Computability Theory and Foundations of Mathematics

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Preface

Welcome to CTFM (Computability Theory and Foundations of Mathematics)!

CTFM 2014 is the seventh annual conference started and advanced by a group of logicians in Tohoku University (Sendai) and their collaborators, whose aim is to provide participants with the opportunity to exchange ideas, information and experiences on active and emerging topics in logic, including but not limited to: Computability Theory, Reverse Mathematics, Proof Theory, Constructive Mathematics, Theory of Randomness and Computational Complexity Theory. Previous meetings have taken place in Matsushima (2008, 2009), Inawashiro (2010), Akiu (2011), Harumi in Tokyo (2012), Tokyo Tech (2013).

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February 2014

Kazuyuki Tanaka
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Conference Venue

Ookayama campus of Tokyo Institute of Technology,

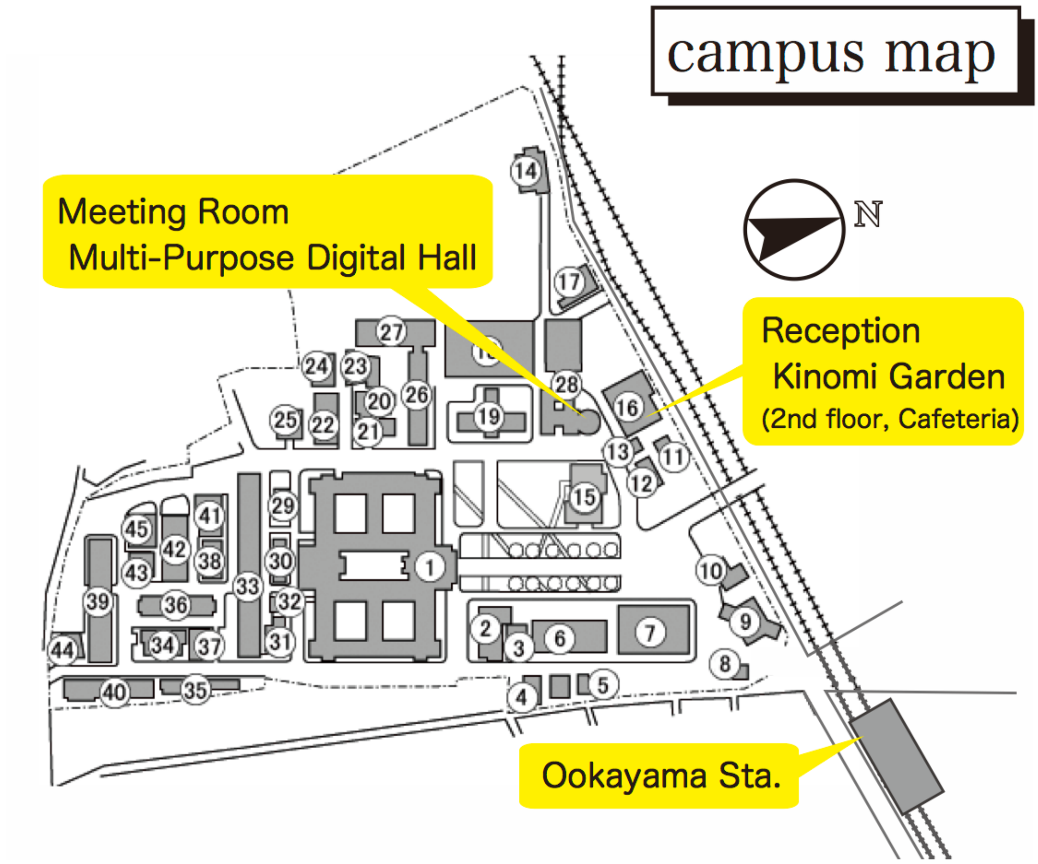
2-12-1 Ookayama, Meguro-ku, Tokyo, 152-8550, Japan.

Meeting Room

Multi-Purpose Digital Hall (West building 9)

Reception Party

February 17, Monday. Kinomi Garden (2nd floor, Cafeteria)



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Applications of nonstandard models in reverse mathematics

Chi Tat Chong
National University of Singapore

This talk concerns the use of nonstandard models of (fragments of) arithmetic to study problems in reverse mathematics, in particular relating to Ramsey type combinatorial principles. We discuss the key idea and philosophy behind this approach, taking as examples Ramsey's Theorem for pairs and principles motivated by this theorem.

Weak Königs lemma is not all that robust

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Abstract. It has been suggested that one explanation for the so-called "big five" phenomenon in reverse mathematics is that the main subsystems of second-order arithmetic used in this endeavor are *robust*, meaning, equivalent to small perturbations of themselves. But some recent results suggest that the notion of robustness may be more subtle, at least as far as the important subsystem WKL_0 is concerned. For example, Day [1] studied two closely-related principles from topological dynamics, and showed that while one is equivalent to WKL_0, the other lies strictly in-between WKL_0 and ACA_0. I will discuss some newer work, joint with Lerman and Solomon, looking at the principles SADLE, ADLE, and WADLE, which are minor combinatorial variations on the assertion that every infinite partial order with an infinite anti-chain admits a non-well-founded linearization. While SADLE is easily seen to be equivalent to WKL_0, we show that it is strictly stronger than ADLE, which in turn is strictly stronger than WADLE. This answers in part an old question from Cholak, Marcone, and Solomon [2] about definitions of well-quasi-orders. The proofs of the separations use iterative forcing techniques to fully approximate a model of the relevant principle, and as such are substantially different from more usual "iterate-and-dovetail" arguments. I will describe some of the ideas behind these methods, and mention some of their applications to other problems in reverse mathematics.

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Computable invariant measures and algorithmically random structures

Cameron E. Freer

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Given a countable structure, when is a presentation of it algorithmically random? Computable invariant measures concentrated on the isomorphism class of the structure provide one possible approach to this question, as suggested by Fouché and Nies (Logic Blog 2012). But when there are many such invariant measures, there may not be a single natural choice — leading to the question of when there is a unique such invariant measure.

In joint work with Ackerman, Kwiatkowska, and Patel, we show that the isomorphism class of a countable structure in a countable language admits a unique S_∞-invariant probability measure if and only if, for each n, it realizes a unique n-type up to permutation. Such a structure is called *highly homogeneous*; this notion arose in Cameron's 1976 classification of the reducts of the rational linear order $(\mathbb{Q}, <)$. In particular, there are five such structures, up to interdefinability, each of whose unique invariant measures has a computable presentation. Furthermore, we show that any countable structure admitting more than one invariant measure must admit continuum-many ergodic invariant measures.

Invariant measures on relational structures can be naturally described in terms of sampling procedures from certain measurable objects, as essentially shown by Aldous and Hoover. This representation is used in the proof of the above result about unique invariant measures, and also plays an important role in Bayesian nonparametric statistics. In joint work with Avigad, Roy, and Rute, we also address the question of when the sampling procedure corresponding to a computable invariant measure can be given in terms of a computable such object.

The combinatory algebras as a monad

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A *monad* $\langle T, \eta, \mu \rangle$ in a category $\mathbf{C}$ consists of a functor $T : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\eta : I_{\mathbf{C}} \dot{\to} T$ and $\mu : T^2 \dot{\to} T$ such that

1. $\mu_D \circ T\mu_D = \mu_D \circ \mu_{TD}$,
2. $\mu_D \circ \eta_{TD} = \mu_D \circ T\mu_D = 1_{TD}$.

A T*-algebra* $\langle D, h \rangle$ is a pair consisting of an object D (the underlying object of the algebra) and an arrow $h : TD \to D$ (the structure map of the algebra) such that

1. $h \circ \mu_D = h \circ Th$,
2. $h \circ \eta_D = 1_D$.

A *combinatory algebra* is a structure $D = \langle \underline{D}, \cdot, k_D, s_D \rangle$ consists of a set $\underline{D}$, a binary operation $\cdot$ on $\underline{D}$ and elements k_D, s_D of $\underline{D}$ such that

1. $\neg(k_D = s_D)$,
2. $k_D \cdot x \cdot y = x$,
3. $s_D \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$.

A *homomorphism* between combinatory algebras $D = \langle \underline{D}, \cdot, k_D, s_D \rangle$ and $D' = \langle \underline{D}', \cdot', k_{D'}, s_{D'} \rangle$ is a mapping $f : \underline{D} \to \underline{D}'$ such that

1. $f(x \cdot y) = f(x) \cdot' f(y)$,
2. $f(k_D) = k_{D'}$,
3. $f(s_D) = s_{D'}$.

We will show that the category of combinatory algebras forms a monad, and for each object D there exists a bijection between the set $\underline{D}$ and the set of T-algebras with the underlying object D.

Circuit Complexity and Derandomization

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Abstract

Randomness is quite useful for real-world computation, for example, to design fast algorithms, efficient communication protocols, et cetera. From a theoretical perspective, it is natural to ask how much randomness makes computation powerful. This question has been investigated in the computational complexity theory for more than three decades.

In 1998, Impagliazzo and Wigderson established a surprising connection between circuit complexity and power of randomness [1]. They proved that if there exists a decision problem that can be computed by a $2^{O(n)}$-time deterministic Turing machine but cannot by any Boolean circuit of size $2^{.1n}$ on every input length $n \in \mathbb{N}$, every problem computed by a polynomial-time probabilistic Turing machine that errs with a small probability can be computed even by a polynomial-time deterministic Turing machine.

Put simply, their result demonstrates that proving hardness against exponentially large circuits in some exponential-time class implies derandomizing probabilistic polynomial-time machines in a bounded-error setting. Therefore, we are able to derandomize any bounded-error polynomial-time algorithms only with polynomial-time overheads if we could prove the hardness assumption.

Following their result, a number of interesting connections have been discovered between the two notions. In this tutorial talk, starting with basic notions for circuit complexity and derandomization, I review recent trends of the connections between them.

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[1] Russell Impagliazzo and Avi Wigderson. Randomness vs. time: de-randomization under a uniform assumption. In *Proceedings of the 39th Annual IEEE Symposium on Foundations of Computer Science*, pages 734–743, 1998.

Resource-bounded randomness and differentiability

Akitoshi Kawamura

Brattka, Miller, and Nies showed that a real number is computably random if and only if every nondecreasing computable real function is differentiable at it. They asked whether the same thing can be said for polynomial-time randomness and polynomial-time computability. We point out that much of the ideas in their argument can be organized into several computational steps related to martingales over different measures. We then show that a simple modification in one of the steps makes the computation efficient and yields the polynomial-time version of the theorem. We also discuss some issues about the formulation of resource-bounded randomness in this context.

Partly based on a joint work with Kenshi Miyabe.

Effective Methods in Descriptive Set Theory

Takayuki Kihara*

Joint work with Vassilis Gregoriades†

A. Louveau [4] says,

> "*Effective descriptive set theory is not only a refinement of classical descriptive set theory, but also a powerful method able to solve problems of classical type.*"

We provide a new concrete example that justifies Louveau's claim. We employ the Louveau separation theorem [4] in *effective descriptive set theory* and the Shore-Slaman join theorem [8] in *Turing degree theory* to give a partial solution to a descriptive set theoretic problem proposed by Andretta [1], Semmes [7], Pawlikowski-Sabok [6], and Motto Ros [5].

The first effective result, *the Louveau separation theorem*, was proved by Louveau [4] to solve the section problem of Borel sets. Louveau's main idea was to use the topology generated by *effective* Suslin sets (i.e., *lightface* Σ^1_1 sets). Today, this topology is known as the Gandy-Harrington topology, which is originally introduced by Gandy, and had been used by Harrington to give an alternative proof of Silver's dichotomy for co-analytic equivalence relations. Later, Harrington-Kechris-Louveau [2] used this topology to show the Glimm-Effros dichotomy for Borel equivalence relations, where they said,

> "*Despite the totally classical descriptive set-theoretic nature of our result, our proof requires the employment of methods of effective descriptive set theory and thus ultimately makes crucial use of computability (or recursion) theory on the integers.*"

The second effective result, *the Shore-Slaman join theorem*, which was proved by Shore and Slaman [8] using so-called Kumabe-Slaman forcing, is a transfinite extension of the Posner-Robinson join theorem. By combining it with the Slaman-Woodin double jump definability theorem, they showed that the Turing jump is first-order definable in the partial ordering $(\mathcal{D}_T, \leq)$ of the Turing degrees.

Now, we start introducing a descriptive set theoretic problem that we solve by using the above effective methods. It dates back to the early 20th century

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when Nikolai Luzin asked whether every Borel function on the real line can be decomposed into countably many continuous functions. The Luzin problem was negatively answered in the 1930s. Then, which Borel functions are decomposable into continuous functions?

A remarkable theorem proved by Jayne and Rogers [3] states that for every function from an absolute Suslin-$\mathcal{F}$ space into a separable metrizable space, the preimage of each F_σ set under it is again F_σ if and only if it is decomposable into countably many continuous functions with closed domains (i.e., closed-piecewise continuous). Subsequently, Solecki [9] showed a dichotomy for Borel functions to sharpen the Jayne-Rogers theorem by using the *Louveau separation theorem*.

More recently, a significant breakthrough was made by Semmes [7]. He used Wadge-like infinite two-player games with *priority argument* to show that for every function on a zero-dimensional Polish space, the preimage of each $G_{\delta\sigma}$ set under it is again $G_{\delta\sigma}$ if and only if it is G_δ-piecewise continuous, and that the preimage of each F_σ set under a function is $G_{\delta\sigma}$ if and only if it is G_δ-piecewise F_σ-measurable.

The countable decomposability at all finite levels of the Borel hierarchy have been studied by Pawlikowski-Sabok [6] and Motto Ros [5]. Naturally, many researchers [1, 7, 6, 5] expected that the Jayne-Rogers theorem and the Semmes theorem could be generalized to all finite levels of the hierarchy of Borel functions.

Question ([1, 7, 6, 5])**.** Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Polish spaces, and $1 < m \leq n$. Are the following two assertions are equivalent for a function $f : \mathcal{X} \to \mathcal{Y}$?

1. The preimage $f^{-1}[A]$ of each $\mathbf{\Sigma}^0_m$ set $A \subseteq \mathcal{Y}$ under f is $\mathbf{\Sigma}^0_n$ in $\mathcal{X}$.
2. f is decomposable into countably many $\mathbf{\Sigma}^0_{n-m+1}$-measurable functions with $\mathbf{\Pi}^0_{n-1}$ domains (i.e., f is $\mathbf{\Pi}^0_{n-1}$-piecewise $\mathbf{\Sigma}^0_{n-m+1}$-measurable).

Note that the implication from (2) to (1) is obvious. For $m = n = 2$, the problem has been solved by Jayne-Rogers [3] and for $m = n = 3$ and $2 = m < n = 3$, the problem has been solved by Semmes [7] as mentioned above.

However, we encounter difficulties when we try to generalize their theorems. We again barrow a word of Louveau [4] in his study of the section problem of Borel sets.

> "*The proofs of Dellacherie for case $\xi = 1$, Saint-Raymond for case $\xi = 2$ and Bourgain for case $\xi = 3$ have in common to be of classical type, ... they are unfortunately very different from one another—and of course more and more difficult—and it does not seem possible to extract from them a general method for solving the section problem*"

We face the same situation. The proofs of Jayne-Rogers [3] for case $m = n = 2$ and Semmes [7] for case $m = n = 3$ and $2 = m < n = 3$ are unfortunately very different from one another—and of course more and more difficult—and it does not seem possible to extract from them a general method for solving the decomposability problem.

To overcome this severe difficulty, Louveau employed an effective method. We follow Louveau's line. Our main theorem is following:

Theorem. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Polish spaces with topological dimension $\neq \infty$, and α and β are countable ordinals with $\alpha \leq \beta$. Then, the assertion (1) implies the assertion (2).*

1. *The preimage $f^{-1}[A]$ of each $\mathbf{\Sigma}^0_{\alpha+1}$ set $A \subseteq \mathcal{Y}$ under f is $\mathbf{\Sigma}^0_{\beta+1}$ in $\mathcal{X}$.*
2. *f is decomposable into countably many functions $\{g_i\}_{i\in\omega}$ such that for every i, g_i is $\mathbf{\Sigma}^0_{\gamma+1}$-measurable for some γ with $\gamma + \alpha \leq \beta$.*

Furthermore, if $\beta < \alpha \cdot 2$, one can choose the domain of each g_i as a $\mathbf{\Pi}^0_\beta$ set.

As a consequence, the main question is solved for every $m \leq n < 2m$.

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Turing determinacy within Second Order Arithmetic

Antonio Montalbán

We study the strength of the various levels of Turing Determinacy that can be proved within Second Order Arithmetic. We obtain a good deal of partial results and leave some questions open. We will start the talk by reviewing previous results on the strength of determinacy at the limits of SOA. This is joint work with Richard A. Shore.

Comparing sets of natural numbers using randomness and lowness properties

Keng Meng Ng

The study of algorithmic randomness, in particular of lowness properties in randomness has led to many exciting recent developments in the area. In particular this has provided us a tool to compare the amount of inherent "information" present in sets of natural numbers, using what is known as "weak reducibilities". This interacts well with notions of randomness, for instance, the "Hungry Set" Theorem shows that the sets A which are Turing-bases for randomness are exactly the K-trivial sets. In this talk we will mention some related results, and show that there exists a non-computable set A which is an LR-base for randomness and is not K-trivial. This is joint work with Johanna Franklin and Reed Solomon.

Effective Multifractal Analysis

Jan Reimann

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Multifractal measures play an important role in the study of point processes and strange attractors. A central component of the theory is the multifractal formalism, which connects local properties of a measure (pointwise dimensions) with its global properties (average scaling behavior).

In this talk I will introduce a new, effective multifractal spectrum, where we replace pointwise dimension by asymptotic compression ratio. It turns out that the underlying measure can be seen as a universal object for the multifractal analysis of computable measures. The multifractal spectrum of a computable measure can be expressed as a"deficiency of multifractality" spectrum with respect to the universal measure. This in turn allows for developing a quantitative theory of dimension estimators based on Kolmogorov complexity. I will discuss some applications to seismological dynamics.

▸ PAUL SHAFER, *Separating the uniformly computably true from the computably true via strong Weihrauch reducibility.*
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We propose strong Weihrauch reducibility, a central concept in computable analysis, as a means by which to separate combinatorial statements that are equivalent in RCA_0. The material discussed in this talk appears in [1] and is joint work with François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, and Joseph R. Mileti.

Let RT^n_k denote Ramsey's theorem for n-tuples and k-colorings, and consider the well-known implication $\mathrm{RT}^2_2 \to \mathrm{RT}^2_3$ in RCA_0. Given a coloring $f\colon [\mathbb{N}]^2 \to 3$, RT^2_2 implies that there is an infinite $H' \subseteq \mathbb{N}$ such that either $\mathrm{ran}(f \restriction [H']^2) \subseteq \{0\}$ or $\mathrm{ran}(f \restriction [H']^2) \subseteq \{1, 2\}$. If $\mathrm{ran}(f \restriction [H']^2) \subseteq \{0\}$, then H' is homogeneous for f. Otherwise, by applying RT^2_2 to $f \restriction [H']^2$, there is an infinite $H \subseteq H'$ that is homogeneous for f.

Notice that this argument makes use of non-uniformities (is $\mathrm{ran}(f \restriction [H']^2) \subseteq \{0\}$ or is $\mathrm{ran}(f \restriction [H']^2) \subseteq \{1, 2\}$?) and multiple applications of RT^2_2. We show that the implication $\mathrm{RT}^2_2 \to \mathrm{RT}^2_3$ cannot be uniformized in the sense of strong Weihrauch reducibility. That is, we show that there is no pair of Turing functionals Φ and Ψ such that Φ^f is an RT^2_2 instance whenever f is an RT^2_3 instance and that Ψ^H is a solution to f whenever H is a solution to Φ^f. Thus one may describe the implication $\mathrm{RT}^2_2 \to \mathrm{RT}^2_3$ as *computably true* but not *uniformly computably true*. We also give further examples of such phenomena by considering statements related to weak weak Konig's lemma and the thin set theorem.

[1] François G. Dorais, Damir D. Dzhafarov, Jeffry L. Hirst, Joseph R. Mileti, and Paul Shafer, *On uniform relationships between combinatorial problems*, ***preprint***.

Reverse mathematics and the ACC

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January 24, 2014

This is my abstract for Computability Theory and Foundations of Mathematics (CTFM 2014), Tokyo Institute of Technology, February 17–20, 2014.

In abstract algebra, a ring is said to satisfy the ACC (ascending chain condition) if it has no infinite ascending sequence of ideals. A famous theorem of Hilbert, 1890, says that polynomial rings with finitely many indeterminates satisfy the ACC. There is also a similar theorem for noncommuting indeterminates, due to J. C. Robson, 1978. In 1988 I performed a reverse-mathematical analysis of the theorems of Hilbert and Robson, proving that they are equivalent over RCA_0 to the well-orderedness of ω^ω and ω^{ω^ω} respectively. Now I perform a similar analysis of a theorem of E. Formanek and J. Lawrence, 1976. Let S be the group of finitely supported permutations of the natural numbers. Let $K[S]$ be the group ring of S over a countable field K of characteristic 0. Formanek and Lawrence proved that $K[S]$ satisfies the ACC. I now prove that the Formanek/Lawrence theorem is equivalent over RCA_0 to the well-orderedness of ω^ω. I also show that, in all of these reverse-mathematical results, RCA_0 can be weakened to RCA_0^*. This recent work was done jointly with Kostas Hatzikiriakou.

In addition, I make some remarks concerning reverse mathematics as it applies to Hilbert's foundational program of finitistic reductionism. It is significant that RCA_0 and WKL_0 and even $\mathsf{WKL}_0 + \Sigma^0_2$-bounding are conservative for Π^0_2 sentences over PRA, while Σ_2-induction and the well-orderedness of ω^ω are not.

(Non-)Reductions in Reverse Mathematics

Henry Towsner

Much of reverse mathe-matics is concerned with implications among theorems of the form

$$\forall X \exists Y \phi(X, Y).$$

We think of such a ϕ as a *problem*, each X as an *instance* of this problem, and each Y making $\phi(X, Y)$ hold a *solution* to the problem X. For instance, Ramsey's Theorem for Pairs says that each 2-coloring of the integers has an infinite homogeneous set, so such a coloring is an instance of the corresponding problem while an infinite homogeneous set is a solution to that instance.

Many results in reverse mathematics provide computable reductions from one such problem to a second. Most (though not all) such reductions have a particularly simple form: $\mathcal{P}$ reduces to $\mathcal{Q}$ because for each instance X of $\mathcal{P}$ there is an instance $\Gamma(X)$ of $\mathcal{Q}$ computable from X so that for each solution Y of $\Gamma(X)$, there is a $\Delta(Y)$, computable from Y, which is a solution to X. This simple type of reduction is sometimes called Weihrauch reducibility.

To prove a non-implication in the sense of reverse mathematics, however, one must rule out much more complicated reductions, even though no such reductions have been observed in practice. In this talk we will describe the technique developed in [1] for upgrading the failure of Weihrauch reducibility to a proof of a non-implication.

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COMBINATORIAL SOLUTIONS PRESERVING THE ARITHMETIC HIERARCHY

WEI WANG

Abstract

The story begins from a theorem of Hirschfeldt and Shore [1] that every recursive stable linear ordering admits an infinite ascending or descending sequence of low degree. Jockusch observed an interesting application of this theorem to degree theory that every degree below the halting problem is of recursively enumerable degree relative to a low degree. In terms of arithmetic hierarchy, there are recursive stable linear orderings $<_L$ such that every infinite $<_L$-ascending or $<_L$-descending sequence turns a fixed properly Δ^0_2 set into a relatively Σ^0_1 or Π^0_1 set. So we may roughly say that SADS helps simplifying some Δ^0_2 definitions.

However, we prove that none of WKL_0, COH and EM has the same strength. More precisely, every recursive infinite binary tree (sequence of sets, tournament) has an infinite path (cohesive set, transitive set) G such that every properly Δ^0_2 set is properly Δ^{G}_2.

So we have alternative proofs of the following known non-implications: $\mathrm{WKL}_0 \nvdash$ SADS, COH $\nvdash$ SADS and EM $\nvdash$ SADS ([2]).

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Isolation: with applications considered

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Abstract. Isolation phenomenon was first proposed in a unpublished paper of Cooper and Yi, and later developed by several other recursion theorists. In this talk, I will present our recent work of applying this structural phenomenon to diamond embeddings, and cupping properties. Our result implies some knowing results, including Downey's diamond theorem and Li-Yi's cupping theorem. Some work of Ishmukhametov in 1999 will be introduced.

Infinite Games in the Cantor Space over Admissible Set Theories

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Abstract. Specific fragments of the axiom of determinacy of Δ^0_2 definable infinite games in the Cantor space will be discussed. In particular, we will discuss relationships between the fragments of axiom of determinacy and fragments of a system of set theory corresponding to a system of second order arithmetic which is known as Arithmetical Transfinite Recursion ATR_0.

In terms of reverse mathematics, various existence axioms of reals can be characterised by axioms of determinacy of infinite games cf. [6]. It is known that many interesting axioms of determinacy lie between Δ^0_1- and Δ^0_3-definable games in the Baire space, i.e., in $\mathbb{N}^{\mathbb{N}}$. For example, over a weak base system, the axiom of *arithmetical transfinite recursion* ATR_0 is equivalent to the determinacy of Δ^0_1-definable games. In contrast, in [4], it is shown that over the same base system the axiom ATR_0 is equivalent to the determinacy of Δ^0_2-definable games in the *Cantor space*, i.e., in $2^{\mathbb{N}}$. Notationally, for a class Φ of formulas, the axiom of determinacy of Φ-definable games in the Baire space is denoted as Φ-Det whereas the one in the Cantor space is denoted as Φ-Det*.

Theorem 1 (Nemoto-Ould MedSalem-Tanaka [4]). *Over the system* RCA_0 *of recursive comprehension axiom,* ATR_0 *and* Δ^0_2-Det* *are equivalent.*

On the other side, an interesting characterisation of Δ^0_2-definable sets is given in [7]. For an element a of Kleene's ordinal notation system $\mathcal{O}$ (for the definition, see, e.g., [5]), an *a-r.e. set* is defined as the symmetric difference of a many recursively enumerable sets.

Theorem 2 (Stephan-Yang-Yu [7]). *For any* Δ^0_2 *set, there exists an element* $a \in \mathcal{O}$ *such that* a *is a notation for* ω^2 *and that* A *is an* a*-r.e. set.*

This fact motivates us to investigate the logical strength of the axiom Φ-Det* for the class Φ of formulas corresponding to a-r.e. sets where a is a notation for $\omega \cdot n$ $(n < \omega)$. In accordance with a-r.e. sets, we will define specific classes of formulas as follows.

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Definition 1. *Let $a \in \mathcal{O}$. Assume that the restriction of the standard well-founded partial order $<_{\mathcal{O}}$ on $\mathcal{O}$ less than a can be expressed in an underlying formal system. Then we say a formula is* $(\Sigma^0_1)_a$-formula *if it is of the form $(\exists b <_{\mathcal{O}} a)\,[\varphi(b) \wedge (\forall c <_{\mathcal{O}} b)\neg\varphi(c)]$ for some Σ^0_1-formula φ.*

The purpose of this work is to investigate the logical strength of the axiom $(\Sigma^0_1)_a$-Det* for a notation a for $\omega \cdot n$ $(n < \omega)$. To discuss the strength of these axioms, it seems natural to reason over fragments of *admissible set theories*. In [1,2], a system $\mathrm{KPu}^0 + (\mathcal{U}_n)$ of set theory is defined to be a base system KPu^0 augmented with an axiom which formalises the existence of an increasing sequence of n admissible sets. In [1], it is shown that ATR_0 holds in $\bigcup_{n<\omega} \mathrm{KPu}^0 + (\mathcal{U}_n)$.

Lemma 1 (Jäger [1]). *The axiom* ATR_0 *of arithmetical transfinite recursion holds in $\bigcup_{n<\omega} \mathrm{KPu}^0 + (\mathcal{U}_n)$. More precisely, given an arithmetical formula φ, in $\bigcup_{n<\omega} \mathrm{KPu}^0 + (\mathcal{U}_n)$, if a well order $<$ together with all the set parameters appearing in φ belong to an admissible set d, then the set defined by transfinite recursion via φ along the well order $<$ also belongs to d.*

Moreover, these two systems are proof-theoretically equivalent, i.e., they can prove well-ordering of the same recursive well orders. From Lemma 1, one can show the following theorem employing ideas presented in [3].

Theorem 3. *Let $1 \leq n$. Suppose that $a \in \mathcal{O}$ is a notation for $\omega \cdot n$. Then $(\Sigma^0_1)_a$-Det* holds in $\mathrm{KPu}^0 + (\mathcal{U}_n)$.*

This work is not completed, but the speaker believes that Theorem 3 holds. For further investigations, it is natural to ask whether the system $\mathrm{KPu}^0 + (\mathcal{U}_n)$ precisely corresponds to the axiom $(\Sigma^0_1)_a$-Det* in a certain sense if $a \in \mathcal{O}$ is a notation for $\omega \cdot n$.

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Reverse Mathematics: well-scattered partial orders and Erdös-Rado

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Abstract. I will discuss the reverse mathematics of *well-scattered partial orders* and the relation with a Ramsey-like partition theorem for rationals due to Erdös and Rado.

We say that P is a *well-scattered partial order* if for every function $f\colon \mathbb{Q} \to P$ there exist $x <_{\mathbb{Q}} y$ such that $f(x) \leq_P f(y)$. The starting point is the following theorem:

Theorem 1 (Bonnet, Pouzet 1969). *Let P be a partial order. Then the following are equivalent:*

1. *P is a well-scattered partial order;*
2. *P is scattered and has no infinite antichains;*
3. *every linear extension of P is scattered;*
4. *for every function $f\colon \mathbb{Q} \to P$ there exists an infinite set $A \subseteq \mathbb{Q}$ such that $x <_{\mathbb{Q}} y$ implies $f(x) \leq_P f(y)$ for all $x, y \in A$.*

It is interesting to notice that exactly the same conditions hold for well-partial orders once scattered is replaced with well-founded. The reverse mathematics of all possible implications (two for any equivalence) was studied in [1]. We do the same for well-scattered partial orders.
It turns out that, except for the implications already provable in RCA_0, there are roughly two families of implications: the ones provable in WKL_0 but not in RCA_0 (neither in WWKL_0) and the ones provable in ACA_0 but not in WKL_0. So far, the situation is similar to that for well-partial orders. With regard to the second family of implications, though, the following partition theorem for rationals plays the role of RT^2_2 in the reverse mathematics of well-scattered partial orders.

Theorem 2 (Erdös, Rado 1952). *The partition relation $\mathbb{Q} \to (\aleph_0, \mathbb{Q})^2$ holds, that is for every coloring $c\colon [\mathbb{Q}]^2 \to 2$ there exists either an infinite 0-homogeneous set or a dense 1-homogeneous set.*

Let ER^2_2 (after Erdös-Rado) be the formal statement corresponding to Erdös-Rado theorem. The reverse mathematics of ER^2_2 is interesting by itself as it lies between ACA_0 and RT^2_2. I still do not know whether it is strictly between them. As for the relation with well-scattered partial orders, I will discuss "semitransitive" versions of ER^2_2.

Keywords: Reverse Mathematics, partial order, scattered, colorings, rationals

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Classical provability of uniform versions and intuitionistic provability

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Abstract. In the practice of reverse mathematics, sequential versions of Π^1_2-statements, which assert to solve infinitely many instances of a particular problem simultaneously, have been investigated in order to reveal the lack of uniformity of some proofs in RCA. On the other hand, there is a stronger form to capture uniform provability than just sequentialization. For a statement $\mathrm{S} := \forall X\,(A(X) \to \exists Y B(X,Y))$, one can consider a statement

$$\mathrm{Uni(S)} := \exists F \forall X\,(A(X) \to B(X, F(X)))\,,$$

which asserts the existence of a uniform procedure F to construct a solution for each problem X. This is the fully uniform version of S compared to the sequential version which is only a weaker representation of uniformity. However, this uniform version of Π^1_2-statement is not naturally represented in the language of second order arithmetic since F is a third order object. To investigate the strength of uniform versions, systems of arithmetic in all finite types are employed in higher order reverse mathematics ([4], [5]).

Recently, it has been established in [3] and [1] that for every Π^1_2-statement of some syntactical form, its provability in certain (semi-)intuitionistic systems guarantees the provability of its sequential version in RCA (or +WKL). Motivated by these previous works, we analyze the relationship between the intuitionistic provability of Π_2-statements and the classical provability of their uniform versions with the use of systems of arithmetic in all finite types. The crucial tool for our analysis is an application of the Dialectica interpretation. Our main result ([2]) is that for every Π_2-statement S of some syntactical form, if its uniform version Uni(S) derives the uniform variant of ACA over a classical system of arithmetic in all finite types with weak extensionality, then S is not provable in strong semi-intuitionistic systems including bar induction BI in all finite types but also nonconstructive principles such as König's lemma KL and uniform weak König's lemma UWKL. This metatheorem is applicable to many mathematical principles whose sequential versions imply ACA. Roughly speaking, the metatheorem often allows one to detect using classical reasoning on Uni(S) that S intuitionistically implies at least the Π^0_1-law-of-excluded-middle principle Π^0_1-LEM (and so - in the presence of Markov's principle - Σ^0_1-LEM) rather than only the strictly weaker principle Σ^0_1-LLPO (as WKL already does).

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Order Dimensions of Degree Structures

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Abstract. It is known that for all partial ordered set P there exists a set $\{Q_i\}_{i\in I}$ of linear extensions of P such that P is embeddable into the product order $\prod_{i\in I} Q_i$ of $\{Q_i\}_{i\in I}$. The least cardinality of such I is called the *order dimension* of P (written $\dim(P)$). In this talk, we investigate the order dimensions of the partial orders of Turing degrees $\mathcal{D}_{\mathrm{T}}$, Medvedev degrees $\mathcal{D}_{\mathrm{s}}$ and Muchnik degrees $\mathcal{D}_{\mathrm{w}}$. We see that $\aleph_1 \leq \dim(\mathcal{D}_{\mathrm{T}}) \leq 2^{\aleph_0}$, $2^{\aleph_0} \leq \dim(\mathcal{D}_{\mathrm{s}}) \leq 2^{2^{\aleph_0}}$ and $\dim(\mathcal{D}_{\mathrm{w}}) = 2^{\aleph_0}$.

Point-free characterisation of Bishop compact metric spaces

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Bishop [2] developed a large body of analysis constructively, but he did not develop general topology beyond the theory of metric spaces. He found it difficult to find a useful topological notion of compactness which is compatible with the corresponding metric notion defined by completeness and totally boundedness. If the classical notion of compactness by open cover is adopted, there would be no nontrivial examples of compact spaces constructively. In fact, the main examples of compact metric spaces, the unit interval and Cantor space, cannot be proved to be compact in the sense of open cover without recourse to Fan theorem, which is constructively unacceptable.

General topology in constructive setting was initiated by Sambin [5], when he introduced a constructive notion of point-free topology, called formal topology. Formal topology has been successful in constructivising many results of classical topology, however, the connection between Bishop's metric space and formal topology has been somewhat neglected. In particular, the notion of compactness of formal topology, which is defined by open cover, seems to conflict with that of Bishop metric space via completeness and totally boundedness.

Palmgren [4], in his pioneering work in this direction, constructed a full and faithful functor, called localic completion, from the category of Bishop locally compact metric spaces to that of locally compact formal topologies. The functor can be restricted to the full subcategory of compact metric spaces and that of compact formal topologies, proving that two seemingly conflicting notions of compactness are actually compatible. Later, Spitters [6] and Coquand et al. [3] found a connection between the compact subspaces of a Bishop locally compact metric space and the compact overt subtopologies of its localic completion.

Building on these previous works, we characterise the image of compact metric spaces under the localic completion in terms of formal topology. We identify overt compact enumerably completely regular formal topologies as point-free counterpart of Bishop compact metric spaces. Specifically, our main result states the equivalence of the following conditions for a formal topology $\mathcal{S}$.

1. $\mathcal{S}$ is isomorphic to an overt compact enumerably completely regular formal topology.
2. $\mathcal{S}$ is isomorphic to a compact overt subtopology of the countable product $\prod_{n\in\mathbb{N}} \mathcal{I}[0,1]$ of the formal unit interval $\mathcal{I}[0,1]$.
3. $\mathcal{S}$ is isomorphic to a localic completion of some compact metric space.

The result gives a purely point-free characterisation of Bishop compact metric spaces, and it allows us to prove results about Bishop compact metric spaces in a purely point-free (and possibly choice free) way.

We work in Bishop style constructive mathematics, including the axiom of Dependent Choice. Our work can be carried out in any major constructive framework such as Aczel's constructive set theory CZF [1] with suitable extensions (e.g. we require Dependent Choice and the Regular Extension Axiom in CZF).

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Rosser-type Henkin sentences and local reflection principles

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1 Henkin sentences

Let T be a recursive theory containing Peano arithmetic PA. Gödel's first incompleteness theorem was proved by constructing a sentence π satisfying $\mathsf{PA} \vdash \pi \leftrightarrow \neg \mathsf{Pr}_T(\ulcorner \pi \urcorner)$ where $\mathsf{Pr}_T(x)$ is a provability predicate of T, namely, a formula satisfying that for any sentence φ, $T \vdash \varphi$ if and only if $\mathsf{PA} \vdash \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$. A provability predicate of T is said to be standard if it satisfies the following two conditions:

1. $\mathsf{PA} \vdash \mathsf{Pr}_T(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Pr}_T(\ulcorner \varphi \urcorner) \to \mathsf{Pr}_T(\ulcorner \psi \urcorner))$;
2. If φ is a Σ_1 sentence, then $\mathsf{PA} \vdash \varphi \to \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$.

In 1952, Henkin [5] asked the question whether each sentence asserting its own provability in a theory T is provable or not. A sentence φ satisfying $T \vdash \varphi \leftrightarrow \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$ is called a Henkin sentence of T. In 1955, Löb [4] answered to this question by proving the following theorem:

Theorem 1 (Löb (1955)). *Let $\mathsf{Pr}_T(x)$ be a standard provability predicate of T. Then for any sentence φ, $T \vdash \varphi$ whenever $T \vdash \mathsf{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi$.*

Therefore each Henkin sentence of T is provable in T. However, Kreisel [2] pointed out that the situation of the provability of Henkin sentences of non-standard provability predicates can vary.

2 Rosser-type Henkin sentences

A Δ_1 formula $\mathsf{Prf}_T(x, y)$ is called a standard proof predicate of T if the formula $\mathsf{Pr}_T(x)$ defined as $\exists y \mathsf{Pr}_T(x, y)$ is a standard provability predicate.

For any standard proof predicate $\mathsf{Prf}_T(x, y)$ of T, we define its Rosser provability predicate $\mathsf{Pr}_T^R(x)$ as the formula $\exists y(\mathsf{Prf}_T(x, y) \wedge \forall z \leq y \neg \mathsf{Prf}_T(\neg x, z))$. It is known that for every sentence φ refutable in T, $\neg \mathsf{Pr}_T^R(\ulcorner \varphi \urcorner)$ is provable in PA. Then $\varphi \leftrightarrow \mathsf{Pr}_T^R(\ulcorner \varphi \urcorner)$ is provable in PA, and thus φ is a Henkin sentence of $\mathsf{Pr}_T^R(x)$. Therefore every provable or refutable sentence is a Henkin sentence of any Rosser provability predicate.

A natural question arises: Is there an independent Henkin sentence based on $\mathsf{Pr}_T^R(x)$? We answered in [3] to this question that whether Rosser provability predicate has an independent Henkin sentence is dependent on the choice of a predicate. This is a consequence of the following two theorem.

Theorem 2. *1. For any Rosser sentence π of T, there is a Rosser provability predicate $\mathsf{Pr}_T^R(x)$ such that $\neg\pi$ is a Henkin sentence of $\mathsf{Pr}_T^R(x)$.*

2. There is a Rosser provability predicate $\mathsf{Pr}_T(x)$ such that for any sentence φ, if $T \vdash \mathsf{Pr}_T^R(\ulcorner\varphi\urcorner) \to \varphi$, then either $T \vdash \varphi$ or $T \vdash \neg\varphi$.

3 Rosser-type local reflection principles

Local reflection principle $\mathsf{Rfn}(T)$ for T is the set $\{\mathsf{Pr}_T(\ulcorner\varphi\urcorner) \to \varphi : \varphi \text{ is a sentence}\}$ which can be seen as a schema expressing the soundness of T. Goryachev investigated local reflection principles based on Rosser provability predicates. Let $\mathsf{Rfn}^R(T)$ be the set $\{\mathsf{Pr}_T^R(\ulcorner\varphi\urcorner) \to \varphi : \varphi \text{ is a sentence}\}$.

Theorem 3 (Goryachev [1]). *There is a Rosser provability predicate of T such that the theories $T + \mathsf{Rfn}(T)$ and $T + \mathsf{Rfn}^R(T)$ are equivalent.*

Shavrukov [6] raised a question concerning a Rosser provability predicate in which the order of non-standard proofs of unprovable sentences cannot be captured, and pointed out an affirmative answer to his question gives a Rosser provability predicate whose local reflection principle is strictly weaker than the usual one. We gave an affirmative answer to his question, and thus we obtained the following theorem.

Theorem 4. *There is a Rosser provability predicate of T such that the theories $T + \mathsf{Rfn}(T)$ and $T + \mathsf{Rfn}^R(T)$ are not equivalent.*

We also investigated the hierarchy of partial local reflection principles based on Rosser provability predicates. Let Γ be a class of formulas, and let $\mathsf{Rfn}_\Gamma(T)$ be the set $\{\mathsf{Pr}_T(\ulcorner\varphi\urcorner) \to \varphi : \varphi \text{ is a } \Gamma \text{ sentence}\}$. We define $\mathsf{Rfn}_\Gamma^R(T)$ in the same way. Then we proved the following theorem.

Theorem 5. *1. For any $n \geq 1$, $T + \mathsf{Rfn}_{\Pi_n}^R(T)$ does not contain $T + \mathsf{Rfn}_{\Sigma_n}^R(T)$.*

2. For any $n \geq 2$, $T + \mathsf{Rfn}_{\Sigma_n}^R(T)$ does not contain $T + \mathsf{Rfn}_{\Pi_n}^R(T)$.

3. Whether $T + \mathsf{Rfn}_{\Sigma_1}^R(T)$ contains $T + \mathsf{Rfn}_{\Pi_1}^R(T)$ is dependent on the choice of a Rosser provability predicate.

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What non-standard analysis is about?

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Abstract. After Robinson ended his presentation of the non-standard analysis he adds this very interesting statement: "However, from a formalist point of view we may look at our theory syntactically and may consider that what we have done is to introduce new deductive procedures rather than mathematical entities "(Robinson, 1966:282) Robinson did not add any explanation in his book about what he meant regarding this new deduction, namely, what are the basic assumptions, the following conclusions and the rules of deduction. In my talk I would like to deal with these issues and show that the purpose of his new deduction was to separate the non-standard objects from the standard ones. Robinson made a connection between ontology and epistemology. I will therefore also discuss whether the logic proof is also an ontological proof according to Robinson's philosophical point of view.

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The interplay between computability and logic in Kurt Goedel's thought

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Abstract. Since Kurt Goedel's work helped to bring together computability theory and recursion theory, there has been some wonder as to why he did not immediately accept Turing's results. Eventually, however, Goedel completely acknowledged that Turing's celebrated paper (1937) was unquestionably convincing. Goedel himself thereafter commenced a research project that culminated in his Dialectica interpretation of intuitionistic arithmetic in a theory of computable functionals of finite type. This, combined with his reduction of classical to intuitionistic arithmetic, provides an interpretation of classical arithmetic. There are many similarities between proofs and functionals. All this is well-understood, yet it is another manifestation of the under-appreciated point that Goedel throughout his philosophical development remained a friend of proof.

Gap phenomenon for Schnorr randomness

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Randomness means incompressibility. One mathematical formulation of this is Levin-Schnorr's theorem saying that $A \in 2^\omega$ is Matin-Löf random if and only if $K(A \upharpoonright n) > n - O(1)$ where K is the prefix-free Kolmogorov complexity. Actually, we have some stronger results. Chaitin [2] already observed that A is ML-random if and only if $\lim_n K(A \upharpoonright n) - n = \infty$. Thus, there is a "gap phenomeno" that the initial segment complexity of A must be higher than the lower bound.

A more refined theorem is Ample Excess Lemma by Miller and Yu [4], which states that A is ML-random if and only if $\sum_n 2^{n-K(A\upharpoonright n)} < \infty$. This is a nice result and has many nice corollaries. For instance, we have $K(A \upharpoonright n) \geq n + K^A(n) - O(1)$ for a ML-random set A.

We can ask whether an analogous property holds for Schnorr randomness. Schnorr randomness can be characterized via incompressibility with respect to computable measure machines [3] and decidable machines [1]. Formally, a set A is Schnorr random if and only if $K_M(A \upharpoonright n) > n - O(1)$ for every computable measure machine. We will show the Ample Excess Lemma for Schnorr randomness, which states that A is Schnorr random if and only if $\sum_n 2^{n-K_M(A\upharpoonright n)} < \infty$ for every computable measure machine. Thus, we can observe the gap phenomenon for Schnorr randomness too. The main tool in the proof is an integral test. Actually, $f(A) = \sum_n 2^{n-K(A\upharpoonright n)}$ is an integral test and $f_M(A) = \sum_n 2^{n-K_M(A\upharpoonright n)}$ is a Schnorr integral test.

Ample Excess Lemma for ML-randomness has many corolleries and so does the lemma for Schnorr randomness. For instance, Miller and Yu [4] showed that, for a ML-random set Z, $X \oplus Z$ is ML-random if and only if $C(X \upharpoonright n) + K(Z \upharpoonright n) \geq 2n - O(1)$. We will show that, for a Schnorr random set Z, $X \oplus Z$ is Schnorr random if and only if $C_N(X \upharpoonright n) + K_M(Z \upharpoonright n) \geq 2n - O(1)$ for every decidable machine N and computable measure machine M. This strongly suggests that K_M for computable measure machines are K for Schnorr randomness and C_N for decidable machines are C for Schnorr randomness.

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Ramseyan factorization theorem in reverse mathematics

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(Joint work with Takeshi Yamazaki[1] and Keita Yokoyama[2])

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We study, in the context of reverse mathematics, the strength of Ramseyan factorization theorem, a Ramsey-type theorem used in automata theory.

Ramseyan factorization theorem is the following statement:

Definition 1 (Ramseyan factorization theorem). *For any $A \subseteq \mathbb{N}$ and finite $B \subseteq \mathbb{N}$, the following statement (RF^A_B) holds:*

> *For any $u \in A^{\mathbb{N}}$ and $f : A^{<\mathbb{N}} \to B$, there exists $v \in (A^{<\mathbb{N}})^{\mathbb{N}}$ such that $u = v_0{}^\frown v_1{}^\frown \cdots$ and for any $j \geq i > 0$ and $j' \geq i' > 0$, $f(v_i{}^\frown v_{i+1}{}^\frown \cdots {}^\frown v_j) = f(v_{i'}{}^\frown v_{i'+1}{}^\frown \cdots {}^\frown v_{j'})$.*

We prove that RF^s_k is equivalent to RT^2_2 for all $s, k \geq 2$, $k \in \omega$ over RCA_0. We also consider a weak version of Ramseyan factorization theorem (WRF^s_k) and prove that $\mathrm{WRF}^{\mathbb{N}}_2$ is in between ADS and CAC.

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The strength of determinacy between $\boldsymbol{\Sigma}_1^0$ and $\boldsymbol{\Delta}_2^0$

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It is known that $(\Sigma_1^0)_2(=\Sigma_1^0 \wedge \Sigma_2^0)$ and Δ_2^0 determinacies in the Cantor space are equivalent to Π_1^0 comprehension and Π_1^0 transfinite recursion, respectively (cf. [1]). Since the class Δ_2^0 is described as the union of difference hierarchy above Σ_1^0, by investigating $(\Sigma_1^0)_\alpha$ determinacy, we can see how determinacy schemata gain the strength. While the strength in the sense of logical implication has been mainly investigated in the study of reverse mathematics, we consider also the strength in the sense of consistency. We observe that the strengths of determinacy in the two senses behave in completely different ways below and above Γ_0, the proof theoretic ordinal of Π_1^0 transfinite recursion, as follows:

(Results for the strength in the sense of logical implication)

- If $\alpha \cdot \omega \leq \beta$, $(\Sigma_1^0)_\beta\text{-}\mathsf{Det}^* \vdash (\Sigma_1^0)_\alpha\text{-}\mathsf{Det}^*$ and $(\Sigma_1^0)_\alpha\text{-}\mathsf{Det}^* \nvdash (\Sigma_1^0)_\beta\text{-}\mathsf{Det}^*$.
- If $\alpha < \Gamma_0$, $(\Sigma_1^0)_{1+\alpha}\text{-}\mathsf{Det}^* \vdash (\Pi_1^0\text{-}\mathsf{CA}_0)_\alpha$ and $(\Pi_1^0\text{-}\mathsf{CA}_0)_\alpha \vdash (\Sigma_1^0)_{1+\alpha}\text{-}\mathsf{Det}^*$.
- If $\alpha \geq \Gamma_0$, $(\Sigma_1^0)_{1+\alpha}\text{-}\mathsf{Det}^* \nvdash (\Pi_1^0\text{-}\mathsf{CA}_0)_\alpha$ and $(\Pi_1^0\text{-}\mathsf{CA}_0)_\alpha \vdash (\Sigma_1^0)_\alpha\text{-}\mathsf{Det}^*$.

(Results for the strength in the sense of consistency)

- If $\alpha < \Gamma_0$ and $\alpha \cdot \omega \leq \beta$, the proof theoretic strength of $(\Sigma_1^0)_\alpha\text{-}\mathsf{Det}^*$ is strictly weaker than that of $(\Sigma_1^0)_\beta\text{-}\mathsf{Det}^*$
- If $\Gamma_0 \leq \alpha \leq \beta$, $(\Sigma_1^0)_{1+\alpha}\text{-}\mathsf{Det}^*$ and $(\Sigma_1^0)_\beta\text{-}\mathsf{Det}^*$ have the same proof theoretic strength as $\Delta_2^0\text{-}\mathsf{Det}^*$.

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Connecting the provable with the unprovable

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Phase transitions are a recent development in unprovability. The programme, started by Andreas Weiermann, aims to classify parameter values (functions) $f: \mathbb{N} \to \mathbb{N}$ according to the provability of statements φ_f. Here φ_f is some parametrised variant of a theorem which is not provable in a theory T. Examples of theorems for which such f are classified include the Paris–Harrington, Kanamori–McAloon, Adjacent Ramsey theorems, Paris–Kirby Hydra battles, Kruskal's tree theorem, Higman's and Dickson's lemma.

These results follow certain heuristics: suppose we have $\psi_f \equiv \forall x \exists y \varphi_f(x, y)$, $M_f(x) = \min\{y : \varphi(x, y)\}$ and $T \nvdash \psi_{\mathrm{id}}$. Furthermore let $l: \mathbb{N} \to \mathbb{N}$ be provably total in T, increasing and unbounded such that there exists x for which:

$$l < k \mapsto M_k(x),$$

where $<$ indicates ordering by eventual domination and M_k denotes M_f with as parameter value the function $f(i) = k$. Define $l^{-1}(i) = \max\{j : l(j) \leq i\}$. Then:

$$T \nvdash \psi_{l^{-1}}.$$

Using the same notations as above suppose $u: \mathbb{N} \to \mathbb{N}$ is provably total in T, increasing and unbounded such that for all x:

$$k \mapsto M_k(x) < u,$$

then:

$$T \vdash \psi_{u^{-1}}.$$

For existing phase transition results the latter part can be proven using an *upper bounds lemma*.

The above principle can be surmised informally: As soon as $\psi_{l^{-1}}$ cannot be proven using an upper bounds lemma (because l is a lower bound for $k \mapsto M_k(x)$) one can show that $T \nvdash \psi_{l^{-1}}$.

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Blackwell Games with a Constraint Function

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Abstract. Blackwell games are two-person zero-sum infinite games of imperfect information. In each round, two players simultaneously make their own moves, and then they are informed of each other's moves. The payoff of these games is determined by a Borel measurable function on the set of possible resulting sequences of moves. Assuming AD, Martin [1] proved that all Blackwell games are determined. However, considering the restrictions on Blackwell games, the selected strategies may be constrainted. We introduce new games as Blackwell games with some constraints, and investigate the determinacy of these games.

Keywords: Blackwell games, determinacy, constraint function

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Proof Theory of Projective Geometry: Orevkov's speed up result in strange surroundings

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We present a prooftheoretic analysis of projective geometry, a system which, although very simple, till now didn't find an extensive analysis besides a few comments in books on Euclidean geometry.

An extension of Gentzens **LK** for the special language of projective geometry is given and some results on **LK** are extended to $\mathbf{L_{PG}K}$.

Of special interest for our analysis was the concept of sketches. They are widely used but nobody thought on accepting them as a proofing tool. They have to be accompanied by a formal proof. To our mind this was an underestimation of the strength of sketches. We thought that in certain cases sketches can be considered as a proof by itself.

A new formulisation of sketches based on Herbrand disjunctions [Her71] is developed and the equivalence of sketches and proofs is shown. These results are similar to those in [Pre96] or [Pre97], but the new result is, since it is based on the general concept of the Herbrand disjunction, not dependent on the particular formulisation in $\mathbf{L_{PG}K}$.

The undecidability of projective geometry, together with an analysis of Herbrand disjunctions will lead us to new results on non-elementary speedups, based on Statman [Sta79] and Orevkovs [Ore79] results, from sketches to proofs.

To achieve this we transform Orevkovs formula into the language or projective geometry. Some very old results on the undecidability in the arithmetic of integers and rationals and in the theory of fields by Julia Robinson [Rob49] together with the concept of representability from Gödels [Göd31] historic work let us define a formula representing the predicate P from Orevkovs paper, where $P(a,b,c)$ holds iff $a + 2^b = c$. We make a detailed analysis of the Herbrand disjunction and obtain a lower bound for the cut-free proof of the modified Orevkov formula. Together with the short proof from Orevkovs paper we obtain the mentioned result.

As an interesting consequence of these analyses we will see that sketches are not constructive in the logical sense.

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UNIVERSAL PROPERTIES IN HIGHER-ORDER REVERSE MATHEMATICS

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We study higher-order Reverse Mathematics ([1]) from the point of view of Nonstandard Analysis. We discuss a general theme which is 'universal' in the sense that it holds for both classical and intuitionistic theorems. Intuitively speaking, this theme expresses that the (classical) existence of a standard object with the same standard and nonstandard properties, is equivalent to the existence of a standard functional computing said object. In symbols:

$$(\forall^{st} x^\tau)[A^{st}(x) \to (\exists^{st} y^\rho)B^*(x,y)) \tag{I}$$

is equivalent to

$$(\exists^{st}\Phi^{\tau\to\rho})(\forall^{st} x^\tau)[A^{st}(x) \to B^{st}(x,\Phi(x)). \tag{II}$$

Examples of theorems behaving in this fashion include uniform versions of WKL, the fan functional, the sup functional, the fan theorem etc. As it turns out, the logical strength of the uniform (I) and nonstandard (II) versions is directly determined by the constructive content of the non-uniform/standard counterpart. Finally, as a contribution to Hilbert's program of finitistic mathematics, the functional Φ from (II) has an elementary recursive *nonstandard* approximation $\Psi(\cdot, M)$, which is independent of the choice of infinite number M, given (I).

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Reverse Mathematics and Isbell's Zig-Zag Theorem

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Abstract. We contribute to the program of reverse mathematics by determining the exact logical strength of Isbell's zig-zag theorem for countable monoids. The zig-zag theorem is of fundamental importance in the theory of monoids whose statement is the following.

Definition 1 (dominions). *Let $B \supset A$ be a monoid extension and $b \in B$. b is dominated by A if for any monoid C and for any pair of homomorphisms $\alpha : B \to C$ and $\beta : B \to C$, if $\forall a \in A(\alpha(a) = \beta(a))$, then $\alpha(b) = \beta(b)$. The dominion of A is a set of all elements of B that is dominated by A.*

Definition 2 (zig-zags). *Let $B \supset A$ be a monoid extension and $b \in B$. A zig-zag of b over A is a triple of sequences*

$$\langle\langle a_0, a_1, \ldots, a_{2m}\rangle, \langle x_1, x_2, \ldots, x_m\rangle, \langle y_1, y_2, \ldots, y_m\rangle\rangle$$

such that

1. $a_i \in A$ *and* $x_j, y_j \in B$ $(0 \le i \le 2m, 1 \le j \le m)$,
2. $b = x_1 a_0 = a_{2m} y_m$,
3. $a_0 = a_1 y_1$, $a_{2i} y_i = a_{2i+1} y_{i+1} (1 \le i < m)$,
4. $x_i a_{2i-1} = x_{i+1} a_{2i} (1 \le i < m)$, $x_m a_{2m-1} = a_{2m}$.

Note that the assertion "b is dominated by A" is Π^1_1 while "b has a zig-zag over A" is Σ^0_1.

Theorem 1 (Isbell's Zig-Zag Theorem). *For a monoid extension $B \supset A$, $b \in B$ is an element of the dominion of A if and only if b has a zig-zag over A.*

The theorem was first stated by Isbell [3], and Philip [4] completed the proof. Since then, many simpler proofs have been published for about half a century, including those of Storre [8], Higgins [1], Renshaw [5] or Hoffman [2].
We carefully examine these proofs and explore the exact logical strength of Isbell's zig-zag theorem using the framework of reverse mathematics. Working in RCA_0, we show that WKL_0 is equivalent to the zig-zag theorem while ACA_0 is equivalent to the existence of dominions.
Typical proofs of the zig-zag theorem involve construction of an algebraic structure (e.g. a tensor product or a monoid of finite words) with a Σ^0_1

definable equality relation. These proofs can be formalized in ACA_0, but our proof in WKL_0 sidesteps this unnecessarily strong use of set comprehension, that is to say that along the idea of Hoffman [2] (he constructs the monoid of finite words with some nice property) we prepare a lemma about binary relations and modify his proof to be able to carried out in WKL_0.

The reversals that Isbell's zig-zag theorem implies WKL_0 or the existence of dominions implies ACA_0 are established by showing Σ^0_1 separation or existence of the image of arbitrarily chosen one-to-one function respectively, for details of these methods as well as the overview of reverse mathematics of second order arithmetic, see [7].

Keywords: Reverse Mathematics, Second Order Arithmetic, Isbell's Zig-Zag Theorem, Dominions.

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Transfinite Recursion in Higher Reverse Mathematics*

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Given a theorem Φ, phrased in the language of second-order arithmetic, reverse mathematics — initiated by Harvey Friedman [Fri75b] — examines the consequences of Φ over the base theory RCA_0, which captures computable mathematics in the sense that the ω-models of RCA_0 are precisely the Turing ideals. (See [Sim99] for an overview, as well as a precise definition of RCA_0.) The success of reverse mathematics has been partly due to the fact that many theorems of classical mathematics are naturally expressible as statements about sets of natural numbers; however, there is a significant amount of classical mathematics, including parts of measure theory and most of general topology, which resists any natural coding into this language. This was already recognized by Friedman in [Fri75a], and has recently become the subject of renewed interest following Ulrich Kohlenbach's development [Koh05] of a framework for reverse mathematics in arbitrary finite types; specifically, Kohlenbach defined a higher-type base theory, RCA_0^ω, which is a proof-theoretically natural conservative extension of RCA_0 in the language of finite-order arithmetic. Since then, reverse mathematics over RCA_0^ω has been further studied in [SY04], [Hun08], [Tow11], [Kre12], and others.

We will use reverse mathematics over RCA_0^ω as a vehicle for *comparative reverse mathematics*: the study of what patterns, present in standard reverse mathematics, hold or fail when classical statements are replaced with analogous statements in other settings, such as in this case higher types. In particular, we focus on the question, "To what extent is there a robust — that is, equivalent to a wide array of distinct theorems — theory corresponding to a type-2 version of ATR_0?" The key observation which drives this question is that much of the robustness of the classical system ATR_0 is due to the fact that "being well-ordered" is a Π^1_1-complete property of relations on $\mathbb{N}$, but "being well-ordered" is not a Π^2_1-complete property of relations on $\mathbb{R}$. This discrepancy causes many standard proofs of implications between versions of ATR_0 to fail, and raises doubt that there is any robust higher-type analogue of ATR_0.

Kohlenbach's RCA_0^ω is presented in language familiar to proof theorists, yet very distinct from the theory RCA_0 of which it is an analogue; we will begin by presenting a new theory, RCA_0^3, which captures precisely the third-order part of RCA_0^ω while being as similar as possible to RCA_0. We will then present a number of higher-type versions of principles equivalent to ATR_0 — specifically, clopen and open determinacy, Σ^1_1-separation, and comparability of well-orderings — as well as choice principles arising naturally in their proofs;

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some implications and separations will then be sketched. In particular, we will show that the comparability of well-orderings is exceptionally weak at higher types, and that open determinacy for reals is implied by the higher-type separation principle together with sufficient choice; and with regard to the relevant choice principles, we will show:

Theorem 01 *Over RCA_0^ω, the statements "The reals are well-orderable" and "Every real-indexed sequence of nonempty sets of reals has a choice function" are incomparable in strength.*

We will then turn to the main result: the relationship between open and clopen determinacy for reals. We consider games of length ω on $\mathbb{R}$. Such a game is *clopen* (*open*) if its payoff set is clopen (open) as a subset of $\mathbb{R}^\omega$, endowed with the product topology coming from the discrete topology on each factor of $\mathbb{R}$. We show:

Theorem 02 *Over RCA_0^ω, clopen determinacy for reals is strictly weaker than open determinacy for reals.*

The proof of this result uses a countably closed forcing similar to Steel's tagged tree forcing; the separating model is defined hierarchically, in the same way that Steel forcing builds models broken into layers corresponding to the levels of the hyperarithmetic hierarchy.

Finally, we will address several questions — both technical and foundational — arising from this work.

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Resource-bounded Forcing Theorem and Randomness

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Forcing complexity = The minimum size of a forcing condition: Forcing complexity [5] is the minimum size of a forcing condition that forces a given propositional formula. The origin of forcing complexity is in Dowd's study on NP=? coNP question [2].

Forcing complexity ≠ Time-bound of extension strategy: Ambos-Spies et al. [1] introduced the concept of resource-bounded random sets by extending the works of Schnorr and Lutz. They show that resource-bounded randomness implies resource-bounded genericity. While the genericity of Ambos-Spies is based on time-bound of finite-extension strategy, the genericity of Dowd, the main topic of this talk, is based on an analogy of forcing theorem.

Resource-bounded forcing theorem holds almost everywhere: It is widely known that 1-randomness and 1-genericity are incompatible. Interestingly, Dowd found that the following holds for a randomly chosen $X : \omega \to \{0, 1\}$. A property of an exponential-sized portion of X is forced by a polynomial-sized portion of X. To be more precise, for a positive integer r, an oracle D is r-generic in the sense of Dowd (r-Dowd, for short) if the following holds: If a certain formula F on an exponential-sized portion of D is a tautology then a polynomial-sized sub-function of D forces F to be a tautology. Here, r is the number of occurrences of query symbols in F. Dowd showed that the class of all r-Dowd sets has measure 1 (See [2, 5, 6]).

Does resource-bounded randomness imply the resource-bounded forcing theorem?: The answer is yes.

Main theorem [4] There exists an elementary recursive function $t(n)$ with the following property: "For every set X, if X is $t(n)$-random (random for $O(t(n))$-computable martingales) then for every positive integer r, the resource-bounded forcing theorem with respect to r-query tautologies holds for X (in other words, X is r-Dowd)."

⋆ Corresponding author. This work was partially supported by Japan Society for the Promotion of Science (JSPS) KAKENHI (C) 22540146, (B) 19340019 and (B) 23340020.

Outline of the proof: The key to our proof is a construction of a martingale that succeeds on every "non-Dowd" set. A basic idea is as follows.

Suppose that a forcing condition S is given and we want to define the value $d(S)$ of the martingale. Assume that a polynomial p is given at the node S. In the two basic open sets given by $S0$ (S concatenated by 0) and $S1$, we investigate the following conditional probabilities. We randomly chose an oracle T; To be more precise, we chose a finite initial segment of it. Then we investigate a probability of T having the followin property $(*)$, under the condition that T extends $S0$ (or $S1$, respectively). Here, a function $f(n)$ is chosen so that $f(n)$ is sufficiently larger than n.

$(*)$ For some i such that $n+1 \leq i \leq f(n)$, the following holds. We restrict the domain of T to the first 2^i strings in the length-lexicographic order. Then, the restriction of T fails a test for "the forcing theorem at stage i with respect to r and p".

We denote these conditional probabilities by $\varrho(S0)$ and $\varrho(S1)$. We define the martingale values $d(S0)$ and $d(S1)$ in proportion to $\varrho(S0)$ and $\varrho(S1)$. In other words, we shall define them so that the following equation holds.

$$d(S0)/\varrho(S0) = d(S1)/\varrho(S1)$$

Then, in many nodes, the ratio of d to ϱ shall be the same as that of the parent node. For example, the following holds.

$$d(S0)/\varrho(S0) = d(S)/\varrho(S)$$

By means of this property, we show that d succeeds on every "non-Dowd" oracle. In other words, for every "non-Dowd" oracle X, it holds that lim sup of $d(X \upharpoonright n)$ is infinite.

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On Gács' quantum algorithmic entropy

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In early 2000s, several different definitions of quantum Kolmogorov complexity have been proposed [1-3]. In [3], Gács introduced the notion of *lower-semicomputable semi-density matrices*, a quantum analogue of lower-semicomputable semimeasures. In the paper, though, proofs of two crucial theorems have some flaw:

Theorem 1. *There is a lower-semicomputable semi-density matrix* μ *dominating all other such matrices in the sense that for every other such matrix* ρ *there is a constant* $c > 0$ *with* $\rho \leq c\mu$.

Theorem 2. *Let* $|1\rangle, |2\rangle, \ldots$ *be a computable orthogonal sequence of states. Then for* $H = \overline{H}$ *or* $H = \underline{H}$ *we have*

$$H(|i\rangle) = K(i) + O(1).$$

Here, $K(i)$ *is the prefix Kolmogorov complexity of* i.

The former is indispensable to define quantum algorithmic entropy, and the latter is expected to be true when we wish to compare Gács' quantum algorithmic entropy and the qubit complexity defined by Berthiaume et al [2]. We introduce an infinite dimensional modification of Gács' lower-semicomputable matrices, and discuss the problem. We also see some properties and examples which stimulate further research.

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Gap-sequences

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Kruskal's theorem is a well-known statement in mathematics and logic. It is for example used in computer science with regard to termination orderings. The theorem states that the set of finite rooted trees is a *well-partial-ordering*. More specifically, if one takes an infinite sequence of finite rooted trees $(T_i)_{i<\omega}$, then Kruskal's theorem says that there exists two indices i and j such that $i < j$ and $T_i \trianglelefteq T_j$. Here, one tree is embeddable ($\trianglelefteq$) in the other tree if there exists an injective order- and infimum-preserving mapping from the first tree into the other.

In 1985, Harvey Friedman [4] proved that Kruskal's theorem is not provable in ATR_0. Furthermore, in this article, he introduced a new kind of embeddability relation between finite (labelled) rooted trees, namely the gap-embeddability relation. Using this new embeddability relation, he got a statement not provable in Π^1_1-CA_0, the strongest theory of the big-five in reverse mathematics! So by going from a 'normal' ordering to an ordering with a gap-condition, one obtains stronger statements. Therefore, it seems natural to study structures with a gap-embeddability relation.

In the same year, Schütte and Simpson [3] published a paper about the linearized version of Friedmans gap-embeddability relation for trees. There, they proved that this structure gives rise to an independence result for PA. More specifically, they proved that the statement 'for every natural number n, the set of finite sequences over $\{0, \dots, n-1\}$ is a well-partial-ordering under the gap-embeddability relation' is not provable in PA. This talk will deal with these sequences, which we will call the *gap-sequences*.

A partial order $(X, \leq_X)$ is a *well-partial-ordering* if it is well-founded and does not admit an infinite antichain. The *maximal order type* of a well-partial-ordering $(X, \leq_X)$ is an important characteristic of that well-partial-ordering. It is defined as the order type of the largest possible extension of $\leq_X$ to a well-ordering and denoted as $o(X, \leq_X)$. (For more information about well-partial-orderings, see [1].) This largest extension is called the *maximal linear extension* of the well-partial-ordering. The maximal order type captures a lot of information about that ordering. For example, there is a relation between the maximal order type of a 'natural' well-partial-ordering and the provability of its well-partial-orderedness in a specific theory by comparing the maximal order type with the proof-theoretical ordinal of that theory. Knowing a maximal linear extension of a well-partial-ordering is even better: we then know what the maximal order type is (the order type of this maximal linear extension) and furthermore, we know how elements of the partial ordering behave with regard to the maximal order type!

There is a general believe that the standard theta-functions (for a good overview of this function and its connection with Buchholz' Ψ-function, see [2]) capture well the maximal order type of trees with the gap-embeddability relation [5]. More specifically, the believe is that there is correspondence between the collapsing functions θ_i and a maximal linear extension of this famous well-partial-ordering. In this talk, we want to investigate the question if this is true for the sequence version.

Let θ_i be the theta-functions defined without addition. This give rise to an ordinal representation system of ε_0. One can see that if for example $01210 <^{gap} 012210$, then $\theta_0\theta_1\theta_2\theta_1(0) < \theta_0\theta_1\theta_2\theta_2\theta_1(0)$. Therefore, the theta-functions give rise to a linear extension of the gap-embeddability relation on finite sequences. We were wondering if this extension is maximal. More specifically, is

$$\sup_{m_1,\dots,m_k} \theta_0\theta_1^{m_1}\dots\theta_{n-1}^{m_{n-1}}\theta_0\theta_1^{m_n}\dots\theta_{n-1}^{m_{2n-2}}\theta_0\dots\theta_{n-1}^{m_k}(0) \stackrel{?}{=} o(S_n),$$

where S_n is the set of finite sequences over $\{0,\dots,n-1\}$ with the gap-embeddability relation. We will show that this is true for the case $n = 2$, but not for the case $n > 2$. This last fact is somewhat surprising.

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Finite iterations of infinite and finite Ramsey's theorem

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It is well-known that several finite variations of Ramsey's theorem provide independent statements from Peano Arithmetic (PA). The first such example was found by Paris[7] by using an iteration of finite Ramsey's theorem plus relatively largeness condition based on the idea of "indicator functions" by Kirby/Paris[5]. Later, that statement is simplified by Harrington, and nowadays, it is known as the famous Paris-Harrington principle [6]. However, the original iteration version, which can be considered as the "iterated Paris-Harrington principle", has the advantage that it can approximate the infinite version of Ramsey's theorem (see Bovykin/Weiermann[1]). More precisely, the Π^0_2-part (or equivalently, the class of provably recursive functions) of infinite Ramsey's theorem can be expressed by n-th iteration of Paris-Harrington principle for standard natural number $n \in \omega$. This fact also shows the limitation of the power of infinite Ramsey's theorem, in other words, infinite Ramsey's theorem as itself cannot prove the statement "for any m, m-th iteration of Paris-Harrington principle holds". This happens because of the lack of Σ^1_1-induction, but infinite Ramsey's theorem as itself does not prove such a strong induction. (Note that the study of the strength of induction provided by Ramsey's theorem is one of the most important topics in the field of reverse mathematics for combinatorial principles. See, e.g., [2, 3].)

In this talk, we try to fill this gap. Now a natural question arising from the above argument is "*what is a version of infinite Ramsey's theorem which implies iterated Paris-Harrington principle?*" Of course a naive answer to this question would be a (finite) iteration of infinite Ramsey's theorem. However, this does not succeed, since the iterated version of infinite Ramsey's theorem is just equivalent to the original one (in case the number of coloring is arbitrary). Thus, we will introduce a slightly strengthened version of infinite Ramsey's theorem, which is still equivalent to the original one over $\mathsf{WKL_0}$, but the iterated version is stronger. This, new iterated version turn to imply iterated Paris-Harrington principle, and in fact, it implies the consistency of original infinite Ramsey's theorem.

The strengthened version of Ramsey's theorem here is actually a natural generalization of Ramsey type König's lemma (RKL) introduced by Flood[4].

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Interpretations into Weihrauch Lattice

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Weihrauch lattice is a degree structure whose underlying reducibility requires uniform computability. It has originally been investigated under purely computable analytic motivations; however recently they started to claim that there is a close relationship between (constructive) reverse mathematics and the classification of Weihrauch degrees.

Such relationship was first suggested by G. Gherardi and A. Marcone in their paper [5]. Afterwards V. Brattka and G. Gherardi started their synthetic project of clasifying non-constructive principles in Weihrauch lattice [2]; the project will be referred as *BGM-program* here. Many analogies have been found between results of constructive reverse mathematics and those of BGM-program [3], [4], [1].

This research aims to find a formal connection between constructive reverse mathematics and BGM-program. Main results are partial soundness theorems, i.e. soundness holds only under several technical conditions, respectively for the following semantics:

(i) semantics of (theories over) SILL given by Weihrauch lattice
(ii) semantics of (theories over) SIL given by finitely parallelized Weihrauch lattice
(iii) semantics of a specific theory over SIL given by countably parallelized Weihrauch lattice

where SILL stands for *simply typed intuitionistic linear logic* and SIL stands for *simply typed intuitionistic logic*. Those soundness results are shown by combinations of two works respectively on syntactic and semantic aspects.

As the work on syntactic aspect we show a version of *inversion* for a meta-level translated left rule of universal quantification. In general an inference rule (resp. a meta-inference rule) is said to be *invertible* provided that the upper sequent(s) (resp. statement(s)) is derivable if and only if the lower sequent (resp. statement) is derivable; an inversion is a statement which asserts invertibility of an inference rule (resp. a meta-inference rule) [7]. We translate left rule of universal quantification to the meta-level and show invertibility of the resulting meta-inference rule over SILL, permitting several technical side conditions. We also show invertibility of a slight modification of the meta-inference rule over SIL, again, permitting several technical side conditions.

As the work on semantic aspect we introduce an operator which generates a degree structure, called an *abstract Weihrauch degree structure*, for given a fibration. Fibrations are fundamental structures from catetorical logic which are frequently used to define semantics for various type theories [6]. If the given fibration is under a technical assumption, the generated abstract Weihrauch degree structure forms a bounded distributive lattice; in such a case we call it an *abstract Weihrauch lattice*. In particular Weihrauch lattice is embeddable into a suitable instance of abstract Weihrauch lattice.

There is a well-known semantics of SIL given by first order fibration [6]. The semantics can be regarded as a sophisticated abstraction of realizability interpretation. We utilize its soundness and abstract Weihrauch lattices to define the semantics listed as (i)-(iii) above.

We also give a discussion about an application of our results to constructive reverse mathematics.

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