Fluid Mechanics II

Viscosity and shear stresses

Shear stresses in a Newtonian fluid

A fluid at rest can not resist shearing forces. Under the action of such forces it deforms continuously, however small they are. The resistance to the action of shearing forces in a fluid appears only when the fluid is in motion. This implies the principal difference between fluids and solids. For solids the resistance to a shear deformation depends on the deformation itself, that is the shear stress τ is a function of the shear strain γ. For fluids the shear stress τ is a function of the **rate of strain** $d\gamma/dt$. The property of a fluid to resist the growth of shear deformation is called *viscosity*. The form of the relation between shear stress and rate of strain depends on a fluid, and most common fluids obey Newton's law of viscosity, which states that the shear stress is proportional to the strain rate:

$$\tau = \mu \frac{d\gamma}{dt} \, .$$

Such fluids are called NEWTONIAN FLUIDS. The coefficient of proportionality μ is known as *dynamic viscosity* and its value depends on the particular fluid. The ratio of dynamic viscosity to density is called *kinematic viscosity*

$$\nu = \frac{\mu}{\rho} \, .$$

Let us consider the parallel motion of fluid where all particles are moving in the same direction, but different layers have different velocities. After a small time Δt the fluid volume $abcd$ moves to $a'b'c'd'$ (figure 1), where $|aa'| = |bb'| = u(y+\delta y)\Delta t$ and $|cc'| = |dd'| = u(y)\Delta t$. The corresponding shear strain is

$$\gamma = \frac{\Delta x}{\delta y} = \frac{(\, u(y+\delta y) - u(y)\,)\, \Delta t}{\delta y} \, .$$

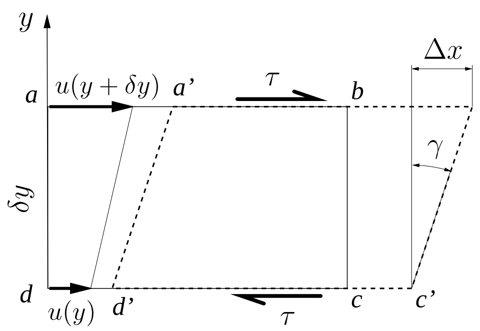

Fig. 1

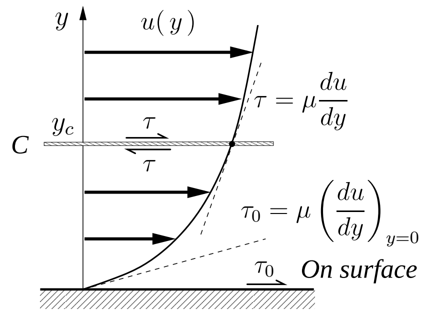

Fig. 2

For small Δt the strain can be expressed via its rate of change as

$$\gamma = \frac{d\gamma}{dt}\Delta t\,.$$

Then we can write

$$\frac{d\gamma}{dt} = \frac{u(y+\delta y) - u(y)}{\delta y}$$

and for small δy this gives

$$\frac{d\gamma}{dt} = \frac{du}{dy}\,.$$

Therefore, for a parallel flow of a Newtonian fluid shear stress is proportional to the gradient of velocity in the direction perpendicular to the flow, that is

$$\tau = \mu\frac{du}{dy}\,.$$

Let a surface $C:\ y = y_c$ is parallel to the flow and the velocity gradient is positive (figure 2). The flow above $y = y_c$ will apply the positive shear force on the upper surface of C, and the equal negative shear force will act on the lower surface of C from the fluid behind $y = y_c$. Both of these forces are due to the same shear stress τ, which is considered as positive in this case. On a rigid surface ($y = 0$, figure 2) the fluid velocity is equal the surface velocity (*no-slip condition*), and the shear force on a solid wall can be found from the value of the velocity gradient on the wall (figure 2). For a uniform flow τ is constant along the wall, and the value of the shear force acting on area A of the wall is:

$$F_\tau = A\,\mu\left(\frac{du}{dy}\right)_{y=0}\,.$$

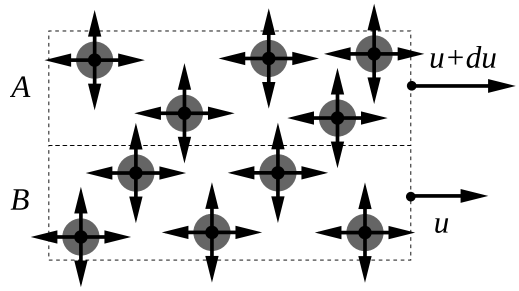

Fig. 3

The causes of viscosity

One of the causes of viscosity is the interchange of momentum between layers of fluid moving with different velocities due to the chaotic molecular motion. The motion of molecules in a flowing fluid is the superposition of the motion of the fluid media and of the chaotic motion of molecules due to thermal agitation (figure 3). Molecules are transferred from one layer to another. Thus molecules with higher momentum (layer A on figure 3) moving into layer with lower momentum (layer B). There they mix with slower molecules and increase the overall momentum of the layer B. Similarly, molecules with lower momentum moving from the layer B moving into the higher momentum layer A mix and reduce the overall momentum of the layer A. Both processes tend to reduce the relative velocity between the layers. This momentum exchange generates an effective shear force between the two layers. This mechanism is the principal viscosity mechanism for gases, when forces between molecules are small. When the temperature of a gas increases the chaotic molecular motion becomes more intensive and the molecular interchange increases. As the result, the viscosity μ of a gas will increase as the temperature increases, and it is practically independent on pressure. For liquids molecules are closer packed and therefore intermolecular forces also play a part in generation of viscosity in addition to molecular interchange. Viscosity μ is then reduces with the temperature, and is essentially independent of pressure.

Examples of laminar viscous flows

1. Couette flow

A gap h between two parallel horizontal plates is filled by a viscous fluid, and the upper plate moves with velocity V (figure 4). The dimensions of the plates are much larger then the distance h between them. Find the velocity

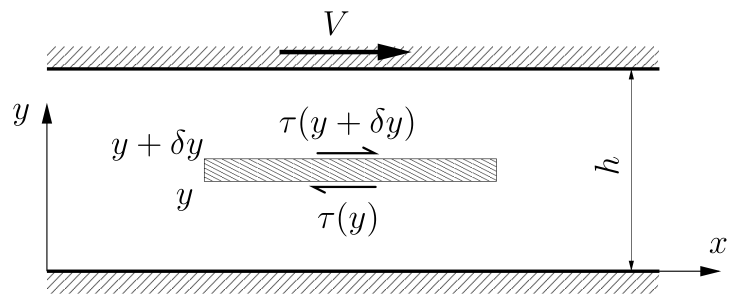

Fig. 4

distribution and the force applied to the upper plate if its area is A.

Solution:

Edges of the plates will influence the flow on a finite distance from the edges, comparable with h. If the distance between the plates is much smaller then their other dimensions the main part of the flow will be parallel, and velocity will not depend on the horizontal coordinate x: $u = u(y)$. For constant pressure on the plate edges the pressure remains the same everywhere between the plates. Then the only force acting on a fluid element shown on figure 4 is due to shear stresses on its boundaries: $F = A\,(\tau(y + \delta y) - \tau(y))$, and for a steady flow this force should be zero and the shear stress τ is constant. For a Newtonian fluid we have:

$$\tau = \mu \frac{du}{dy} = const$$

and after differentiation

$$\frac{d^2u}{dy^2} = 0.$$

The solution of this differential equation gives the linear velocity profile

$$u(y) = C_1 y + C_2,$$

where constants C_1 and C_2 to be found from the no-slip conditions on the plates:

$$u(0) = 0; \quad u(h) = V\,,$$

which gives $C_1 = V/h$ and $C_2 = 0$, and the velocity profile is

$$u(y) = V\,\frac{y}{h}\,.$$

The corresponding shear stress is

$$\tau = \mu \frac{V}{h},$$

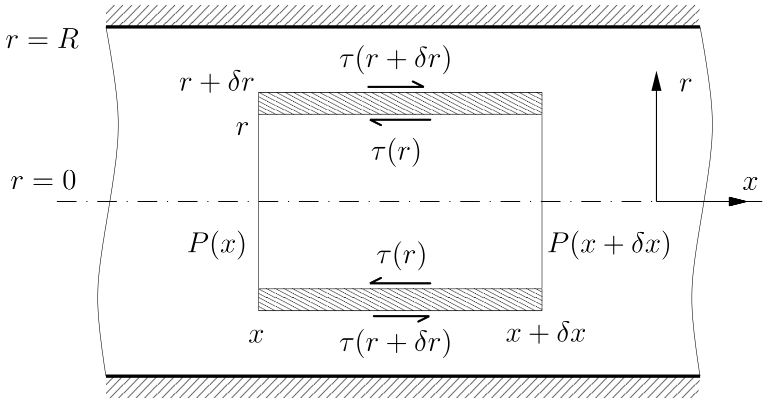

Fig. 5

and the force required to move the plate is

$$F = \tau\, A = \frac{\mu\, V\, A}{h}\,.$$

2. Laminar flow in a circular pipe

Calculate the velocity profile and the friction factor for a fully developed steady flow in a circular pipe.

Solution:

For a fully developed steady parallel flow pressure does not change across the flow, and the velocity profile (and shear stress) does not change along the pipe. That is:

$$P = P(x); \quad u = u(y); \quad \tau = \tau(y)\,.$$

Let us consider a cylindrical fluid volume with the internal radius r, external radius $r + \delta r$ and length δx (figure 5). For a steady flow the total horizontal force acting on the fluid in this volume should be zero:

$$(P(x) - P(x + \delta x))\, 2\pi r\, \delta r - \tau(r)\, 2\pi r\, \delta x + \tau(r + \delta r)\, 2\pi (r + \delta r)\, \delta x = 0\,.$$

This gives:

$$\frac{1}{r}\frac{(r + \delta r)\, \tau(r + \delta r) - r\, \tau(r)}{\delta r} = \frac{P(x + \delta x) - P(x)}{\delta x}$$

and for small δr and δx we obtain

$$\frac{1}{r}\,\frac{d\,(r\tau)}{dr} = \frac{dP}{dx}\,.$$

For a Newtonian fluid and a prescribed pressure gradient $dP/dx = B$ this leads to the following differential equation for the velocity profile:

$$\frac{d}{dr}\left(r\,\frac{du}{dr}\right) = r\,\frac{B}{\mu}\,.$$

The boundary conditions are:

$$u(R) = 0; \quad \left.\frac{du}{dr}\right|_{r=0} = 0\,,$$

where the first conditions is the no-slip condition on the pipe wall $r = R$, and the second one expresses the flow symmetry on the central axis $r = 0$. First integration gives:

$$r\frac{du}{dr} = \frac{1}{2}\,\frac{B}{\mu}r^2 + C_1\,,$$

and from the boundary condition at $r = 0$ $C_1 = 0$. After the second integration we obtain:

$$u(r) = \frac{1}{4}\frac{B}{\mu}\,r^2 + C_2\,,$$

and the solution satisfying the wall no-slip condition is

$$u(r) = -\frac{1}{4}\frac{B}{\mu}\,(R^2 - r^2)\,.$$

For pressure decreasing in the x-direction $B < 0$ and u is positive, that is fluid flows from higher to lower pressure. The mean velocity of the flow is

$$U = \frac{Q}{\pi R^2} = \frac{1}{\pi R^2}\int_0^R u(r)\,2\pi r\,dr = \frac{-B}{2\mu}\int_0^R (1 - \frac{r^2}{R^2})\,r\,dr = \frac{-B\,R^2}{8\,\mu}\,.$$

Expressing B via the mean velocity as $B = -8\mu\,U/R^2$ we rewrite the velocity profile as

$$u(r) = 2\,U\,(1 - \frac{r^2}{R^2})\,.$$

The wall shear stress:

$$\tau_{\mathrm{w}} = \mu\left.\frac{du}{dr}\right|_{r=R} = -4\mu\frac{U}{R}\,.$$

The sign is irrelevant and we will take plus thereafter, keeping in mind that the shear stress acts against the flow. The friction coefficient:

$$f = \frac{\tau_{\mathrm{w}}}{\rho\,U^2/2} = \frac{8\,\mu}{\rho\,U\,R} = \frac{16\,\mu}{\rho\,U\,d} = \frac{16}{Re_d}\,.$$

Reading:

MASSEY,B.S. Mechanics of Fluids

Chapter 1: Fundamental Concepts Relating to Fluids

1.5 Viscosity

Chapter 6: Laminar Flow

6.1 Introduction

6.2 Steady Laminar Flow in Circular pipes. The Hagen-Poiseuille Law

6.3 Steady Laminar Flow between Parallel Planes

6.4 Steady Laminar Flow between Parallel Planes, one of which is moving

6.5 Stokes' Law

6.6 The Measurement of Viscosity

DOUGLAS,J.F., *et al.* Fluid Mechanics

Chapter 1: Fluids and their Properties

1.1 Fluids

1.9 Viscosity

1.10 Causes of viscosity in gases

1.11 Causes of viscosity in a liquid

Chapter 10: Laminar and Turbulent Flows in Bounded Systems

10.1 Incompressible, steady and uniform laminar flow between parallel plates

10.2 Incompressible, steady and uniform laminar flow in circular cross-section pipes

Problems:

1. A uniform film of oil 0.13mm thick separates two circular discs, each 150mm diameter and mounted coaxially. Find the torque required to rotate one disc relative to the other at a steady speed of 400 rev/min if the oil has viscosity of 0.14Pa s. Ignore edge effects at the rim of the disc.

2. (a) Consider forces acting on a suitably chosen fluid element in a steady fully developed laminar flow between two parallel plates with a pressure gradient applied to the fluid along the plates. Justifying the necessary assumptions derive a differential equation describing the velocity profile of such flow. Discuss the conditions on flow boundaries.

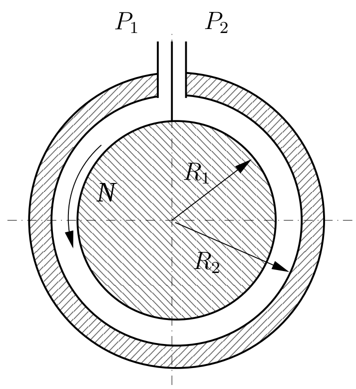

Fig. 6

   (b) A cylinder with an outer radius R_1 rotates inside a tube of an internal radius R_2 with the rate N rev/s. The cylinder and the tube are coaxial, and the uniform gap h between them is filled with oil, with viscosity μ. The system is used to pump the oil against the pressure difference ΔP (see figure 6). Find the relation between the head and the flow rate of the pump (characteristic curve). Find the efficiency of the pump and its optimal operational parameters.

3. A uniform layer h of viscous fluid with viscosity μ flows along a plate inclined to 30° to the horizon.

   (a) Consider the condition of steady motion of a fluid element and derive the differential equation for velocity profile.

   (b) Formulate conditions on fluid boundaries and write the formula for the velocity profile.

   (c) Calculate the flow rate

   (d) Discuss conditions when your solution can be applied.