A Connection Between Score Matching and Denoising Autoencoders

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Abstract

Denoising autoencoders have been previously shown to be competitive alternatives to Restricted Boltzmann Machines for unsupervised pre-training of each layer of a deep architecture. We show that a simple denoising autoencoder training criterion is equivalent to matching the score (with respect to the data) of a specific energy based model to that of a non-parametric Parzen density estimator of the data. This yields several useful insights. It defines a proper probabilistic model for the denoising autoencoder technique which makes it in principle possible to sample from them or to rank examples by their energy. It suggests a different way to apply score matching that is related to learning to denoise and does not require computing second derivatives. It justifies the use of tied weights between the encoder and decoder, and suggests ways to extend the success of denoising autoencoders to a larger family of energy-based models.

1 Introduction

This note uncovers an unsuspected link between the *score matching* technique (Hyvärinen, 2005; Hyvärinen, 2008) for learning the parameters of unnormalized density models over continuous-valued data, and the training of *denoising autoencoders* (Vincent *et al.*, 2008, 2010).

Score matching (SM) is an alternative to the maximum likelihood principle suitable for unnormalized probability density models whose partition function is intractable. Its

relationship to maximum likelihood has been investigated by Lyu (2009) who formally relates the Fisher divergence that yields score matching and the Kullback-Leibler divergence that yields maximum likelihood. Interestingly, his formal analysis indicates that score matching searches for parameters that are more robust to small-noise perturbations of the training data (Lyu, 2009). Score matching has also been recast as a special case under the more general frameworks of *generalized score matching* (Lyu, 2009; Marlin *et al.*, 2009) and *minimum probability flow* (Sohl-Dickstein *et al.*, 2009), allowing generalizations of score matching to discrete distributions (Hyvärinen, 2007b; Lyu, 2009; Sohl-Dickstein *et al.*, 2009). The *minimum probability flow* paradigm is particularly interesting as it unifies several recent alternative parameter-estimation methods, both for continuous and discrete data, under a single unified view[1]. Recently, Kingma and LeCun (2010) investigated a *regularized* form of score matching which adds a specific regularization term to the original score matching objective. Its relationship to the present work will be discussed in detail in Section 5.

Denoising Autoencoders (DAE) were proposed by Vincent *et al.* (2008) as a simple and competitive alternative to the Contrastive-Divergence-trained Restricted Boltzmann Machines (RBM) used by Hinton *et al.* (2006) for pretraining deep networks (Erhan *et al.*, 2010; Vincent *et al.*, 2010). Previous studies have already pointed out connections between SM and Contrastive Divergence (Hyvärinen, 2007a; Sohl-Dickstein *et al.*, 2009), have connected SM to optimal denoising for Gaussian noise with infinitesimal variance (Hyvärinen, 2008) and have shown that training Gaussian Binary RBM with SM is equivalent to training a regular (non-denoising) autoencoder with an additional regularization term (Swersky, 2010). The present study however is the first to recast the training of a DAE as a form of regularized score matching. This connection yields insights relevant to both research directions and suggests a novel parameter estimation technique that has its roots in both DAE and SM.

We begin by a very brief presentation of the DAE architecture for continuous-valued inputs in Section 2 and of the SM technique in Section 3. This allows us to introduce our formalism and precise terminology. Then, in section 4, we connect the denoising autoencoder objective to score matching. We conclude by a discussion on how our findings advance our understanding of both approaches.

Notation

We are interested in techniques that learn the parameters θ of a model by minimizing some objective function $J(\theta)$. For uniformity of notation, all distributions will be represented by their probability density functions (pdf) on $\mathbb{R}^d$. The pdf for discrete distributions will be expressed with Dirac-deltas δ.

$$\begin{aligned} q(\mathbf{x}) &\quad \text{Unknown } \textit{true} \text{ pdf. } \mathbf{x} \in \mathbb{R}^d. \\ D_n = \{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\} &\quad \text{Training set: i.i.d. sample from } q. \end{aligned}$$

[1] Specifically score matching (Hyvärinen, 2005), minimum velocity learning (Movellan, 2008), and certain forms of contrastive divergence (Hinton, 2002; Welling and Hinton, 2002) are all recast as minimizing the Kullback-Leibler divergence between the data distribution and the distribution obtained after running, for infinitesimal time, a dynamic that would transform it into the model distribution (Sohl-Dickstein *et al.*, 2009).

$q_0(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} \delta(\|\mathbf{x} - \mathbf{x}^{(i)}\|)$	Empirical pdf associated with D_n.
$q_\sigma(\tilde{\mathbf{x}}\|\mathbf{x}) = \frac{1}{(2\pi)^{d/2}\sigma^d} e^{-\frac{1}{2\sigma^2}\|\tilde{\mathbf{x}}-\mathbf{x}\|^2}$	Smoothing kernel or noise model: isotropic Gaussian of variance σ^2.
$q_\sigma(\tilde{\mathbf{x}}, \mathbf{x}) = q_\sigma(\tilde{\mathbf{x}}\|\mathbf{x})q_0(\mathbf{x})$	Joint pdf.
$q_\sigma(\tilde{\mathbf{x}}) = \frac{1}{n}\sum_{t=1}^{n} q_\sigma(\tilde{\mathbf{x}}\|\mathbf{x}^{(t)})$	Parzen density estimate based on D_n obtainable by marginalizing $q_\sigma(\tilde{\mathbf{x}}, \mathbf{x})$.
$p(\mathbf{x}; \theta)$	Density *model* with parameters θ.
$J_1 \backsim J_2$	Means $J_1(\theta)$ and $J_2(\theta)$ are equivalent optimization objectives [2].
$\mathbb{E}_{q(\mathbf{x})}[g(\mathbf{x})] = \int_{\mathbf{x}} q(\mathbf{x})g(\mathbf{x})d\mathbf{x}$	Expectation with respect to distribution q.
$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_i \mathbf{u}_i \mathbf{v}_i$	Dot product between two vectors.
$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$	Euclidean norm of vector $\mathbf{u}$.
$\text{softplus}(x) = \log(1 + e^x)$	Will be applied elementwise to vectors.
$\text{sigmoid}(x) = \frac{1}{1+e^{-x}} = \text{softplus}'(x)$	Will be applied elementwise to vectors.
$\mathbf{I}$	Identity matrix.
$\mathbf{W}^T$	Transpose of matrix $\mathbf{W}$.
$\mathbf{W}_i$	Vector for i^{th} row of $\mathbf{W}$.
$\mathbf{W}_{\cdot,j}$	Vector for j^{th} column of $\mathbf{W}$.

2 Denoising Autoencoders

Denoising Autoencoders (DAEs) are a simple modification of classical autoencoder neural networks that are trained, not to reconstruct their input, but rather to denoise an artificially corrupted version of their input (Vincent *et al.*, 2008, 2010). Whereas an over-complete regular autoencoder can easily learn a useless identity mapping, a DAE must extract more useful features in order to solve the much harder denoising problem. DAEs have proven to be an empirically successful alternative to Restricted Boltzmann Machines (RBM) for pre-training deep networks (Vincent *et al.*, 2008, 2010; Erhan *et al.*, 2010). Denoising autoencoders have also been used in different contexts in the earlier works of LeCun (1987); Gallinari *et al.* (1987); Seung (1998).

In this study, we will consider the denoising version of a simple classical autoencoder that uses a single sigmoidal hidden layer. Since data points originate from a continuous real valued distribution it is natural to use a linear decoder with a squared reconstruction loss[3]. We will be using *tied weights* whereby encoder and decoder share the same linear transformation parameters. The considered corruption is additive isotropic Gaussian noise. A detailed description of the architecture follows:

- A training input $\mathbf{x} \in D_n$ is first corrupted by additive Gaussian noise of covariance $\sigma^2\mathbf{I}$ yielding corrupted input $\tilde{\mathbf{x}} = \mathbf{x} + \epsilon$, $\epsilon \sim \mathcal{N}(0, \sigma^2\mathbf{I})$. This corresponds to

[2]Equivalence will be asserted when $J_2 = \alpha J_1 + \beta$ with $\alpha > 0$, $\beta \in \mathbb{R}$. Indeed a gradient-based optimization algorithm, when starting from some initial θ value, should land in the exact same minimum whether optimizing J_1 or J_2 (this may however require learning rate adjustment to compensate for scaling factor α).

[3]as opposed to a linear+sigmoid decoder with a Bernoulli cross-entropy loss, which would be the preferred choice for binary input.

conditional density $q_\sigma(\tilde{\mathbf{x}}|\mathbf{x}) = \frac{1}{(2\pi)^{d/2}\sigma^d} e^{-\frac{1}{2\sigma^2}\|\tilde{\mathbf{x}}-\mathbf{x}\|^2}$.

- The corrupted version $\tilde{\mathbf{x}}$ is *encoded* into a hidden representation $\mathbf{h} \in \mathbb{R}^{d_h}$ through an affine mapping followed by a nonlinearity:
  $\mathbf{h} = \text{encode}(\tilde{\mathbf{x}}) = \text{sigmoid}(\mathbf{W}\tilde{\mathbf{x}} + \mathbf{b})$, where $\tilde{\mathbf{x}} \in \mathbb{R}^d$, $\mathbf{h} \in (0,1)^{d_h}$, $\mathbf{W}$ is a $d_h \times d$ matrix and $\mathbf{b} \in \mathbb{R}^{d_h}$.

- The hidden representation $\mathbf{h}$ is *decoded* into reconstruction $\mathbf{x}^\mathbf{r}$ through affine mapping: $\mathbf{x}^\mathbf{r} = \text{decode}(\mathbf{h}) = \mathbf{W}^T\mathbf{h} + \mathbf{c}$, where $\mathbf{c} \in \mathbb{R}^d$.

- The parameters $\theta = \{\mathbf{W}, \mathbf{b}, \mathbf{c}\}$ are optimized so that the expected squared reconstruction error $\|\mathbf{x}^\mathbf{r} - \mathbf{x}\|^2$ is minimized, i.e. the objective function being minimized by such a denoising autoencoder (DAE) is:

$$\begin{aligned} J_{DAE\sigma}(\theta) &= \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}},\mathbf{x})}\left[\|\text{decode}(\text{encode}(\tilde{\mathbf{x}})) - \mathbf{x}\|^2\right] \\ &= \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}},\mathbf{x})}\left[\left\|\mathbf{W}^T\text{sigmoid}(\mathbf{W}\tilde{\mathbf{x}} + \mathbf{b})) + \mathbf{c} - \mathbf{x}\right\|^2\right]. \end{aligned} \quad (1)$$

3 Score Matching

3.1 Explicit Score Matching (ESM)

Score Matching was introduced by Hyvärinen (2005) as a technique to learn the parameters θ of probability density models $p(\mathbf{x};\theta)$ with intractable partition function $Z(\theta)$, where p can be written as

$$p(\mathbf{x};\theta) = \frac{1}{Z(\theta)}\exp(-E(\mathbf{x};\theta)).$$

E is called the energy function. Following Hyvärinen (2005), we will call *score* the gradient of the log density with respect to the data vector: $\psi(\mathbf{x};\theta) = \frac{\partial \log p(\mathbf{x};\theta)}{\partial \mathbf{x}}$. Beware that this usage differs slightly from traditional statistics terminology where *score* usually refers to the derivative of the log likelihood with respect to *parameters,* whereas here we are talking about a *score* with respect to the *data*. The core principle of *score matching* (Hyvärinen, 2005) is to learn θ so that $\psi(\mathbf{x};\theta) = \frac{\partial \log p(\mathbf{x};\theta)}{\partial \mathbf{x}}$ best matches the corresponding *score* of the true distribution, i.e. $\frac{\partial \log q(\mathbf{x})}{\partial x}$. The corresponding objective function to be minimized is the expected squared error between these two vectors, i.e.

$$J_{ESMq}(\theta) = \mathbb{E}_{q(\mathbf{x})}\left[\frac{1}{2}\left\|\psi(\mathbf{x};\theta) - \frac{\partial \log q(\mathbf{x})}{\partial \mathbf{x}}\right\|^2\right].$$

We refer to this formulation as *explicit score matching* (ESM).

Note that the score $\psi(\mathbf{x};\theta)$ does not depend on troublesome $Z(\theta)$. But since q is unknown, we do not have explicit regression targets $\frac{\partial \log q(\mathbf{x})}{\partial \mathbf{x}}$. Hyvärinen (2005) mentions in passing that non-parametric methods might be used to estimate those, and we shall later pay closer attention to this possibility.

3.2 Implicit Score Matching (ISM)

Hyvärinen (2005) instead proceeds by proving the following remarkable property:

$$\underbrace{\mathbb{E}_{q(\mathbf{x})}\left[\frac{1}{2}\left\|\psi(\mathbf{x};\theta)-\frac{\partial \log q(\mathbf{x})}{\partial \mathbf{x}}\right\|^2\right]}_{J_{ESMq}(\theta)} = \underbrace{\mathbb{E}_{q(\mathbf{x})}\left[\frac{1}{2}\left\|\psi(\mathbf{x};\theta)\right\|^2+\sum_{i=1}^{d}\frac{\partial \psi_i(\mathbf{x};\theta)}{\partial \mathbf{x}_i}\right]}_{J_{ISMq}(\theta)} + C_1 \tag{2}$$

where $\psi_i(\mathbf{x};\theta) = \psi(\mathbf{x};\theta)_i = \frac{\partial \log p(\mathbf{x};\theta)}{\partial \mathbf{x}_i}$, and C_1 is a constant that does not depend on θ. This yields an *implicit score matching* objective J_{ISMq} that no longer requires having an explicit score target for q but is nevertheless equivalent to J_{ESMq}. Hyvärinen (2005) formally shows that, provided $q(\mathbf{x})$ and $\psi(\mathbf{x};\theta)$ satisfy some weak regularity conditions[4], we have

$$J_{ESMq} \backsim J_{ISMq} \tag{3}$$

3.3 Finite Sample Version of Implicit Score Matching

Since we only have samples D_n from q, Hyvärinen proposes to optimize the finite sample version of J_{ISMq} which, following our notation, we shall write J_{ISMq_0}:

$$\begin{aligned} J_{ISMq_0}(\theta) &= \mathbb{E}_{q_0(\mathbf{x})}\left[\frac{1}{2}\left\|\psi(\mathbf{x};\theta)\right\|^2+\sum_{i=1}^{d}\frac{\partial \psi_i(\mathbf{x};\theta)}{\partial \mathbf{x}_i}\right] \\ &= \frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{2}\left\|\psi(\mathbf{x}^{(t)};\theta)\right\|^2+\sum_{i=1}^{d}\frac{\partial \psi_i(\mathbf{x}^{(t)};\theta)}{\partial \mathbf{x}_i}\right). \end{aligned} \tag{4}$$

J_{ISMq_0} is *asymptotically* equivalent to J_{ISMq} when $n \to \infty$ and hence asymptotically equivalent to objective J_{ESMq}. This can be summarized as

$$J_{ESMq} \backsim J_{ISMq} \backsim \lim_{n\to\infty} J_{ISMq_0}. \tag{5}$$

What happens in the transition from J_{ISMq} to finite sample version J_{ISMq_0} is however not entirely clear. Concerns regarding the stability of the resulting criterion were raised by Kingma and LeCun (2010), who propose instead to optimize a regularized version of J_{ISMq_0}:

$$J_{ISMreg}(\theta) = J_{ISMq_0}(\theta) + \lambda\sum_{i=1}^{d}\left(\frac{\partial \psi_i(\mathbf{x}^{(t)};\theta)}{\partial \mathbf{x}_i}\right)^2, \tag{6}$$

where the strength of the additional regularization term is controlled by hyperparameter λ. The relationship between this criterion and the criteria we propose below will be further discussed in Section 5.

[4]Namely: $q(\mathbf{x})$ and $\psi(\mathbf{x};\theta)$ are differentiable, $\mathbb{E}_{q(\mathbf{x})}\left[\left\|\frac{\partial \log q(\mathbf{x})}{\partial \mathbf{x}}\right\|^2\right]$ is finite, and for any θ, $\mathbb{E}_{q(\mathbf{x})}\left[\left\|\psi(\mathbf{x};\theta)\right\|^2\right]$ is finite and $\lim_{\|\mathbf{x}\|\to\infty} q(\mathbf{x})\psi(\mathbf{x};\theta) = 0$.

4 Linking Score Matching to the Denoising Autoencoder Objective

4.1 Matching the Score of a Non-Parametric Estimator

As previously stated, the possibility of matching the score $\psi(\mathbf{x};\theta)$ with an explicit target score for q obtained through non-parametric estimation was mentioned but not pursued in Hyvärinen (2005). We now examine this possibility more closely. Explicitly matching $\psi(\mathbf{x};\theta)$ with the score of Parzen windows density estimator $q_\sigma(\tilde{\mathbf{x}})$ yields the following objective:

$$J_{ESMq_\sigma}(\theta) = \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta) - \frac{\partial \log q_\sigma(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}}\right\|^2\right]. \tag{7}$$

For $\sigma > 0$, q_σ is differentiable, decreases to 0 at infinity, and $\mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\left\|\frac{\partial \log q_\sigma(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}}\right\|^2\right]$ is finite. All regularity conditions are satisfied, so the same equivalence with ISM as in Eq. 3 holds, i.e.

$$J_{ESMq_\sigma} \backsim J_{ISMq_\sigma}. \tag{8}$$

Note that this equivalence however breaks in the limit $\sigma \to 0$, because q_σ no longer satisfies these regularity conditions, and $J_{ESMq\sigma}$ can no longer be computed (whereas J_{ISMq_σ} remains well-behaved).

4.2 Denoising Score Matching (DSM)

Let us now consider a slightly different objective, that is inspired by both the Score Matching principle and by the Denoising Autoencoder approach of using pairs of clean and corrupted examples $(\mathbf{x}, \tilde{\mathbf{x}})$. For joint density $q_\sigma(\tilde{\mathbf{x}}, \mathbf{x}) = q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})q_0(\mathbf{x})$, we define the following *denoising score matching* (DSM) objective:

$$J_{DSMq_\sigma}(\theta) = \mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta) - \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\|^2\right]. \tag{9}$$

The underlying intuition is that following the gradient ψ of the log density at some corrupted point $\tilde{\mathbf{x}}$ should ideally move us towards the clean sample $\mathbf{x}$. Note that with the considered Gaussian kernel we have

$$\frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}} = \frac{1}{\sigma^2}(\mathbf{x} - \tilde{\mathbf{x}}). \tag{10}$$

Direction $\frac{1}{\sigma^2}(\mathbf{x} - \tilde{\mathbf{x}})$ clearly corresponds to moving from $\tilde{\mathbf{x}}$ back towards clean sample $\mathbf{x}$, and we want ψ to match that as best it can.

Now this alternate objective, inspired by denoising autoencoders, is equivalent to explicit score matching. Formally,

$$J_{ESMq_\sigma} \backsim J_{DSMq_\sigma} \tag{11}$$

The proof is in the appendix, and does not depend on the particular form of $q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})$ or $q(\mathbf{x})$ as long as $\log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})$ is differentiable with respect to $\tilde{\mathbf{x}}$.

4.3 An Energy Function that Yields the Denoising Autoencoder Objective

Let us now choose for model p the form

$$\begin{aligned} p(\mathbf{x};\theta) &= \frac{1}{Z(\theta)}\exp(-E(x;\theta)) \\ E(\mathbf{x};\underbrace{\mathbf{W},\mathbf{b},\mathbf{c}}_{\theta}) &= -\frac{\langle \mathbf{c},\mathbf{x}\rangle - \frac{1}{2}\left\|\mathbf{x}\right\|^2 + \sum_{j=1}^{d_h}\mathrm{softplus}\left(\langle \mathbf{W}_j,\mathbf{x}\rangle + \mathbf{b}_j\right)}{\sigma^2}. \end{aligned} \quad (12)$$

We then have

$$\begin{aligned} \psi_i(\mathbf{x};\theta) &= \frac{\partial \log p(\mathbf{x};\theta)}{\partial \mathbf{x}_i} \\ &= -\frac{\partial E}{\partial \mathbf{x}_i} \\ &= \frac{1}{\sigma^2}\left(\mathbf{c}_i - \mathbf{x}_i + \sum_{j=1}^{d_h}\mathrm{softplus}'\left(\langle \mathbf{W}_j,\mathbf{x}\rangle + \mathbf{b}_j\right)\frac{\partial\left(\langle \mathbf{W}_j,\mathbf{x}\rangle + \mathbf{b}_j\right)}{\partial \mathbf{x}_i}\right) \\ &= \frac{1}{\sigma^2}\left(\mathbf{c}_i - \mathbf{x}_i + \sum_{j=1}^{d_h}\mathrm{sigmoid}\left(\langle \mathbf{W}_j,\mathbf{x}\rangle + \mathbf{b}_j\right)\mathbf{W}_{ji}\right) \\ &= \frac{1}{\sigma^2}\left(\mathbf{c}_i - \mathbf{x}_i + \langle \mathbf{W}_{\cdot i},\, \mathrm{sigmoid}(\mathbf{W}\mathbf{x}+\mathbf{b})\rangle\right), \end{aligned}$$

which we can write as the single equation

$$\psi(\mathbf{x};\theta) = \frac{1}{\sigma^2}\left(\mathbf{W}^T\mathrm{sigmoid}(\mathbf{W}\mathbf{x}+\mathbf{b}) + \mathbf{c} - \mathbf{x}\right). \quad (13)$$

Substituting Eq. 10 and Eq. 13 in the expression for J_{DSMq_σ} (Eq. 9) we get, for this choice of Parzen kernel and density model,

$$\begin{aligned} J_{DSMq_\sigma}(\theta) &= \mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta) - \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\|^2\right] \\ &= \mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\frac{1}{\sigma^2}\left(\mathbf{W}^T\mathrm{sigmoid}(\mathbf{W}\tilde{\mathbf{x}}+\mathbf{b}) + \mathbf{c} - \tilde{\mathbf{x}}\right) - \frac{1}{\sigma^2}(\mathbf{x}-\tilde{\mathbf{x}})\right\|^2\right] \\ &= \frac{1}{2\sigma^4}\mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\left\|\mathbf{W}^T\mathrm{sigmoid}(\mathbf{W}\tilde{\mathbf{x}}+\mathbf{b}) + \mathbf{c} - \mathbf{x}\right\|^2\right] \\ &= \frac{1}{2\sigma^4}J_{DAE\sigma}(\theta). \end{aligned}$$

We have thus shown that

$$J_{DSMq_\sigma} \backsim J_{DAE\sigma} \quad (14)$$

5 Discussion

Putting together equations 8, 11 and 14, we can write, for $\sigma > 0$,

$$J_{ISMq_\sigma} \backsim J_{ESMq_\sigma} \backsim J_{DSMq_\sigma} \backsim J_{DAE\sigma}. \tag{15}$$

In summary, **training the denoising autoencoder defined in section 2 is equivalent to performing score matching (explicit or implicit) with the energy function of Eq. 12 on Parzen density estimate** q_σ. Such a training would typically use stochastic gradient descent, whereby samples from q_σ are obtained by corrupting samples from D_n. And it may be carried out with any of these four optimization objective formulations[5].

We introduced the kernel-smoothed empirical distribution q_σ to show a connection between score matching and a simple denoising autoencoder. Interestingly, the regularized score matching criterion J_{ISMreg} (Eq. 6) that Kingma and LeCun (2010) recently introduced with the very different motivation of curing possible instabilities, was derived by approximating[6] what amounts to J_{ISMq_σ}. From this perspective our four q_σ-based criteria in Eq. 15, including the denoising autoencoder, may be seen as alternative approximation-free forms of regularized score matching. A key difference is that, as is done with denoising autoencoder training, we would optimize *stochastic* versions of these approximation-free regularized criteria by corrupting training examples (i.e. sampling form q_σ), whereas Kingma and LeCun (2010) optimize an *approximation* of J_{ISMq_σ}, centered on the training samples only (i.e. sampling from q_0). Also, whereas J_{ISMreg}, like the other ISM criteria, requires computing second derivatives, the stochastic version of our novel J_{DSMq_σ} criterion does not, and thus appears much simpler to implement.

Note that the energy function in Eq. 12 is particular in that its scaling, which we may call its *temperature,* is chosen to match the corrupting noise level σ^2. This is required only to establish the last equivalence with the specific DAE we considered. But regarding the generic objectives $J_{ISMq_\sigma} \backsim J_{ESMq_\sigma} \backsim J_{DSMq_\sigma}$, their σ may in principle be chosen irrespective of the form or temperature of whatever energy function is to be learnt. Interestingly, the energy function in Eq. 12, that we designed to yield the equivalence with our denoising autoencoder objective, happens to be very similar to the free energy of a Restricted Boltzmann Machine with binary hidden units and Gaussian visible units (Welling *et al.*, 2005; Bengio *et al.*, 2007; Swersky, 2010). The major difference is that this latter free energy does not have a *global* temperature scaling of the whole expression[7]. We designed Eq. 12 to exactly yield the denoising version of the classic autoencoder described in Section 2. But with *tied* weights, it may be preferable to allow for an extra positive scaling parameter α for the reconstruction, so that there at least exists an equivalent reparametrization of the model for scaled input

[5]Note however that while these q_σ-based objectives are formally equivalent, their stochastic gradient descent optimization, based on sampling a limited number of corrupted examples, is likely to behave differently for each objective.

[6]A first order Taylor expansion and a diagonal Hessian approximation are used.

[7]Specifically, in the free energy of a Gaussian-binary RBM, the $\mathrm{softplus}$ terms are *not* divided by σ^2 nor scaled in any way.

data[8]. This is easily obtained in the energy function by multiplying the sum of $\mathrm{softplus}$ terms in Eq. 12 by α. We may even allow an arbitrary rescaling factor α_j for each hidden layer dimension independently by multiplying each $\mathrm{softplus}$ term by its own rescaling parameter α_j, which yields the following more flexible energy function:

$$E(\mathbf{x}; \underbrace{\mathbf{W}, \mathbf{b}, \mathbf{c}, \alpha, \sigma_m}_{\theta}) = -\frac{1}{\sigma_m^2}\left(\langle \mathbf{c}, \mathbf{x}\rangle - \frac{1}{2}\|\mathbf{x}\|^2 + \sum_{j=1}^{d_h} \alpha_{\mathrm{j}} \mathrm{softplus}\left(\langle \mathbf{W}_j, \mathbf{x}\rangle + \mathbf{b}_j\right)\right).$$

Here we have also included, as model parameter, a σ_m (where m stands for model) distinct from the noise σ of the training objective[9].

Our q_σ-based objectives J_{ISMq_σ}, J_{ESMq_σ}, or J_{DSMq_σ} can be used as alternatives to the finite sample objective J_{ISMq_0} (Eq. 4) advocated in Hyvärinen (2005) for learning unnormalized densities. Note that J_{ISMq_0} is a special case of J_{ISMq_σ} obtained in the limit of $\sigma \to 0$. Also, since Kingma and LeCun (2010) showed that it may be preferable to use a regularized criterion (that they derived form smoothed empirical distribution q_σ), it is likely that our q_σ-based criteria may, for $\sigma > 0$, yield better generalization performance than the J_{ISMq_0} advocated in Hyvärinen (2005)[10]. It seems that σ could allow one to choose an optimal bias-variance tradeoff for the finite-sample estimation of the true score matching gradient of interest $\nabla_\theta J_{ESMq} = \nabla_\theta J_{ISMq}$. While $\nabla_\theta J_{ISMq_0}$ is an unbiased estimator of it, $\nabla_\theta J_{ISMq_\sigma} = \nabla_\theta J_{ESMq_\sigma} = \nabla_\theta J_{DSMq_\sigma}$ will generally be biased when $\sigma > 0$ but are also likely to have a lower variance.

Among the three equivalent score matching objectives based on q_σ, objective J_{DSMq_σ} appears particularly interesting as a novel alternative formulation. It was motivated by both the SM principle and the DAE principle. From DAE it borrows the idea of learning to denoise artificially corrupted samples, and from SM it borrows the idea of learning a score function derived from an unnormalized density. J_{DSMq_σ}may prove simpler and more efficient in practice than the mathematically equivalent J_{ISMq_σ} , as it does not require computing second derivatives.

Our result is also a significant advance for DAEs. First, we have defined a proper energy function for the considered DAE through Eq. 12. This will enable many previously impossible or ill-defined operations on a trained DAE, for example deciding which is the more likely among several inputs, or sampling from a trained DAE using Hybrid Monte-Carlo (Duane *et al.*, 1987). Second, whereas using the same weight matrix ("tied weights") for the encoder and decoder is justified for RBMs, the encoder-decoder framework does not constrain that choice. Previous work on DAEs (Vincent *et al.*, 2008; Erhan *et al.*, 2010; Vincent *et al.*, 2010) explored both options, often finding tied weights to yield better empirical results. Within the SM framework presented here, using tied weights between encoder and decoder now has a proper justification,

[8] If for example one multiplies the input values by 100, one can obtain the same hidden representation as before by dividing $\mathbf{W}$ by 100. But because of the tied weights this means that the reconstruction would also be divided by 100 (i.e. there is no equivalent reparametrization), unless it can be compensated by an additional scaling of the reconstruction by a parameter α.

[9] We would however have to set $\sigma_m = \sigma$ to recover a recognizable denoising autoencoder objective.

[10] It is also noteworthy that the experimental results of Vincent *et al.* (2008, 2010) on DAE showed that the best models, judged by their ability to extract useful features, were obtained for non negligible values of the noise parameters. Moreover this way of controlling the model's capacity worked much better than either reducing the hidden layer size or than traditional weight decay.

since it follows naturally from differentiating the energy. Third, this framework opens the door to new variants that would naturally fall out from other choices of the energy function.

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Appendix

Proof that $J_{ESMq_\sigma} \backsim J_{DSMq_\sigma}$ (11)

The explicit score matching criterion using the Parzen density estimator is defined in Eq. 7 as

$$J_{ESMq_\sigma}(\theta) = \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta) - \frac{\partial \log q_\sigma(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}}\right\|^2\right]$$

which we can develop as

$$J_{ESMq_\sigma}(\theta) \;=\; \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta)\right\|^2\right] - S(\theta) + C_2 \tag{16}$$

where $C_2 = \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\frac{\partial \log q_\sigma(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}}\right\|^2\right]$ is a constant that does not depend on θ, and

$$\begin{aligned}
S(\theta) &= \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}}\right\rangle\right] \\
&= \int_{\tilde{\mathbf{x}}} q_\sigma(\tilde{\mathbf{x}}) \left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}}\right\rangle d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} q_\sigma(\tilde{\mathbf{x}}) \left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\frac{\partial}{\partial \tilde{\mathbf{x}}} q_\sigma(\tilde{\mathbf{x}})}{q_\sigma(\tilde{\mathbf{x}})}\right\rangle d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} \left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial}{\partial \tilde{\mathbf{x}}} q_\sigma(\tilde{\mathbf{x}})\right\rangle d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} \left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial}{\partial \tilde{\mathbf{x}}} \int_{\mathbf{x}} q_0(\mathbf{x}) q_\sigma(\tilde{\mathbf{x}}|\mathbf{x}) d\mathbf{x}\right\rangle d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} \left\langle \psi(\tilde{\mathbf{x}};\theta), \int_{\mathbf{x}} q_0(\mathbf{x}) \frac{\partial q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}} d\mathbf{x}\right\rangle d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} \left\langle \psi(\tilde{\mathbf{x}};\theta), \int_{\mathbf{x}} q_0(\mathbf{x}) q_\sigma(\tilde{\mathbf{x}}|\mathbf{x}) \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}} d\mathbf{x}\right\rangle d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} \int_{\mathbf{x}} q_0(\mathbf{x}) q_\sigma(\tilde{\mathbf{x}}|\mathbf{x}) \left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\rangle d\mathbf{x} d\tilde{\mathbf{x}} \\
&= \int_{\tilde{\mathbf{x}}} \int_{\mathbf{x}} q_\sigma(\tilde{\mathbf{x}}, \mathbf{x}) \left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\rangle d\mathbf{x} d\tilde{\mathbf{x}} \\
&= \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}},\mathbf{x})}\left[\left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\rangle\right].
\end{aligned}$$

Substituting this expression for $S(\theta)$ in Eq. 16 yields

$$\begin{aligned}
J_{ESMq_\sigma}(\theta) \;=\; & \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta)\right\|^2\right] \\
& -\mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\rangle\right] + C_2.
\end{aligned} \tag{17}$$

We also have defined in Eq. 9,

$$J_{DSMq_\sigma}(\theta) = \mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta) - \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\|^2\right],$$

which we can develop as

$$\begin{aligned} J_{DSMq_\sigma}(\theta) &= \mathbb{E}_{q_\sigma(\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\psi(\tilde{\mathbf{x}};\theta)\right\|^2\right] \\ &\quad -\mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\left\langle \psi(\tilde{\mathbf{x}};\theta), \frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\rangle\right] + C_3 \end{aligned} \tag{18}$$

where $C_3 = \mathbb{E}_{q_\sigma(\mathbf{x},\tilde{\mathbf{x}})}\left[\frac{1}{2}\left\|\frac{\partial \log q_\sigma(\tilde{\mathbf{x}}|\mathbf{x})}{\partial \tilde{\mathbf{x}}}\right\|^2\right]$ is a constant that does not depend on θ.

Looking at equations 17 and 18 we see that $J_{ESMq_\sigma}(\theta) = J_{DSMq_\sigma}(\theta) + C_2 - C_3$. We have thus shown that the two optimization objectives are equivalent.