9 More on differentiation

9a Finite Taylor expansion

An infinitely differentiable function $\mathbb{R} \to \mathbb{R}$ need not be analytic. It has a formal Taylor expansion, but maybe of zero radius of convergence, or maybe converging to a different function. An example:

$$f(x) = \mathrm{e}^{-1/x} \quad \text{for } x > 0\,, \qquad f(x) = 0 \quad \text{for } x \le 0\,.$$

9a1 Theorem. [1] [2] If an infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ is not a polynomial then there exists $x \in \mathbb{R}$ such that $f^{(n)}(x)$ is irrational for all n.

Thus, $\exists x \; \forall n \;\; f^{(n)}(x) \ne 0$.

The set of rational numbers may be replaced with any other countable set.

We'll prove the theorem via iterated Baire category theorem.

9a2 Lemma. If f is a polynomial on $[a, b]$ and $\forall n \;\; f(b+\varepsilon_n) = f(b)$ for some $\varepsilon_n \to 0+$ then f is constant on $[a, b]$.

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[1] Exercise 10.2.9 in book: B. Thomson, J. Bruckner, A. Bruckner, "Real analysis", second edition, 2008.

[2] The theorem:

Theorem: Let $f(x)$ be C^∞ on (c, d) such that for every point x in the interval there exists an integer N_x for which $f^{(N_x)}(x) = 0$; then $f(x)$ is a polynomial.

is due to two Catalan mathematicians:

F. Sunyer i Balaguer, E. Corominas, Sur des conditions pour qu'une fonction infiniment dérivable soit un polynôme. Comptes Rendues Acad. Sci. Paris, 238 (1954), 558-559.

F. Sunyer i Balaguer, E. Corominas, Condiciones para que una función infinitamente derivable sea un polinomio. Rev. Mat. Hispano Americana, (4), 14 (1954).

The proof can also be found in the book (p. 53):

W. F. Donoghue, Distributions and Fourier Transforms, Academic Press, New York, 1969.

I will never forget it because in an "Exercise" of the "Opposition" to became "Full Professor" I was posed the following problem:

What are the real functions indefinitely differentiable on an interval such that a derivative vanish at each point?

Juan Arias de Reyna; see Question 34059 on Mathoverflow.

Proof. We have $f^{(n)}(b) = 0$ for $n = 1, 2, \dots$ since otherwise $f(b+\varepsilon) = f(b) + c\varepsilon^k + o(\varepsilon^k)$ for some $k \ge 1$ and $c \ne 0$. $\square$

The same holds for $f(a - \varepsilon_n)$, of course.

Assume that f is a counterexample to Theorem 9a1.

Consider a (maybe empty) set P_f of all maximal nondegenerate intervals $I \subset \mathbb{R}$ such that f is a polynomial on I. Note that intervals of P_f are closed and pairwise disjoint.

9a3 Lemma. The open set

$$G_f = \bigcup_{I \in P_f} \operatorname{Int} I$$

is dense (in $\mathbb{R}$).

Proof. Closed sets

$$F_{n,r} = \{x : f^{(n)}(x) = r\} \quad \text{for } r \in \mathbb{Q} \text{ and } n = 0, 1, 2, \dots$$

cover $\mathbb{R}$. By (5b7), $\cup_{n,r} \operatorname{Int} F_{n,r}$ is dense. Clearly, f is a polynomial on each interval contained in this dense open set. $\square$

It follows that P_f, treated as a totally (in other words, linearly) ordered set, is dense (that is, if $I_1, I_2 \in P_f$, $I_1 < I_2$ then $\exists I \in P_f \;\; I_1 < I < I_2$). It may contain minimal and/or maximal element (unbounded intervals), but the rest of P_f, being an unbounded dense countable totally ordered set, is order isomorphic to $\mathbb{Q} \cap (0, 1)$ (the proof is similar to the proof of Lemma 2d4; so-called back-and-forth method).

Now we want to contract each interval of P_f into a point. (We could consider a topological quotient space...)

We take an order isomorphism $\varphi : P_f \to \mathbb{Q}$ between P_f and one of $\mathbb{Q} \cap (0, 1)$, $\mathbb{Q} \cap [0, 1)$, $\mathbb{Q} \cap (0, 1]$, $\mathbb{Q} \cap [0, 1]$, and construct an increasing $\psi : \mathbb{R} \to [0, 1]$ such that $\psi(x) = \varphi(I)$ whenever $x \in I$. Clearly, such ψ exists and is unique. It is continuous. The image $\psi(\mathbb{R})$ is one of $(0, 1)$, $[0, 1)$, $(0, 1]$, $[0, 1]$. In every case $\psi(\mathbb{R})$ is completely metrizable. Note that $\psi^{-1}(\mathbb{Q}) = \cup_{I \in P_f} I$, and ψ is one-to-one on $\mathbb{R} \setminus \cup_{I \in P_f} I$.

We define $E_{n,r} \subset \psi(\mathbb{R})$ for $r \in \mathbb{Q}$ and $n = 0, 1, 2, \dots$ as follows:

$$E_{n,r} = \{x : \psi^{-1}(x) \subset F_{n,r}\}\,.$$

9a4 Lemma. Each $E_{n,r}$ is closed in $\psi(\mathbb{R})$.

Proof. Given $x_1 > x_2 > \dots$, $x_k \in E_{n,r}$, $x_k \downarrow x$ in $\psi(\mathbb{R})$, we take $t_k \in \psi^{-1}(x_k) \subset F_{n,r}$ and note that $t_1 > t_2 > \dots$, $t_k \downarrow t \in \psi^{-1}(x)$, $f^{(n)}(t_k) = r$ for all k, thus $f^{(n)}(t) = r$, that is, $t \in F_{n,r}$.

If x is irrational then $x \in E_{n,r}$ since $\psi^{-1}(x) = \{t\}$.

If x is rational then $\psi^{-1}(x) = [s,t]$, and $f^{(n)}(\cdot) = r$ on $[s,t]$ by Lemma 9a2 (applied to $f^{(n)}$).

The case $x_k \uparrow x$ is similar. $\square$

9a5 Exercise. Each $E_{n,r}$ is nowhere dense in $\psi(\mathbb{R})$.

Prove it.

Now Theorem 9a1 follows from the Baire category theorem (applied the second time).

9a6 Corollary. If an infinitely differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ has only finitely many non-zero partial derivatives at every point then f is a polynomial.

Proof. Let $d = 2$ (the general case is similar).

By Theorem 9a1, for every $x \in \mathbb{R}$ the function $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a polynomial; similarly, each $f(\cdot,y)$ is a polynomial. Introducing the set A_n of all $x \in \mathbb{R}$ such that $f(x,\cdot)$ is a polynomial of degree $\le n$ we have $A_n \uparrow \mathbb{R}$, therefore A_n is infinite (moreover, uncountable) for n large enough. The same holds for $f(\cdot,y)$ and B_n.

For $x \in A_n$ the coefficients $a_0(x), \dots, a_n(x)$ of the polynomial $f(x,\cdot)$ are linear functions of $f(x,y_0), \dots, f(x,y_n)$ provided that $y_0, \dots, y_n \in B_n$ are pairwise different. Therefore these coefficients are polynomials (in x), of degree $\le n$.

We get a polynomial $P : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) = P(x,y)$ for $x \in A_n$, $y \in \mathbb{R}$. For every $y \in \mathbb{R}$ two polynomials $f(\cdot,y)$ and $P(\cdot,y)$ coincide on the infinite set A_n, therefore they coincide on the whole $\mathbb{R}$. $\square$

A very similar (and a bit simpler) argument gives an interesting purely topological result.

9a7 Theorem. [1] If $[0,1]$ is the disjoint union of countably many closed sets then one of the sets is the whole $[0,1]$ (and others are empty).

Proof. (sketch). Assume the contrary: $[0,1] = \uplus_n F_n$, $F_n \ne \emptyset$ are closed. (Finitely many sets cannot do because of connectedness.) Then $\cup_n \operatorname{Int} F_n$ is dense in $[0,1]$.

[1]Exercise 10:2.8 in "Real analysis". Also Problem 13.15.3 in book: B. Thomson, J. Bruckner, A. Bruckner, "Elementary real analysis", second edition, 2008.

Consider a (maybe empty) set P of all maximal nondegenerate intervals $I \subset [0,1]$ such that $\exists n \;\; I \subset F_n$. Note that intervals of P_f are closed and pairwise disjoint. The open set $G = \cup_{I \in P} \operatorname{Int} I$ is dense in $[0,1]$, since it contains $\cup_n \operatorname{Int} F_n$.

It follows that P, treated as a totally ordered set, is dense. Thus, the set $C = [0,1] \setminus G$ is perfect, with no interior (and in fact, homeomorphic to the Cantor set).

As before, each $F_n \cap C$ is nowhere dense in C. (Hint: if an endpoint of an interval $I \in P$ belongs to $F_n \cap C$ then $I \subset F_n$.)

It remains to apply the Baire category theorem (in the second time). $\square$

9a8 Corollary. If the cube $[0,1]^d$ is the disjoint union of countably many closed sets then one of the sets is the whole $[0,1]^d$ (and others are empty).

Proof. Let $d = 2$ (the general case is similar).

Assume the contrary: $[0,1]^2 = \uplus_n F_n$, F_n are closed.

By Theorem 9a7, each $\{x\} \times [0,1]$ is contained in a single F_n. The same holds for each $[0,1] \times \{y\}$. Thus, it is a single n. $\square$

I wonder, is it true for an arbitrary continuum (that is, a compact connected metrizable space)?

9b Continuous and nowhere differentiable

9b1 Theorem. There exists a continuous function $f : [0,1] \to \mathbb{R}$ such that for every $x \in (0,1)$, f is not differentiable at x.

We consider the complete metric space $C[0,1]$ of all continuous $f : [0,1] \to \mathbb{R}$ (separable, in fact). We define continuous functions $\varphi_n : C[0,1] \to \mathbb{R}$ by

$$\varphi_n(f) = \min_{k=1,\dots,n} \Big| f\Big(\frac{k}{n}\Big) - f\Big(\frac{k-1}{n}\Big) \Big| \, .$$

Clearly, $\varphi_n \to 0$ pointwise. What about the rate of convergence? We take arbitrary $\varepsilon_n \to 0$ and examine $\frac{1}{\varepsilon_n} \varphi_n$.

9b2 Exercise. $\displaystyle \limsup_{n \to \infty, g \to f} \frac{1}{\varepsilon_n} \varphi_n(g) = \infty$ for all $f \in C[0,1]$.

Prove it.

By Prop. 5b9,

$$\limsup_{n \to \infty} \frac{1}{\varepsilon_n} \varphi_n(f) = \infty \tag{9b3}$$

for quasi all $f \in C[0,1]$.

On the other hand, if f is differentiable at $x_0 \in (0,1)$ then $f(x) - f(x_0) = O(|x - x_0|)$, that is,

$$\exists C \; \forall x \in [0,1] \;\; |f(x) - f(x_0)| \le C|x - x_0| \, .$$

Taking k such that $\frac{k-1}{n}, \frac{k}{n} \in [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ we get $|f\big(\frac{k}{n}\big) - f\big(\frac{k-1}{n}\big)| \le \frac{2C}{n}$. Thus,

$$\forall n \;\; \varphi_n(f) \le \frac{2C}{n} \, .$$

By (9b3), such f are a meager set, which proves Theorem 9b1.

9b4 Exercise. There exists a continuous function $f : [0,1] \to \mathbb{R}$ such that for every $x \in (0,1)$

$$\limsup_{y \to x-} |f(y) - f(x)| \log\log\log \frac{1}{|y-x|} = \infty \, ,$$
$$\limsup_{y \to x+} |f(y) - f(x)| \log\log\log \frac{1}{|y-x|} = \infty \, .$$

Prove it.

However, $|f(y) - f(x)|$ cannot be replaced with $f(y) - f(x)$. If $C > f(1) - f(0)$ then there exists $x \in (0,1)$ such that

$$\limsup_{y \to x+} \frac{f(y) - f(x)}{y - x} \le C$$

and moreover, $\sup_{y \in (x,1]} \frac{f(y)-f(x)}{y-x} \le C$. Proof (sketch): choose $b \in \big(f(1) - C, f(0)\big)$ and take the greatest x such that $f(x) \ge Cx + b$.

9c Differentiable and nowhere monotone

9c1 Theorem. [1] There exists a differentiable function $f : [0,1] \to \mathbb{R}$ such that for every $(a,b) \subset [0,1]$, f is not monotone on (a,b).

9c2 Lemma. [2] There exists a strictly increasing differentiable function $f : [0,1] \to \mathbb{R}$ such that $f'(\cdot) = 0$ on a dense set.

[1] C.E. Weil (1976) "On nowhere monotone functions", Proc. AMS **56**, 388–389. (Yes, two pages!) See also Sect. 10.7.2 in "Real analysis".

[2] S. Marcus (1963) "Sur les dérivées dont les zéros forment un ensemble frontière partout dense", Rend. Circ. Mat. Palermo (2) **12**, 5–40.

Proof. We'll construct a continuous strictly increasing surjective $g : [0,1] \to [0,1]$ such that the inverse function $f = g^{-1} : [0,1] \to [0,1]$ has the needed properties. It is sufficient to ensure that (finite or infinite) derivative $g'(\cdot) \in (0,\infty]$ exists everywhere (and never vanishes), and is infinite on a dense set.

A function

$$\alpha(x) = x^{1/3}$$

is strictly increasing (on $\mathbb{R}$), with $\alpha'(0) = +\infty$ and $\alpha'(x) \in (0,\infty)$ for $x \ne 0$. We introduce

$$A = \max_{h \ne 0} \frac{\alpha(1+h) - \alpha(1)}{h\alpha'(1)} \in (0,\infty)$$

(this continuous function vanishes on $\pm\infty$; in fact, $A = 4$) and note that

$$\frac{\alpha(x+h) - \alpha(x)}{h\alpha'(x)} \le A \tag{9c3}$$

for all $h \ne 0$ and x (since for $x \ne 0$ it equals $\frac{x^{1/3}(\alpha(1+\frac{h}{x})-\alpha(1))}{hx^{-2/3}\alpha'(1)} = \frac{\alpha(1+\frac{h}{x})-\alpha(1)}{\frac{h}{x}\alpha'(1)}$).

Similarly to Sect. 5a we choose some $a_n, c_n \in (0,1)$ such that a_n are pairwise distinct, dense, and $\sum_n c_n < \infty$. The series

$$\beta(x) = \sum_{n=1}^{\infty} c_n \alpha(x - a_n)$$

converges uniformly on $[0,1]$ (since $|\alpha(\cdot)| \le 1$ and $\sum_n c_n < \infty$). The series $\sum_{n=1}^{\infty} c_n \alpha'(x - a_n)$ converges (to a finite sum) for some x and diverges (to $+\infty$) for other x (in particular, for $x \in \{a_1, a_2, \dots\}$). We consider $\beta_n(x) = \sum_{k=1}^{n} c_k \alpha(x - a_k)$ and $\gamma_n(x) = \beta(x) - \beta_n(x) = \sum_{k=n+1}^{\infty} c_k \alpha(x - a_k)$. By (9c3),

$$0 \le \frac{\gamma_n(x+h) - \gamma_n(x)}{h} \le A \sum_{k=n+1}^{\infty} c_k \alpha'(x - a_k)$$

for all $h \ne 0$ and x. Thus (similarly to Sect. 5a)

$$\underbrace{\beta_n'(x)}_{\xrightarrow[n]{} \sum_{k=1}^{\infty} c_k \alpha'(x-a_k)} \le \liminf_{h \to 0} \frac{\beta(x+h) - \beta(x)}{h} \le$$

$$\le \limsup_{h \to 0} \frac{\beta(x+h) - \beta(x)}{h} \le \beta_n'(x) + A \sum_{k=n+1}^{\infty} c_k \alpha'(x - a_k) \,,$$

therefore

$$\beta'(x) = \sum_{n=1}^{\infty} c_n \alpha'(x - a_n) \in (0,\infty]$$

for all x.

It remains to take $g(x) = \frac{\beta(x)-\beta(0)}{\beta(1)-\beta(0)}$. $\square$

Do not think that $\beta'(\cdot) = \infty$ *only* on the countable set $\{a_1, a_2, \dots\}$. Amazingly, $f'(x) = 0$ for quasi all $x \in [0,1]$ (and therefore $\beta'(x) = \infty$ for quasi all $x \in [0,1]$). Here is why. By 5b2 and 5c5, f' is of Baire class 1, thus, $\{x : f'(x) \ne 0\}$ is an F_σ set, and $\{x : f'(x) = 0\}$ is a G_δ set;[1] being dense it must be comeager (as noted before 5c2).

We introduce the space D of all bounded derivatives on $(0,1)$; that is, of F' for all differentiable $F : (0,1) \to \mathbb{R}$ such that F' is bounded. We endow D with the metric

$$\rho(f,g) = \sup_{x\in(0,1)} |f(x) - g(x)| \,.$$

9c4 Exercise. (a) D is a complete metric space.

(b) D is not separable.

Prove it.

We consider a subspace D_0 of all $f \in D$ such that $f(x) = 0$ for quasi all x. As noted above, this happens if and only if $f(\cdot) = 0$ on a dense set. By 9c2, D_0 is not $\{0\}$; moreover, for every $x \in (0,1)$ there exists $f \in D_0$ such that $f(x) \ne 0$ (try $f(ax+b)$).

9c5 Exercise. (a) D_0 is a vector space; that is, a linear combination of two functions of D_0 is a function of D_0.

(b) D_0 is a closed subset of D.

Prove it.

Given $(a,b) \subset (0,1)$, the set

$$E_{a,b} = \{f \in D_0 : \forall x \in (a,b) \;\; f(x) \ge 0\}$$

is closed (evidently). Given $f \in E_{a,b}$, we take $x \in (a,b)$ such that $f(x) = 0$ and $g \in D_0$ such that $g(x) > 0$. Then $f - \varepsilon g \in D_0$ and $f - \varepsilon g \notin E_{a,b}$ for all $\varepsilon > 0$; thus, f is not an interior point of $E_{a,b}$. We see that $E_{a,b}$ is nowhere dense. Similarly, $-E_{a,b} = \{f \in D_0 : \forall x \in (a,b) \;\; f(x) \le 0\}$ is nowhere dense. It follows that quasi all functions of D_0 change the sign on every interval. Theorem 9c1 is thus proved.

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[1] A straightforward representation

$$f'(x) = 0 \quad\Longleftrightarrow\quad \forall\varepsilon\,\exists\delta\,\forall h\;\big(|h|<\delta \implies |f(x+h)-f(x)| \le \varepsilon|h|\big)$$

gives only $F_{\sigma\delta}$. Taking into account that f is differentiable we have another representation

$$f'(x) = 0 \quad\Longleftrightarrow\quad \forall\varepsilon\,\exists h\;\big(|h|<\varepsilon \wedge |f(x+h)-f(x)| < \varepsilon|h|\big)$$

that gives G_δ.

Hints to exercises

9a5: otherwise, some interval of P_f is not maximal.

9b2: $g(\frac{k}{n}) = f(\frac{k}{n}) \pm \sqrt{\varepsilon_n}$.

9b4: similar to 9b1.

9c4: (a) D is closed in the space of all bounded functions; (b) try shifts of a discontinuous derivative.