ON A CONJECTURE ON RAMANUJAN PRIMES

SHANTA LAISHRAM

Abstract. For $n \geq 1$, the *nth Ramanujan prime* is defined to be the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi(\frac{x}{2}) \geq n$ where $\pi(\nu)$ is the number of primes not exceeding ν for any $\nu > 0$ and $\nu \in \mathbb{R}$. In this paper, we prove a conjecture of Sondow on upper bound for Ramanujan primes. An explicit bound of Ramanujan primes is also given. The proof uses explicit bounds of prime π and θ functions due to Dusart.

1. Introduction

In [3], J. Sondow defined Ramanujan primes and gave some conjectures on the behaviour of Ramanujan primes. For $n \geq 1$, the *nth Ramanujan prime* is defined to be the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi(\frac{x}{2}) \geq n$ where $\pi(\nu)$ is the number of primes not exceeding ν for any $\nu > 0$ and $\nu \in \mathbb{R}$. It is easy to see that R_n is a prime for each n. The first few Ramanujan primes are given by $R_1 = 2, R_2 = 11, R_3 = 17, R_4 = 29, R_5 = 41, \ldots$. Sondow showed that for every $\epsilon > 0$, there exists $\mathcal{N}_0(\epsilon)$ such that $R_n < (2+\epsilon)n \log n$ for $n \geq \mathcal{N}_0(\epsilon)$. In this note, an explicit value of $\mathcal{N}_0(\epsilon)$ for each $\epsilon > 0$ is given. We prove

Theorem 1. *Let $\epsilon > 0$. For $\epsilon \leq 1.08$, let $\mathcal{N}_0 = \mathcal{N}_0(\epsilon) = \exp(\frac{c}{\epsilon} \log \frac{2}{\epsilon})$ where c is given by the following table.*

$\epsilon \in$	$(0, \frac{2}{11}]$	$(\frac{2}{11}, .4]$	$(.4, .6]$	$(.6, .8]$	$(.8, 1]$	$(1, 1.08]$
c	4	5	6	7	8	9

For $\epsilon > 1.08$, let $\mathcal{N}_0 = \mathcal{N}_0(\epsilon)$ be given by

$\epsilon \in$	$(1.08, 1.1]$	$(1.1, 1.21]$	$(1.21, 1.3]$	$(1.3, 2.5]$	$(2.5, 6]$	$(6, \infty)$
$\mathcal{N}_0$	169	101	74	48	6	2

Then

$$R_n < (2+\epsilon)n \log n \quad \text{for} \quad n \geq \mathcal{N}_0(\epsilon).$$

Sondow also showed that $p_{2n} < R_n < p_{4n}$ for $n > 1$ and he conjectured ([3, Conjecture 1]) that $R_n < p_{3n}$ for all $n \geq 1$, where p_i is the ith prime number. We derive the assertion of conjecture as a consequence of Theorem 1. We have

Theorem 2. *For $n > 1$, we have*

$$p_{2n} < R_n < p_{3n}.$$

We prove Theorems 1 and 2 in Section 3. In Section 2, we give preliminaries and lemmas for the proof which depend on explicit and sharp estimates from prime number theory.

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2. Lemmas

We begin with the following estimates from prime number theory. Recall that p_i is the ith prime prime and $\pi(\nu)$ is the number of primes $\leq \nu$. Let $\theta(\nu) = \sum_{p\leq\nu} \log p$ where p is a prime.

Lemma 2.1. *For $\nu \in \mathbb{R}$ and $\nu > 1$, we have*

(*a*) $p_i > i \log i$ *for* $i \geq 1, i \in \mathbb{Z}$.
(*b*) $\nu(1 - \frac{0.006788}{\log \nu}) \leq \theta(\nu) \leq \nu(1 + \frac{0.006788}{\log \nu})$ *for* $\nu \geq 10544111$.
(*c*) $\frac{\nu}{\log \nu - 1} \underset{\nu\geq 5393}{\leq} \pi(\nu) \underset{\nu>1}{\leq} \frac{\nu}{\log \nu}\left(1 + \frac{1.2762}{\log \nu}\right)$.

The estimate (a) is due to Rosser [2] and the estimates (b) and (c) are due to Dusart [1, p. 54]. □

From Lemma 2.1 (b) and (c), we obtain

Lemma 2.2. *Hence for $x \geq 2 \cdot 10544111$, we obtain*

$$\pi(x) - \pi(\frac{x}{2}) \geq \frac{x}{2\log x}\left(1 - \frac{0.020364}{\log x}\right) =: F(x) \text{ for } x \geq 2 \cdot 10544111 \tag{1}$$

and

$$\pi(x) - \pi(\frac{x}{2}) \geq \frac{x}{2(\log x - 1)}\left\{1 - \frac{1}{\log \frac{x}{2}}\left(\delta_1 - \frac{\delta_2}{\log \frac{x}{2}}\right)\right\} =: F_1(x) \text{ for } x \geq 5393 \tag{2}$$

where $\delta_1 = .2762 + \log 2$ and $\delta_2 = 1.2762(1 - \log 2)$.

Proof. For $x \geq 2 \cdot 10544111$, we obtain from Lemma 2.1 (b) that

$$\begin{aligned}
\pi(x) - \pi(\frac{x}{2}) &\geq \frac{\theta(x) - \theta(\frac{x}{2})}{\log x} \\
&\geq \frac{x\left(1 - \frac{0.006788}{\log x}\right) - \frac{x}{2}\left(1 + \frac{0.006788}{\log \frac{x}{2}}\right)}{\log x} \\
&= \frac{x}{2\log x}\left(1 - \frac{0.006788}{\log x}\left(2 + \frac{\log x}{\log \frac{x}{2}}\right)\right) \\
&\geq \frac{x}{2\log x}\left(1 - \frac{0.006788}{\log x}(2+1)\right)
\end{aligned}$$

which imply (1). For $x \geq 5393$, we have from Lemma 2.1 (c) that

$$\begin{aligned}
\pi(x) - \pi(\frac{x}{2}) &\geq \frac{x}{\log x - 1} - \frac{\frac{x}{2}}{\log \frac{x}{2}}\left(1 + \frac{1.2762}{\log \frac{x}{2}}\right) \\
&= \frac{x}{2(\log x - 1)}\left\{2 - \left(1 + \frac{\log 2 - 1}{\log \frac{x}{2}}\right)\left(1 + \frac{1.2762}{\log \frac{x}{2}}\right)\right\} \\
&\geq \frac{x}{2(\log x - 1)}\left\{1 - \frac{1}{\log \frac{x}{2}}\left(\delta_1 - \frac{\delta_2}{\log \frac{x}{2}}\right)\right\}
\end{aligned}$$

implying (2). □

For the proof of Theorem 1 for $\epsilon \leq .4$, we shall use the inequality (1). Then we may assume $n \leq \mathcal{N}_0(.4)$ for $\epsilon > .4$ and we use (2) to prove the assertion.

3. Proof of Theorems 1 and 2

For simplicity, we write $\epsilon_1 = \frac{\epsilon}{2}, \log_2 n := \log\log n$ and

$$f_0(n) := \log n + \log_2 n + \log(1+\epsilon_1) \text{ and } f_1(n) := \frac{\log_2 n + \log(2+2\epsilon_1)}{\log n}. \tag{3}$$

Let $x \geq (2+2\epsilon_1)n\log n$ with $n \geq \mathcal{N}_0(\epsilon)$ =exp$(\frac{c}{2\epsilon_1}\log\frac{1}{\epsilon_1}) =: n_0(\epsilon_1)$. Then $\log x \geq f_0(n) + \log 2$ for $n \geq n_0(\epsilon_1)$.

First we consider $\epsilon_1 \leq .2$. We observe that $F(x)$ is an increasing function of x and $2n_0(.2)\log(n_0(.2)) > 2\cdot 10544111$. Therefore we have from (1) that

$$\frac{\pi(x)-\pi(\frac{x}{2})}{n} \geq \frac{1+\epsilon_1}{1+f_1(n)}\left(1-\frac{0.020364}{f_0(n)+\log 2}\right) =: G(n). \tag{4}$$

$G(n)$ is again an increasing function of n. If $G(n_0(\epsilon_1)) > 1$, then $\pi(x)-\pi(\frac{x}{2}) > n$ for all $x \geq (2+2\epsilon_1)n\log n$ when $n \geq n_0(\epsilon_1)$ and hence $R_n < (2+2\epsilon_1)n\log n$ for $n \geq n_0(\epsilon_1)$. Therefore we show that $G(n_0) > 1$. It suffices to show

$$\epsilon_1 - \frac{0.020364(1+\epsilon_1)}{f_0(n)+\log 2} > f_1(n) = \frac{\log_2 n_0 + \log(2+2\epsilon_1)}{\log n_0}$$

for which it is enough to show

$$\epsilon_1 \geq \frac{\log_2 n_0 + \log(2+2\epsilon_1) + 0.020364(1+\epsilon_1)}{\log n_0}.$$

Since $\log n_0 = \frac{c}{2\epsilon_1}\log\frac{1}{\epsilon_1} = \frac{c_1}{\epsilon_1}\log\frac{1}{\epsilon_1}$ with $c_1 = 2, 2.5$ when $\epsilon_1 \leq \frac{1}{11}, \frac{1}{5}$, respectively, we need to show

$$\frac{(c_1-1)\log\frac{1}{\epsilon_1}}{\log_2\frac{1}{\epsilon_1} + \log c_1 + \log(2+2\epsilon_1) + 0.020364(1+\epsilon_1)} \geq 1.$$

The left hand side of the above expression is an increasing function of $\frac{1}{\epsilon_1}$ and the inequality is valid at $\frac{1}{\epsilon_1} = 11, 5$ implying the assertion for $\epsilon_1 \leq .2$.

Thus we now take $.2 < \epsilon_1 \leq .49$. We may assume that $n < n_0(.2)$. Since $x \geq (2+2\epsilon_1)n_0\log n_0 > 5393$, we have from (2) that

$$\frac{\pi(x)-\pi(\frac{x}{2})}{n} \geq \frac{1+\epsilon_1}{1+f_1(n)-\frac{1}{\log n}}\left\{1-\frac{1}{f_0(n)}\left(\delta_1-\frac{\delta_2}{f_0(n)}\right)\right\}.$$

Note that the right hand side of the above inequality is an increasing function of n since $n < n_0(.2)$. We show that the right hand side of the above inequality is > 1. Since $n \geq n_0(\epsilon_1)$, it suffices to show

$$\begin{aligned}&\log n_0(\epsilon_1 + \frac{1}{\log n_0} - f_1(n_0)) - \frac{1+\epsilon_1}{\frac{f_0(n_0)}{\log n_0}}\left(\delta_1 - \frac{\delta_2}{f_0(n_0)}\right)\\ =&\epsilon_1\log n_0 + 1 - \log_2 n_0 - \log(2+2\epsilon_1) - \frac{1+\epsilon_1}{1+f_1(n_0)-\frac{\log 2}{\log n_0}}\left(\delta_1 - \frac{\delta_2}{f_0(n_0)}\right)\end{aligned}$$

is > 0. Since $n_0(\epsilon_1) = \exp(\frac{c_1}{\epsilon_1}\log\frac{1}{\epsilon_1})$ where $c_1 = 3, 3.5, 4$ if $.2 < \epsilon_1 \leq .3, .3 < \epsilon_1 \leq .4$ and $.4 < \epsilon_1 \leq .49$, respectively, we observe that the right hand side of the above equality is equal to

$$(c_1-1)\log\frac{1}{\epsilon_1} + 1 - \log_2\frac{1}{\epsilon_1} - \log(2c_1+2c_1\epsilon_1) - \frac{1+\epsilon_1}{1+f_1(n_0)-\frac{\log 2}{\log n_0}}\left(\delta_1 - \frac{\delta_2}{f_0(n_0)}\right)$$

This is an increasing function of $\frac{1}{\epsilon_1}$. We find that the above function is > 0 for $\epsilon_1 \in \{.3, .4, .49\}$ implying $R_n < (2 + 2\epsilon_1)n \log n$ for $n \geq n_0(\epsilon_1)$ when $\epsilon_1 \leq .49$. Further we observe that $n_0(.49) \leq 339$. As a consequence, we have

$$R_n < 2.98n \log n \text{ for } n \geq 339.$$

and

$$\pi(x) - \pi(\frac{x}{2}) \geq 339 \text{ for } x \geq 2.98 \cdot 339 \log 339 > 5885.$$

Let $n < 339$. We now compute R_n by computing $\pi(x) - \pi(\frac{x}{2})$ for $p_{2n} < x \leq 5885$. Recall that $R_n > p_{2n}$ for $n > 1$. We find that $\frac{R_n}{n \log n} < 2.98, 3, 3.05, 3.08$ for $n \geq 220, 219, 171, 169$, respectively. Clearly $\frac{R_n}{n \log n} < 2 + \epsilon$ for $n \geq \mathcal{N}_0(\epsilon)$ when $\epsilon \leq 1.08$. Thus $R_n < 3n \log n$ for $n \geq 219$ and $R_n < 3.08n \log n$ for $n \geq 169$. For $\epsilon > 1.08$, we check that the assertion is true by computing R_n for each $n < 169$. This proves Theorem 1.

Now we derive Theorem 2. From the above paragraph, we obtain $R_n < 3n \log n$ for $n \geq 219$. By Lemma 2.1 (a), we have $p_{3n} > 3n \log 3n$ for all $n \geq 1$ implying the assertion of Theorem 2 for $n \geq 219$. For $n < 219$, we check that $R_n < p_{3n}$ and Theorem 2 follows. □

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E-mail address: `slaishram@math.uwaterloo.ca`

Department of Pure Mathematics, University of Waterloo, 200 Univ. Ave. West, Waterloo, Ontario N2L 3G1, Canada