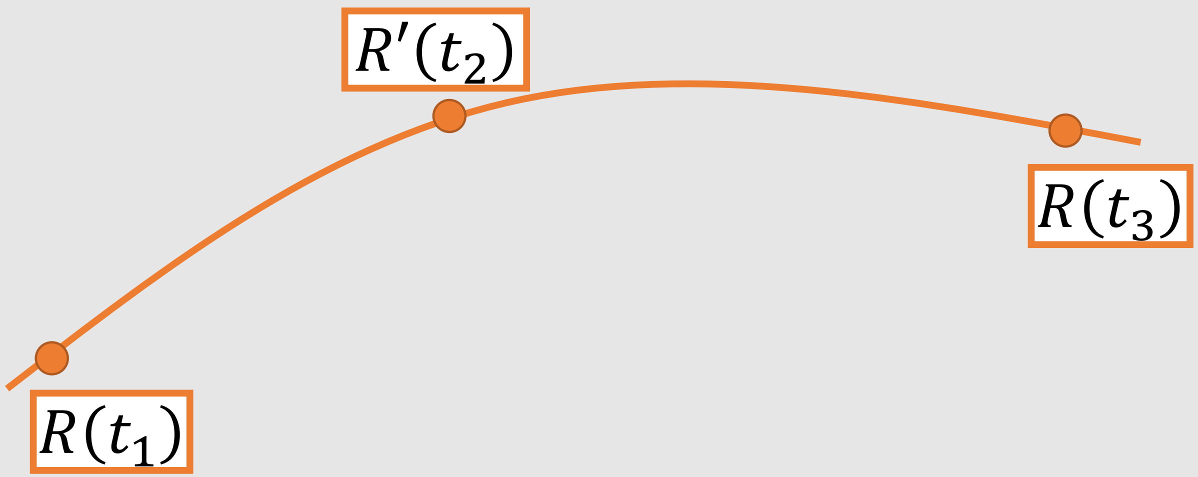
- If $\vec{q}(t)$ is the solution, for arbitrary perturbation $\delta\vec{q}(t)$ it holds:

$$\frac{d}{dt}\left(\frac{\partial \delta \mathcal{L}}{\partial \delta \dot{\vec{q}}}\right) - \frac{\partial \delta \mathcal{L}}{\partial \delta \vec{q}} = 0$$

Parameterization of deviation

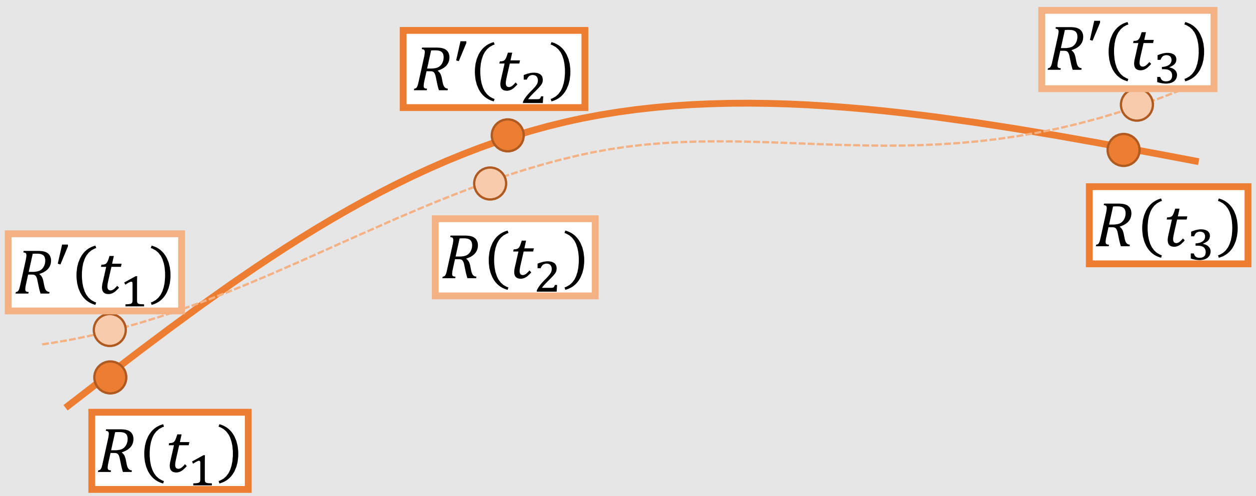
Velocity of the Parameterized deviation
($\vec{\Omega}$ cannot be put in here)

Trajectory of Rotation



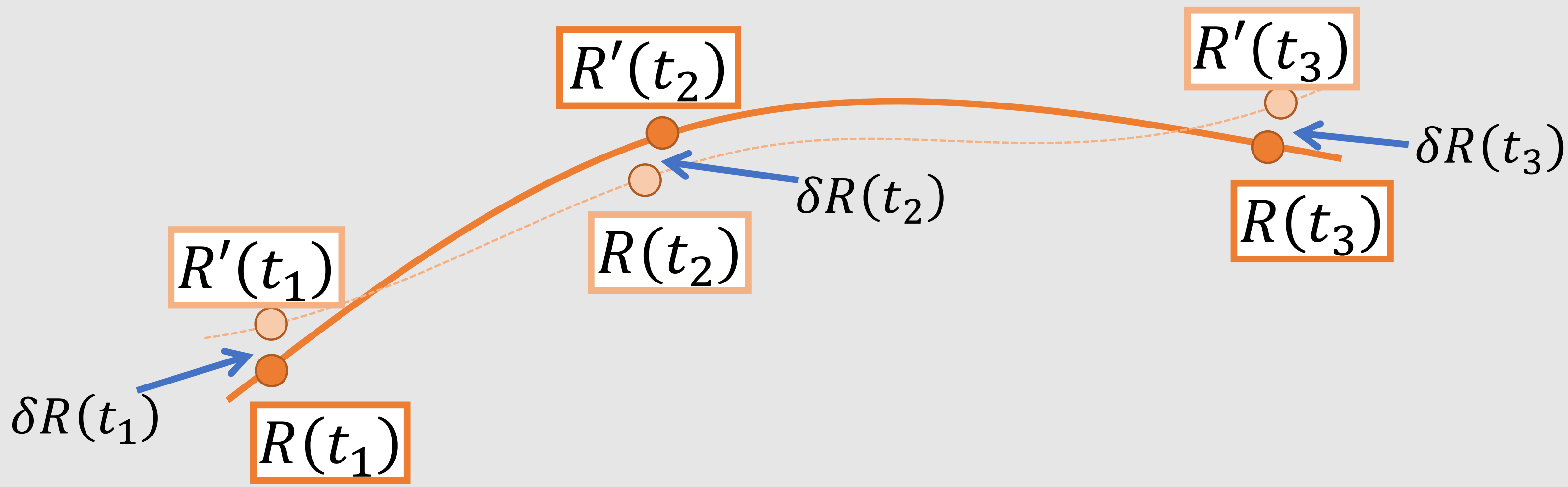
Perturbation of Rotation

Perturbed rotation is constrained as $R'^T R' = I$



Perturbation of Rotation

Perturbation δR get constraint as ${R'}^T R' = (R + \delta R)^T (R + \delta R) = I$



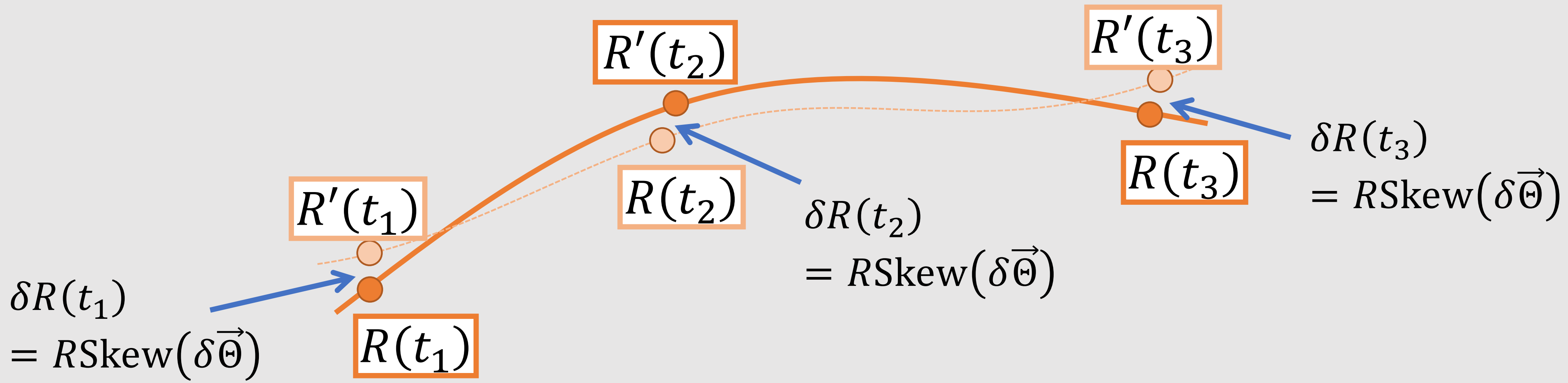
DoF Elimination for Rotation Perturbation

$$R'^T R' = (R + \delta R)^T (R + \delta R) = I$$

$$\text{Skew}(\delta\vec{\Theta}) \equiv R^T \delta R$$

differentiation

$$\text{Skew}\left(\delta\dot{\vec{\Theta}}\right) = \dot{R}^T \delta R + R^T \delta \dot{R}$$



Perturbation of Angular Velocity

perturbation δR that satisfy constraint

$$\mathrm{Skew}(\delta\vec{\Theta}) \equiv R^T \delta R \qquad \mathrm{Skew}(\vec{\Omega}) = R^T \dot{R}$$

$$\mathrm{Skew}\left(\delta\dot{\vec{\Theta}}\right) = \dot{R}^T \delta R + R^T \delta \dot{R} \qquad \mathrm{Skew}(\delta\vec{\Omega}) = \delta R^T \dot{R} + R^T \delta \dot{R}$$

$$= \mathrm{Skew}(\delta\vec{\Theta})\mathrm{Skew}(\vec{\Omega}) - \mathrm{Skew}(\vec{\Omega})\mathrm{Skew}(\delta\vec{\Theta})$$

$$= \mathrm{Skew}(\mathrm{Skew}(\vec{\Omega})\delta\vec{\Theta})$$

$$\mathrm{Skew}(\delta\vec{\Omega}) = \mathrm{Skew}\left(\delta\dot{\vec{\Theta}}\right) + \delta R^T \dot{R} - \dot{R}^T \delta R$$

$$\delta\vec{\Omega} = \delta\dot{\vec{\Theta}} + \mathrm{Skew}(\vec{\Omega})\delta\vec{\Theta}$$

Rigid Body Floating in Space

- No potential energy & no linear velocity

$$\mathcal{L}(R,\dot{R}) = \mathcal{K} - \mathcal{W} = \frac{1}{2}\vec{\Omega}^T I_{in}\vec{\Omega}$$

$$\begin{aligned}\delta\mathcal{L}\left(R,\dot{R},\delta\vec{\Theta},\delta\dot{\vec{\Theta}}\right) &= \delta\vec{\Omega}^T I_{in}\vec{\Omega} \\ &= \left\{\delta\dot{\vec{\Theta}} + \mathrm{Skew}\left(\vec{\Omega}\right)\delta\vec{\Theta}\right\}^T I_{in}\,\vec{\Omega}\end{aligned}$$

Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial\delta\mathcal{L}}{\partial\delta\dot{\vec{\Theta}}}\right) - \frac{\partial\delta\mathcal{L}}{\partial\delta\vec{\Theta}} = 0$$

equation of motion (a.k.a Euler's equation)

$$\frac{d}{dt}\left(I_{in}\vec{\Omega}\right) + \mathrm{Skew}\left(\vec{\Omega}\right)I_{in}\vec{\Omega} = 0$$

Equation of Motion for Rigid Body

equation for reference config

$$I_{in}\dot{\vec{\Omega}} + \text{Skew}\left(\vec{\Omega}\right)I_{in}\vec{\Omega} = 0$$

$$\vec{\omega} = R\vec{\Omega}$$

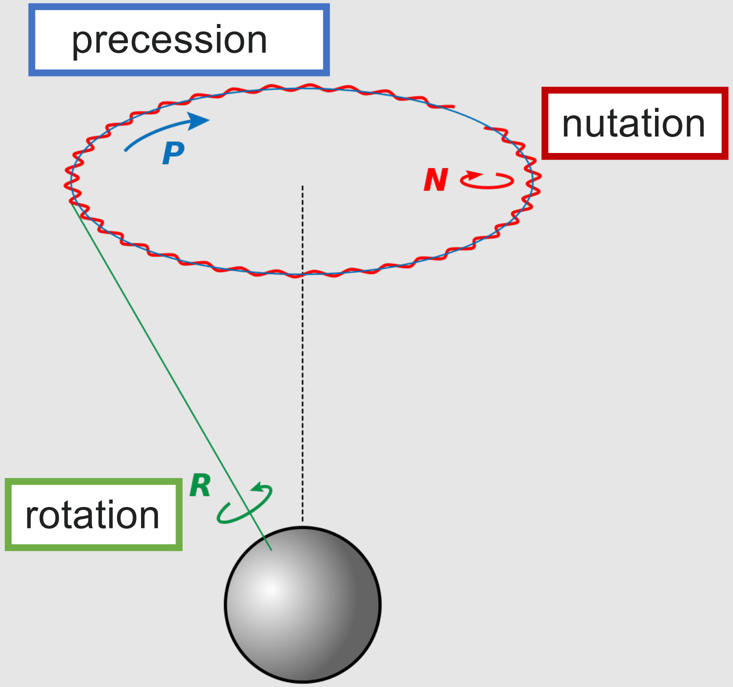
$$\widetilde{I_{in}} = R I_{in} R^T$$

equation for current config

$$\widetilde{I_{in}}\dot{\vec{\omega}} + \text{Skew}(\vec{\omega})\widetilde{I_{in}}\vec{\omega} = 0$$

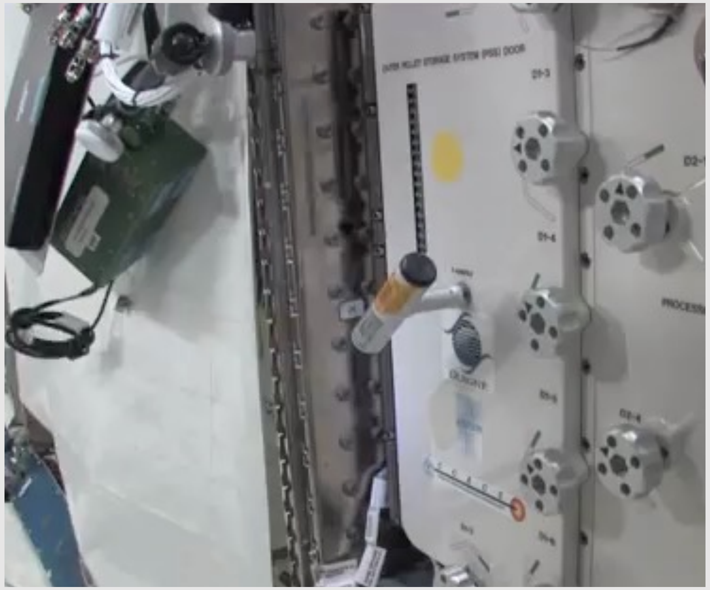
Solution of Euler's Equation

In general, the rotation axis $\vec{\omega}$, $\vec{\Omega}$ also move around



Credit: User Herbye@Wikipedia

Rotating T-shaped object in zero gravity



Dancing T-handle in zero-g

https://www.youtube.com/watch?v=1n-HMSCDYtM

Solution of Euler's Equation: Special Case

When angular velocity $\vec{\Omega}$ lined up with eigen-vector of inertia tensor I_{in}, angular velocities $\vec{\Omega}$, $\vec{\omega}$ are constant

$$I_{in}\dot{\vec{\Omega}} + \text{Skew}(\vec{\Omega})I_{in}\vec{\Omega} = 0$$

$I_{in}\vec{\Omega}$: $\lambda\vec{\Omega}$

$\text{Skew}(\vec{\Omega})I_{in}\vec{\Omega}$: $\lambda\vec{\Omega}\times\vec{\Omega} = 0$

$$\dot{\vec{\Omega}} = 0$$

$$\vec{\Omega} = constant$$

$$\widetilde{I_{in}}\dot{\vec{\omega}} + \text{Skew}(\vec{\omega})\widetilde{I_{in}}\vec{\omega} = 0$$

$\widetilde{I_{in}}\vec{\omega}$: $\lambda\vec{\omega}$

$\text{Skew}(\vec{\omega})\widetilde{I_{in}}\vec{\omega}$: $\lambda\vec{\omega}\times\vec{\omega} = 0$

$$\dot{\vec{\omega}} = 0$$

$$\vec{\omega} = constant$$