XI. *On the Forces, Stresses, and Fluxes of Energy in the Electromagnetic Field.*

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General Remarks, especially on the Flux of Energy.

§ **1.** THE remarkable experimental work of late years has inaugurated a new era in the development of the Faraday-Maxwellian theory of the ether, considered as the primary medium concerned in electrical phenomena—electric, magnetic, and electromagnetic. MAXWELL's theory is no longer entirely a paper theory, bristling with unproved possibilities. The reality of electromagnetic waves has been thoroughly demonstrated by the experiments of HERTZ and LODGE, FITZGERALD and TROUTON, J. J. THOMSON, and others; and it appears to follow that, although MAXWELL's theory may not be fully correct, even as regards the ether (as it is certainly not fully comprehensive as regards material bodies), yet the true theory must be one of the same type, and may probably be merely an extended form of MAXWELL's.

No excuse is therefore now needed for investigations tending to exhibit and elucidate this theory, or to extend it, even though they be of a very abstract nature. Every part of so important a theory deserves to be thoroughly examined, if only to see what is in it, and to take note of its unintelligible parts, with a view to their future explanation or elimination.

§ **2.** Perhaps the simplest view to take of the medium which plays such a necessary part, as the recipient of energy, in this theory, is to regard it as continuously filling all space, and possessing the mobility of a fluid rather than the rigidity of a solid. If whatever possess the property of inertia be matter, then the medium is a form of matter. But away from ordinary matter it is, for obvious reasons, best to call it as usual by a separate name, the ether. Now, a really difficult and highly speculative question, at present, is the connection between matter (in the ordinary sense) and ether. When the medium transmitting the electrical disturbances consists of ether and matter, do they move together, or does the matter only partially carry forward the ether which immediately surrounds it? Optical reasons may lead us to conclude, though only tentatively, that the latter may be the case; but at present, for the purpose of fixing the data, and in the pursuit of investigations not having specially

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optical bearing, it is convenient to assume that the matter and the ether in contact with it move together. This is the working hypothesis made by H. HERTZ in his recent treatment of the electrodynamics of moving bodies; it is, in fact, what we tacitly assume in a straightforward and consistent working out of MAXWELL'S principles without any plainly-expressed statement on the question of the relative motion of matter and ether; for the part played in MAXWELL'S theory by matter is merely (and, of course, roughly) formularised by supposing that it causes the etherial constants to take different values, whilst introducing new properties, that of dissipating energy being the most prominent and important. We may, therefore, think of merely one medium, the most of which is uniform (the ether), whilst certain portions (matter as well) have different powers of supporting electric displacement and magnetic induction from the rest, as well as a host of additional properties; and of these we can include the power of supporting conduction current with dissipation of energy according to JOULE'S law, the change from isotropy to eolotropy in respect to the distribution of the several fluxes, the presence of intrinsic sources of energy, &c.*

§ **3.** We do not in any way form the equations of motion of such a medium, even as regards the uniform simple ether, away from gross matter; we have only to discuss it as regards the electric and magnetic fluxes it supports, and the stresses and fluxes of energy thereby necessitated. First, we suppose the medium to be stationary, and examine the flux of electromagnetic energy. This is the POYNTING flux of energy. Next we set the medium into motion of an unrestricted kind. We have now necessarily a convection of the electric and magnetic energy, as well as the POYNTING flux. Thirdly, there must be a similar convection of the kinetic energy, &c., of the translational motion; and fourthly, since the motion of the medium involves the working of ordinary (Newtonian) force, there is associated with the previous a flux of energy due to the activity of the corresponding stress. The question is therefore a complex one, for we have to properly fit together these various fluxes of energy in harmony with the electromagnetic equations. A side issue is the determination of the proper measure of the activity of intrinsic forces, when the medium moves; in another form, it is the determination of the proper meaning of "true current" in MAXWELL'S sense.

§ **4.** The only general principle that we can bring to our assistance in interpreting electromagnetic results relating to activity and flux of energy, is that of the per-

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* Perhaps it is best to say as little as possible at present about the connection between matter and ether, but to take the electromagnetic equations in an abstract manner. This will leave us greater freedom for future modifications without contradiction. There are, also, cases in which it is obviously impossible to suppose that matter in bulk carries on with it the ether in bulk which permeates it. Either, then, the mathematical machinery must work between the molecules; or else, we must make such alterations in the equations referring to bulk as will be practically equivalent in effect. For example, the motional magnetic force $\mathbf{VDq}$ of equations (88), (92), (93) may be modified either in $\mathbf{q}$ or in $\mathbf{D}$, by use of a smaller effective velocity $\mathbf{q}$, or by the substitution in $\mathbf{D}$ or $c\mathbf{E}$ of a modified reckoning of c for the effective permittivity.

sistence of energy. But it would be quite inadequate in its older sense referring to integral amounts; the definite localisation by MAXWELL, of electric and magnetic energy, and of its waste, necessitates the similar localisation of sources of energy; and in the consideration of the supply of energy at certain places, combined with the continuous transmission of electrical disturbances, and therefore of the associated energy, the idea of a flux of energy through space, and therefore of the continuity of energy in space and in time, becomes forced upon us as a simple, useful, and necessary principle, which cannot be avoided.

When energy goes from place to place, it traverses the intermediate space. Only by the use of this principle can we safely derive the electromagnetic stress from the equations of the field expressing the two laws of circuitation of the electric and magnetic forces; and this again becomes permissible only by the postulation of the definite localisation of the electric and magnetic energies. But we need not go so far as to assume the objectivity of energy. This is an exceedingly difficult notion, and seems to be rendered inadmissible by the mere fact of the relativity of motion, on which kinetic energy depends. We cannot, therefore, definitely individualise energy in the same way as is done with matter.

If ρ be the density of a quantity whose total amount is invariable, and which can change its distribution continuously, by actual motion from place to place, its equation of continuity is

$$\text{conv}\,\mathbf{q}\rho = \dot{\rho}, \quad \ldots \ldots \ldots \quad (1)$$

where $\mathbf{q}$ is its velocity, and $\mathbf{q}\rho$ the flux of ρ. That is, the convergence of the flux of ρ equals the rate of increase of its density. Here ρ may be the density of matter. But it does not appear that we can apply the same method of representation to the flux of energy. We may, indeed, write

$$\text{conv}\,\mathbf{X} = \dot{\mathrm{T}}, \quad \ldots \ldots \ldots \quad (2)$$

if $\mathbf{X}$ be the flux of energy from all causes, and T the density of localisable energy. But the assumption $\mathbf{X} = \mathrm{T}\mathbf{q}$ would involve the assumption that T moved about like matter, with a definite velocity. A part of T may, indeed, do this, viz., when it is confined to, and is carried by matter (or ether); thus we may write

$$\text{conv}\,(\mathbf{q}\mathrm{T} + \mathbf{X}) = \dot{\mathrm{T}}, \quad \ldots \ldots \ldots \quad (3)$$

where T is energy which is simply carried, whilst $\mathbf{X}$ is the total flux of energy from other sources, and which we cannot symbolise in the form $\mathrm{T}\mathbf{q}$; the energy which comes to us from the Sun, for example, or radiated energy. It is, again, often impossible to carry out the principle in this form, from a want of knowledge of how energy gets to a certain place. This is, for example, particularly evident in the case of gravitational energy, the distribution of which, before it is communicated to matter, increasing its kinetic energy, is highly speculative. If it come from the ether (and where else *can* it come from ?), it should be possible to symbolise this in **X**, if not in **q**T; but in default of a knowledge of its distribution in the ether, we cannot do so, and must therefore turn the equation of continuity into

$$\mathrm{S} + \mathrm{conv}\,(\mathbf{q}\mathrm{T} + \mathbf{X}) = \dot{\mathrm{T}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where S indicates the rate of supply of energy per unit volume from the gravitational source, whatever that may be. A similar form is convenient in the case of intrinsic stores of energy, which we have reason to believe are positioned within the element of volume concerned, as when heat gives rise to thermoelectric force. Then S is the activity of the intrinsic sources. Then again, in special applications, T is conveniently divisible into different kinds of energy, potential and kinetic. Energy which is dissipated or wasted comes under the same category, because it may either be regarded as stored, though irrecoverably, or passed out of existence, so far as any immediate useful purpose is performed. Thus we have as a standard practical form of the equation of continuity of energy referred to the unit volume,

$$\mathrm{S} + \mathrm{conv}\,\{\mathbf{X} + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\} = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where S is the energy supply from intrinsic sources, U potential energy and T kinetic energy of localisable kinds, **q** (U + T) its convective flux, Q the rate of waste of energy, and **X** the flux of energy other than convective, *e.g.*, that due to stresses in the medium and representing their activity. In the electromagnetic application we shall see that U and T must split into two kinds, and so must **X**, because there is a flux of energy even when the medium is at rest.

§ **5.** Sometimes we meet with cases in which the flux of energy is either wholly or partly of a circuital character. There is nothing essentially peculiar to electromagnetic problems in this strange and apparently useless result. The electromagnetic instances are paralleled by similar instances in ordinary mechanical science, when a body is in motion and is also strained, especially if it be in rotation. This result is a necessary consequence of our ways of reckoning the activity of forces and of stresses, and serves to still further cast doubt upon the " thinginess " of energy. At the same time, the flux of energy is going on all around us, just as certainly as the flux of matter, and it is impossible to avoid the idea ; we should, therefore, make use of it and formularise it whenever and as long as it is found to be useful, in spite of the occasional failure to obtain readily understandable results.

The idea of the flux of energy, apart from the conservation of energy, is by no means a new one. Had gravitational energy been less obscure than it is, it might have found explicit statement long ago. Professor POYNTING* brought the principle

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* POYNTING, 'Phil. Trans.,' 1884.

into prominence in 1884, by making use of it to determine the electromagnetic flux of energy. Professor LODGE* gave very distinct and emphatic expression of the principle generally, apart from its electromagnetic aspect, in 1885, and pointed out how much more simple and satisfactory it makes the principle of the conservation of energy become. So it would, indeed, could we only understand gravitational energy ; but in that, and similar respects, it is a matter of faith only. But Professor LODGE attached, I think, too much importance to the identity of energy, as well as to another principle he enunciated, that energy cannot be transferred without being transformed, and conversely; the transformation being from potential to kinetic energy or conversely. This obviously cannot apply to the convection of energy, which is a true flux of energy; nor does it seem to apply to cases of wave motion in which the energy, potential and kinetic, of the disturbance, is transferred through a medium unchanged in relative distribution, simply because the disturbance itself travels without change of type; though it may be that in the unexpressed internal actions associated with the wave propagation there might be found a better application.

It is impossible that the ether can be fully represented, even merely in its transmissive functions, by the electromagnetic equations. Gravity is left out in the cold ; and although it is convenient to ignore this fact, it may be sometimes usefully remembered, even in special electromagnetic work; for, if a medium have to contain and transmit gravitational energy as well as electromagnetic, the proper system of equations should show this, and, therefore, include the electromagnetic. It seems, therefore, not unlikely that in discussing purely electromagnetic speculations, one may be within a stone's throw of the explanation of gravitation all the time. The consummation would be a really substantial advance in scientific knowledge.

On the Algebra and Analysis of Vectors without Quaternions. Outline of Author's System.

§ **6.** The proper language of vectors is the algebra of vectors. It is, therefore, quite certain that an extensive use of vector-analysis in mathematical physics generally, and in electromagnetism, which is swarming with vectors, in particular, is coming and may be near at hand. It has, in my opinion, been retarded by the want of special treatises on vector analysis adapted for use in mathematical physics, Professor TAIT's well-known profound treatise being, as its name indicates, a treatise on Quaternions. I have not found the HAMILTON-TAIT notation of vector operations convenient, and have employed, for some years past, a simpler system. It is not, however, entirely a question of notation that is concerned. I reject the quaternionic basis of vector analysis. The anti-quaternionic argument has been recently ably stated by Professor WILLARD GIBBS.† He distinctly separates this from the question

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* LODGE, 'Phil. Mag.,' June, 1885, "On the Identity of Energy."

† Professor GIBBS's letters will be found in 'Nature,' vol. 43, p. 511, and vol. 44, p. 79; and Professor

of notation, and this may be considered fortunate, for whilst I can fully appreciate and (from practical experience) endorse the anti-quaternionic argument, I am unable to appreciate his notation, and think that of HAMILTON and TAIT is, in some respects, preferable, though very inconvenient in others.

In HAMILTON'S system the quaternion is the fundamental idea, and everything revolves round it. This is exceedingly unfortunate, as it renders the establishment of the algebra of vectors without metaphysics a very difficult matter, and in its application to mathematical analysis there is a tendency for the algebra to get more and more complex as the ideas concerned get simpler, and the quaternionic basis forms a real difficulty of a substantial kind in attempting to work in harmony with ordinary Cartesian methods.

Now, I can confidently recommend, as a really practical working system, the modification I have made. It has many advantages, and not the least amongst them is the fact that the quaternion does not appear in it at all (though it may, without much advantage, be brought in sometimes), and also that the notation is arranged so as to harmonise with Cartesian mathematics. It rests entirely upon a few definitions, and may be regarded (from one point of view) as a systematically abbreviated Cartesian method of investigation, and be understood and practically used by any one accustomed to Cartesians, without any study of the difficult science of Quaternions. It is simply the elements of Quaternions without the quaternions, with the notation simplified to the uttermost, and with the very inconvenient *minus* sign before scalar products done away with.*

TAIT'S in vol. 43, pp. 535, 608. This rather one-sided discussion arose out of Professor TAIT stigmatising Professor GIBBS as "a retarder of quaternionic progress." This may be very true; but Professor GIBBS is anything but a retarder of progress in vector analysis and its application to physics.

[*] §§ 7, 8, 9 contain an introduction to vector-analysis (without the quaternion), which is sufficient for the purposes of the present paper, and, I may add, for general use in mathematical physics. It is an expansion of that given in my paper "On the Electromagnetic Wave Surface," 'Phil. Mag.,' June, 1885. The algebra and notation are substantially those employed in all my papers, especially in "Electromagnetic Induction and its Propagation," 'The Electrician,' 1885.

Professor GIBBS'S vectorial work is scarcely known, and deserves to be well known. In June, 1888, I received from him a little book of 85 pages, bearing the singular imprint NOT PUBLISHED, Newhaven, 1881–4. It is indeed odd that the author should not have published what he had been at the trouble of having printed. His treatment of the linear vector operator is specially deserving of notice. Although "for the use of students in physics," I am bound to say that I think the work much too condensed for a first introduction to the subject.

In 'The Electrician' for Nov. 13, 1891, p. 27, I commenced a few articles on elementary vector-algebra and analysis, specially meant to explain to readers of my papers how to work vectors. I am given to understand that the earlier ones, on the algebra, were much appreciated; the later ones, however, are found difficult. But the vector-algebra is identically the same in both, and is of quite a rudimentary kind. The difference is, that the later ones are concerned with analysis, with varying vectors; it is the same as the difference between common algebra and differential calculus. The difficulty, whether real or not, does not indicate any difficulty in the vector-algebra. I mention this on account of the great prejudice which exists against vector-algebra.

§ **7.** Quantities being divided into scalars and vectors, I denote the scalars, as usual, by ordinary letters, and put the vectors in the plain black type, known, I believe, as Clarendon type, rejecting Maxwell's German letters on account of their being hard to read. A special type is certainly not essential, but it facilitates the reading of printed complex vector investigations to be able to see at a glance which quantities are scalars and which are vectors, and eases the strain on the memory. But in MS. work there is no occasion for specially formed letters.

Thus **A** stands for a vector. The tensor of a vector may be denoted by the same letter plain; thus A is the tensor of **A**. (In MS. the tensor is A_0.) Its rectangular scalar components are A_1, A_2, A_3. A unit vector parallel to **A** may be denoted by $\mathbf{A}_1$, so that $\mathbf{A} = A\mathbf{A}_1$. But little things of this sort are very much matters of taste. What is important is to avoid as far as possible the use of letter prefixes, which, when they come two (or even three) together, as in Quaternions, are very confusing.

The scalar product of a pair of vectors **A** and **B** is denoted by **AB**, and is defined to be

$$\mathbf{AB} = A_1B_1 + A_2B_2 + A_3B_3 = AB\cos\widehat{\mathbf{AB}} = \mathbf{BA}. \qquad (6)$$

The addition of vectors being as in the polygon of displacements, or velocities, or forces; *i.e.*, such that the vector length of any closed circuit is zero; either of the vectors **A** and **B** may be split into the sum of any number of others, and the multiplication of the two sums to form **AB** is done as in common algebra; thus

$$\left.\begin{aligned}(\mathbf{a}+\mathbf{b})(\mathbf{c}+\mathbf{d}) &= \mathbf{ac}+\mathbf{ad}+\mathbf{bc}+\mathbf{bd},\\ &= \mathbf{ca}+\mathbf{da}+\mathbf{cb}+\mathbf{db}.\end{aligned}\right\} \qquad (7)$$

If **N** be a unit vector, **NN** or $\mathbf{N}^2 = 1$; similarly $\mathbf{A}^2 = A^2$ for any vector.

The reciprocal of a vector **A** has the same direction; its tensor is the reciprocal of the tensor of **A**. Thus

$$\mathbf{AA}^{-1} = \frac{\mathbf{A}}{\mathbf{A}} = 1;$$

and

$$\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A} = \frac{\mathbf{A}}{\mathbf{B}} = \frac{A}{B}\cos\widehat{\mathbf{AB}}. \qquad (8)$$

The vector product of a pair of vectors is denoted by $\mathrm{V}\mathbf{AB}$, and is defined to be the vector whose tensor is AB sin $\widehat{\mathbf{AB}}$, and whose direction is perpendicular to the plane of **A** and **B**. Or

$$\mathrm{V}\mathbf{AB} = \mathbf{i}(A_2B_3 - A_3B_2) + \mathbf{j}(A_3B_1 - A_1B_3) + \mathbf{k}(A_1B_2 - A_2B_1) = -\mathrm{V}\mathbf{BA}, \qquad (9)$$

where **i**, **j**, **k**, are any three mutually rectangular unit vectors. The tensor of $\mathrm{V}\mathbf{AB}$ is $\mathrm{V}_0\mathbf{AB}$; or

$$\mathrm{V}_0\mathbf{AB} = AB\sin\widehat{\mathbf{AB}}. \qquad (10)$$

Its components are **iVAB**, **jVAB**, **kVAB**.

In accordance with the definitions of the scalar and vector products, we have

$$\left.\begin{array}{lll} \mathbf{i}^2 = 1, & \mathbf{j}^2 = 1, & \mathbf{k}^2 = 1\,; \\ \mathbf{ij} = 0, & \mathbf{jk} = 0, & \mathbf{ki} = 0\,; \\ \mathrm{V}\mathbf{ij} = \mathbf{k}, & \mathrm{V}\mathbf{jk} = \mathbf{i}, & \mathrm{V}\mathbf{ki} = \mathbf{j}\,; \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

and from these we prove at once that

$$\mathrm{V}\,(\mathbf{a} + \mathbf{b})\,(\mathbf{c} + \mathbf{d}) = \mathrm{V}\mathbf{ac} + \mathrm{V}\mathbf{ad} + \mathrm{V}\mathbf{bc} + \mathrm{V}\mathbf{bd},$$

and so on, for any number of component vectors. The order of the letters in each product has to be preserved, since **Vab** = − **Vba**.

Two very useful formulæ of transformation are

$$\begin{aligned} \mathbf{A}\mathrm{V}\mathbf{BC} &= \mathbf{B}\mathrm{V}\mathbf{CA} = \mathbf{C}\mathrm{V}\mathbf{AB} \\ &= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)\,; \quad . \; . \; . \; . \quad (12) \end{aligned}$$

and

$$\left.\begin{aligned} \mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC} &= \mathbf{B.CA} - \mathbf{C.AB}, \\ &= \mathbf{B(CA)} - \mathbf{C(AB)}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

Here the dots, or the brackets in the alternative notation, merely act as separators, separating the scalar products **CA** and **AB** from the vectors they multiply. A space would be equivalent, but would be obviously unpractical.

As $\frac{\mathbf{A}}{\mathbf{B}}$ is a scalar product, so in harmony therewith, there is the vector product $\mathrm{V}\frac{\mathbf{A}}{\mathbf{B}}$. Since **VAB** = − **VBA**, it is now necessary to make a convention as to whether the denominator comes first or last in $\mathrm{V}\frac{\mathbf{A}}{\mathbf{B}}$. Say therefore, $\mathrm{V}\mathbf{AB}^{-1}$. Its tensor is

$$\mathrm{V}_0\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathrm{A}}{\mathrm{B}}\sin \widehat{\mathbf{AB}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

§ **8.** Differentiation of vectors, and of scalar and vector functions of vectors with respect to scalar variables is done as usual. Thus,

$$\left.\begin{aligned} \dot{\mathbf{A}} &= \mathbf{i}\dot{A}_1 + \mathbf{j}\dot{A}_2 + \mathbf{k}\dot{A}_3. \\ \frac{d}{dt}\mathbf{AB} &= \mathbf{A}\dot{\mathbf{B}} + \mathbf{B}\dot{\mathbf{A}}. \\ \frac{d}{dt}\mathbf{A}\mathrm{V}\mathbf{BC} &= \dot{\mathbf{A}}\mathrm{V}\mathbf{BC} + \mathbf{A}\mathrm{V}\dot{\mathbf{B}}\mathbf{C} + \mathbf{A}\mathrm{V}\mathbf{B}\dot{\mathbf{C}}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

The same applies with complex scalar differentiators, *e.g.*, with the differentiator

$$\frac{\partial}{\partial t} = \frac{d}{dt} + \mathbf{q}\nabla,$$

used when a moving particle is followed, $\mathbf{q}$ being its velocity. Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{AB} &= \mathbf{A}\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{A}}{\partial t}\\ &= \mathbf{A\dot{B}} + \mathbf{B\dot{A}} + \mathbf{A.q}\nabla.\mathbf{B} + \mathbf{B.q}\nabla.\mathbf{A}. \qquad (16)\end{aligned}$$

Here $\mathbf{q}\nabla$ is a scalar differentiator given by

$$\mathbf{q}\nabla = q_1\frac{d}{dx} + q_2\frac{d}{dy} + q_3\frac{d}{dz}, \qquad (17)$$

so that $\mathbf{A.q}\nabla.\mathbf{B}$ is the scalar product of $\mathbf{A}$ and the vector $\mathbf{q}\nabla.\mathbf{B}$; the dots here again act essentially as separators. Otherwise, we may write it $\mathbf{A}\,(\mathbf{q}\nabla)\,\mathbf{B}$.

The fictitious vector ∇ given by

$$\nabla = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3 = \mathbf{i}\frac{d}{dx} + \mathbf{j}\frac{d}{dy} + \mathbf{k}\frac{d}{dz} \qquad (18)$$

is *very* important. Physical mathematics is very largely the mathematics of ∇. The name Nabla seems, therefore, ludicrously inefficient. In virtue of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the operator ∇ behaves as a vector. It also, of course, differentiates what follows it.

Acting on a scalar P, the result is the vector

$$\nabla \mathrm{P} = \mathbf{i}\nabla_1\mathrm{P} + \mathbf{j}\nabla_2\mathrm{P} + \mathbf{k}\nabla_3\mathrm{P}, \qquad (19)$$

the vector rate of increase of P with length.

If it act on a vector $\mathbf{A}$, there is first the scalar product

$$\nabla\mathbf{A} = \nabla_1\mathrm{A}_1 + \nabla_2\mathrm{A}_2 + \nabla_3\mathrm{A}_3 = \operatorname{div}\mathbf{A}, \qquad (20)$$

or the divergence of $\mathbf{A}$. Regarding a vector as a flux, the divergence of a vector is the amount leaving the unit volume.

The vector product $\mathrm{V}\nabla\mathbf{A}$ is

$$\begin{aligned}\mathrm{V}\nabla\mathbf{A} &= \mathbf{i}\,(\nabla_2\mathrm{A}_3 - \nabla_3\mathrm{A}_2) + \mathbf{j}\,(\nabla_3\mathrm{A}_1 - \nabla_1\mathrm{A}_3) + \mathbf{k}\,(\nabla_1\mathrm{A}_2 - \nabla_2\mathrm{A}_1),\\ &= \operatorname{curl}\mathbf{A}. \qquad (21)\end{aligned}$$

The line-integral of $\mathbf{A}$ round a unit area equals the component of the curl of $\mathbf{A}$ perpendicular to the area.

We may also have the scalar and vector products $\mathbf{N}\nabla$ and $\mathrm{V}\mathbf{N}\nabla$, where the vector $\mathbf{N}$ is not differentiated. These operators, of course, require a function to follow them on which to operate; the previous $\mathbf{q}\nabla.\mathbf{A}$ of (16) illustrates.

The Laplacean operator is the scalar product ∇^2 or $\nabla\nabla$; or

$$\nabla^2 = \nabla_1^2 + \nabla_2^2 + \nabla_3^2; \qquad (22)$$

and an example of (13) is

$$\mathrm{V}\nabla\mathrm{V}\nabla\mathbf{A} = \nabla.\nabla\mathbf{A} - \nabla^2\mathbf{A},$$

or

$$\mathrm{curl}^2\mathbf{A} = \nabla\ \mathrm{div}\ \mathbf{A} - \nabla^2\mathbf{A}, \quad \ldots \quad (23)$$

which is an important formula.

Other important formulæ are the next three.

$$\mathrm{div}\ \mathrm{P}\mathbf{A} = \mathrm{P}\ \mathrm{div}\ \mathbf{A} + \mathbf{A}\nabla.\mathrm{P}, \quad \ldots \quad (24)$$

P being scalar. Here note that $\mathbf{A}\nabla.\mathrm{P}$ and $\mathbf{A}\nabla\mathrm{P}$ (the latter being the scalar product of $\mathbf{A}$ and $\nabla\mathrm{P}$) are identical. This is not true when for P we substitute a vector. Also

$$\mathrm{div}\ \mathrm{V}\mathbf{AB} = \mathbf{B}\ \mathrm{curl}\ \mathbf{A} - \mathbf{A}\ \mathrm{curl}\ \mathbf{B}; \quad \ldots \quad (25)$$

which is an example of (12), noting that both **A** and **B** have to be differentiated. And

$$\mathrm{curl}\ \mathrm{V}\mathbf{AB} = \mathbf{B}\nabla.\mathbf{A} + \mathbf{A}\ \mathrm{div}\ \mathbf{B} - \mathbf{A}\nabla.\mathbf{B} - \mathbf{B}\ \mathrm{div}\ \mathbf{A}, \quad \ldots \quad (26)$$

This is an example of (13).

§ **9.** When one vector **D** is a *linear* function of another vector **E**, that is, connected by equations of the form

$$\left.\begin{array}{l} \mathrm{D}_1 = c_{11}\mathrm{E}_1 + c_{12}\mathrm{E}_2 + c_{13}\mathrm{E}_3, \\ \mathrm{D}_2 = c_{21}\mathrm{E}_1 + c_{22}\mathrm{E}_2 + c_{23}\mathrm{E}_3, \\ \mathrm{D}_3 = c_{31}\mathrm{E}_1 + c_{32}\mathrm{E}_2 + c_{33}\mathrm{E}_3, \end{array}\right\}, \quad \ldots \quad (27)$$

in terms of the rectangular components, we denote this simply by

$$\mathbf{D} = c\mathbf{E}, \quad \ldots \quad (28)$$

where c is the linear operator. The conjugate function is given by

$$\mathbf{D}' = c'\mathbf{E}, \quad \ldots \quad (29)$$

where $\mathbf{D}'$ is got from **D** by exchanging c_{12} and c_{21}, &c. Should the nine coefficients reduce to six by $c_{12} = c_{21}$, &c., **D** and $\mathbf{D}'$ are identical, or **D** is a self-conjugate or symmetrical linear function of **E**.

But, in general, it is the sum of **D** and $\mathbf{D}'$ which is a symmetrical function of **E**, and the difference is a simple vector-product. Thus

$$\left.\begin{array}{l} \mathbf{D} = c_0\mathbf{E} + \mathrm{V}\boldsymbol{\epsilon}\mathbf{E}, \\ \mathbf{D}' = c_0\mathbf{E} - \mathrm{V}\boldsymbol{\epsilon}\mathbf{E}, \end{array}\right\} \quad \ldots \quad (30)$$

where c_0 is a self-conjugate operator, and $\boldsymbol{\epsilon}$ is the vector given by

$$\boldsymbol{\epsilon} = \mathbf{i}\frac{c_{32} - c_{23}}{2} + \mathbf{j}\frac{c_{13} - c_{31}}{2} + \mathbf{k}\frac{c_{21} - c_{12}}{2}. \qquad (31)$$

The important characteristic of a self-conjugate operator is

$$\left.\begin{array}{rl} & \mathbf{E}_1\mathbf{D}_2 = \mathbf{E}_2\mathbf{D}_1, \\ \text{or} & \mathbf{E}_1c_0\mathbf{E}_2 = \mathbf{E}_2c_0\mathbf{E}_1, \end{array}\right\} \qquad (32)$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are any two $\mathbf{E}$'s, and $\mathbf{D}_1$, $\mathbf{D}_2$ the corresponding $\mathbf{D}$'s. But when there is not symmetry, the corresponding property is

$$\left.\begin{array}{rl} & \mathbf{E}_1\mathbf{D}_2 = \mathbf{E}_2\mathbf{D}'_1, \\ \text{or} & \mathbf{E}_1c\mathbf{E}_2 = \mathbf{E}_2c'\mathbf{E}_1. \end{array}\right\} \qquad (33)$$

Of these operators we have three or four in electromagnetism connecting forces and fluxes, and three more connected with the stresses and strains concerned. As it seems impossible to avoid the consideration of rotational stresses in electromagnetism, and these are not usually considered in works on elasticity, it will be desirable to briefly note their peculiarities here, rather than later on.

On Stresses, irrotational and rotational, and their Activities.

§ **10.** Let $\mathbf{P}_N$ be the vector stress on the $\mathbf{N}$ plane, or the plane whose unit normal is $\mathbf{N}$. It is a linear function of $\mathbf{N}$. This will fully specify the stress on any plane. Thus, if $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ are the stresses on the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ planes, we shall have

$$\left.\begin{array}{l} \mathbf{P}_1 = \mathbf{i}P_{11} + \mathbf{j}P_{12} + \mathbf{k}P_{13}, \\ \mathbf{P}_2 = \mathbf{i}P_{21} + \mathbf{j}P_{22} + \mathbf{k}P_{23}, \\ \mathbf{P}_3 = \mathbf{i}P_{31} + \mathbf{j}P_{32} + \mathbf{k}P_{33}. \end{array}\right\} \qquad (34)$$

Let, also, $\mathbf{Q}_N$ be the conjugate stress; then, similarly,

$$\left.\begin{array}{l} \mathbf{Q}_1 = \mathbf{i}P_{11} + \mathbf{j}P_{21} + \mathbf{k}P_{31}, \\ \mathbf{Q}_2 = \mathbf{i}P_{12} + \mathbf{j}P_{22} + \mathbf{k}P_{32}, \\ \mathbf{Q}_3 = \mathbf{i}P_{13} + \mathbf{j}P_{23} + \mathbf{k}P_{33}. \end{array}\right\} \qquad (35)$$

Half the sum of the stresses $\mathbf{P}_N$ and $\mathbf{Q}_N$ is an ordinary irrotational stress; so that

$$\left.\begin{array}{l} \mathbf{P}_N = \phi_0\mathbf{N} + \mathrm{V}\boldsymbol{\epsilon}\mathbf{N}, \\ \mathbf{Q}_N = \phi_0\mathbf{N} - \mathrm{V}\boldsymbol{\epsilon}\mathbf{N}, \end{array}\right\} \qquad (36)$$

where ϕ_0 is self-conjugate, and

$$2\boldsymbol{\epsilon} = \mathbf{i}(P_{23} - P_{32}) + \mathbf{j}(P_{31} - P_{13}) + \mathbf{k}(P_{12} - P_{21}). \qquad (37)$$

Here $2\boldsymbol{\epsilon}$ is the torque per unit volume arising from the stress **P**.

The translational force, **F**, per unit volume is (by inspection of a unit cube)

$$\mathbf{F} = \nabla_1\mathbf{P}_1 + \nabla_2\mathbf{P}_2 + \nabla_3\mathbf{P}_3 \qquad (38)$$

$$= \mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3; \qquad (39)$$

or, in terms of the self-conjugate stress and the torque,

$$\mathbf{F} = (\mathbf{i}\,\mathrm{div}\,\phi_0\mathbf{i} + \mathbf{j}\,\mathrm{div}\,\phi_0\mathbf{j} + \mathbf{k}\,\mathrm{div}\,\phi_0\mathbf{k}) - \mathrm{curl}\,\boldsymbol{\epsilon}, \qquad (40)$$

where $-\mathrm{curl}\,\boldsymbol{\epsilon}$ is the translational force due to the rotational stress alone, as in Sir W. THOMSON's latest theory of the mechanics of an "ether."*

Next, let **N** be the unit normal drawn outward from any closed surface. Then

$$\Sigma\mathbf{P}_N = \Sigma\mathbf{F}, \qquad (41)$$

where the left summation extends over the surface and the right summation throughout the enclosed region. For

$$\mathbf{P}_N = N_1\mathbf{P}_1 + N_2\mathbf{P}_2 + N_3\mathbf{P}_3$$
$$= \mathbf{i}.\mathbf{N}\mathbf{Q}_1 + \mathbf{j}.\mathbf{N}\mathbf{Q}_2 + \mathbf{k}.\mathbf{N}\mathbf{Q}_3; \qquad (42)$$

so the well-known theorem of divergence gives immediately, by (39),

$$\Sigma\mathbf{P}_N = \Sigma(\mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3) = \Sigma\mathbf{F}. \qquad (43)$$

Next, as regards the equivalence of rotational effect of the surface-stress to that of the internal forces and torques. Let **r** be the vector distance from any fixed origin. Then **VrF** is the vector moment of a force, **F**, at the end of the arm **r**. Another (not so immediate) application of the divergence theorem gives

$$\Sigma\mathbf{V}\mathbf{r}\mathbf{P}_N = \Sigma\mathbf{V}\mathbf{r}\mathbf{F} + \Sigma 2\boldsymbol{\epsilon}, \qquad (44)$$

Thus, any distribution of stress, whether rotational or irrotational, may be regarded as in equilibrium. Given any stress in a body, terminating at its boundary, the body will be in equilibrium both as regards translation and rotation. Of course, the boundary discontinuity in the stress has to be reckoned as the equivalent of internal divergence in the appropriate manner. Or, more simply, let the stress fall off continuously from the finite internal stress to zero through a thin surface-layer. We

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* 'Mathematical and Physical Papers,' vol. 3, Art. 99, p. 436.

then have a distribution of forces and torques in the surface-layer which equilibrate the internal forces and torques.

To illustrate; we know that MAXWELL arrived at a peculiar stress, compounded of a tension parallel to a certain direction, and an equal lateral pressure, which would account for the mechanical actions apparent between electrified bodies, and endeavoured similarly to determine the stress in the interior of a magnetised body to harmonise with the similar external magnetic stress of the simple type mentioned. This stress in a magnetised body I believe to be thoroughly erroneous; nevertheless, so far as accounting for the forcive on a magnetised body is concerned, it will, when properly carried out with due attention to surface-discontinuity, answer perfectly well, not because it *is* the stress, but because *any* stress would do the same, the only essential feature concerned being the external stress in the air.

Here we may also note the very powerful nature of the stress-function, considered merely as a mathematical engine, apart from physical reality. For example, we may account for the forcive on a magnet in many ways, of which the two most prominent are by means of forces on imaginary magnetic matter, and by forces on imaginary electric currents, in the magnet and on its surface. To prove the equivalence of these two methods (and the many others) involves very complex surface- and volume-integrations and transformations in the general case, which may be all avoided by the use of the stress-function instead of the forces.

§ **11.** Next as regards the activity of the stress $\mathbf{P}_N$ and the equivalent translational, distortional, and rotational activities. The activity of $\mathbf{P}_N$ is $\mathbf{P}_N\mathbf{q}$ per unit area, if $\mathbf{q}$ be the velocity. Here

$$\mathbf{P}_N\mathbf{q} = q_1.\mathbf{NQ}_1 + q_2.\mathbf{NQ}_2 + q_3.\mathbf{NQ}_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

by (42); or, re-arranging,

$$\mathbf{P}_N\mathbf{q} = \mathbf{N}\,(q_1\mathbf{Q}_1 + q_2\mathbf{Q}_2 + q_3\mathbf{Q}_3) = \mathbf{N}\Sigma\,\mathbf{Q}q,$$

$$= \mathbf{N}q\mathbf{Q}_q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where $\mathbf{Q}_q$ is the conjugate stress on the $\mathbf{q}$ plane. That is, $q\mathbf{Q}_q$ or $\Sigma\mathbf{Q}q$ is the negative of the vector flux of energy expressing the stress-activity. For we choose P_{NN} so as to mean a pull when it is positive, and when the stress $\mathbf{P}_N$ works in the same sense with $\mathbf{q}$ energy is transferred against the motion, to the matter which is pulled.

The convergence of the energy-flux, or the divergence of $q\mathbf{Q}_q$, is therefore the activity per unit volume. Thus

$$\text{div}\,(\mathbf{Q}_1q_1 + \mathbf{Q}_2q_2 + \mathbf{Q}_3q_3) = \mathbf{q}\,(\mathbf{i}\,\text{div}\,\mathbf{Q}_1 + \mathbf{j}\,\text{div}\,\mathbf{Q}_2 + \mathbf{k}\,\text{div}\,\mathbf{Q}_3) + (\mathbf{Q}_1\nabla q_1 + \mathbf{Q}_2\nabla q_2 + \mathbf{Q}_3\nabla q_3)\,. \quad (47)$$

$$= \mathbf{q}\,(\nabla_1\mathbf{P}_1 + \nabla_2\mathbf{P}_2 + \nabla_3\mathbf{P}_3) + \mathbf{P}_1\nabla_1\mathbf{q} + \mathbf{P}_2\nabla_2\mathbf{q} + \mathbf{P}_3\nabla_3\mathbf{q} \quad . \quad . \quad . \quad . \quad (48)$$

where the first form (47) is generally most useful. Or

$$\mathrm{div}\, \Sigma \mathbf{Q} q = \mathbf{F}\mathbf{q} + \Sigma \mathbf{Q} \nabla q ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

where the first term on the right is the translational activity, and the rest is the sum of the distortional and rotational activities. To separate the latter introduce the strain velocity vectors (analogous to $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$)

$$\mathbf{p}_1 = \tfrac{1}{2} (\nabla q_1 + \nabla_1 \mathbf{q}), \qquad \mathbf{p}_2 = \tfrac{1}{2} (\nabla q_2 + \nabla_2 \mathbf{q}), \qquad \mathbf{p}_3 = \tfrac{1}{2} (\nabla q_3 + \nabla_3 \mathbf{q}) ; \quad . \quad . \quad . \quad . \qquad (50)$$

and generally

$$\mathbf{p}_{\mathrm{N}} = \tfrac{1}{2} (\nabla . \mathbf{q} \mathbf{N} + \mathbf{N} \nabla . \mathbf{q}) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51)$$

Using these we obtain

$$\begin{aligned} \Sigma \mathbf{Q} \nabla \mathbf{q} &= \mathbf{Q}_1 \mathbf{p}_1 + \mathbf{Q}_2 \mathbf{p}_2 + \mathbf{Q}_3 \mathbf{p}_3 + \mathbf{Q}_1 \frac{\nabla q_1 - \nabla_1 \mathbf{q}}{2} + \mathbf{Q}_2 \frac{\nabla q_2 - \nabla_2 \mathbf{q}}{2} + \mathbf{Q}_3 \frac{\nabla q_3 - \nabla_3 \mathbf{q}}{2} \\ &= \Sigma \mathbf{Q} \mathbf{p} + \tfrac{1}{2} \mathbf{Q}_1 \mathrm{V} \mathbf{i} \,\mathrm{curl}\, \mathbf{q} + \tfrac{1}{2} \mathbf{Q}_2 \mathrm{V} \mathbf{j} \,\mathrm{curl}\, \mathbf{q} + \tfrac{1}{2} \mathbf{Q}_3 \mathrm{V} \mathbf{k} \,\mathrm{curl}\, \mathbf{q} \\ &= \Sigma \mathbf{Q} \mathbf{p} + \boldsymbol{\epsilon} \,\mathrm{curl}\, \mathbf{q} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52) \end{aligned}$$

Thus $\Sigma \mathbf{Q} \mathbf{p}$ is the distortional activity and $\boldsymbol{\epsilon}$ curl $\mathbf{q}$ the rotational activity. But since the distortion and the rotation are quite independent, we may put $\Sigma \mathbf{P} \mathbf{p}$ for the distortional activity ; or else use the self-conjugate stress, and write it $\frac{1}{2} \Sigma (\mathbf{P} + \mathbf{Q}) \mathbf{p}$.

§ **12.** In an ordinary "elastic solid," when isotropic, there is elastic resistance to compression and to distortion. We may also imaginably have elastic resistance to translation and to rotation ; nor is there, so far as the mathematics is concerned, any reason for excluding dissipative resistance to translation, distortion, and rotation ; and kinetic energy may be associated with all three as well, instead of with the translation alone, as in the ordinary elastic solid.

Considering only three elastic moduli, we have the old k and n of THOMSON and TAIT (resistance to compression and rigidity), and a new coefficient, say n_1, such that

$$\boldsymbol{\epsilon} = n_1 \,\mathrm{curl}\, \mathbf{D}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

if $\mathbf{D}$ be the displacement and $2\boldsymbol{\epsilon}$ the torque, as before.

The stress on the $\mathbf{i}$ plane (any plane) is

$$\begin{aligned} \mathbf{P}_1 &= n (\nabla \mathrm{D}_1 + \nabla_1 \mathbf{D}) + \mathbf{i} (k - \tfrac{2}{3} n) \,\mathrm{div}\, \mathbf{D} + n_1 \mathrm{V} \,\mathrm{curl}\, \mathbf{D} . \mathbf{i} \\ &= (n + n_1) \nabla_1 \mathbf{D} + (n - n_1) \nabla \mathrm{D}_1 + (k - \tfrac{2}{3} n) \mathbf{i} \,\mathrm{div}\, \mathbf{D} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54) \end{aligned}$$

and its conjugate is

$$\begin{aligned} \mathbf{Q}_1 &= n (\nabla \mathrm{D}_1 + \nabla_1 \mathbf{D}) + \mathbf{i} (k - \tfrac{2}{3} n) \,\mathrm{div}\, \mathbf{D} - n_1 (\nabla_1 \mathbf{D} - \nabla \mathrm{D}_1) \\ &= (n - n_1) \nabla_1 \mathbf{D} + (n + n_1) \nabla \mathrm{D}_1 + \mathbf{i} (k - \tfrac{2}{3} n) \,\mathrm{div}\, \mathbf{D} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55) \end{aligned}$$

from which

$$\mathrm{F}_1 = \mathrm{div}\, \mathbf{Q}_1 = (n - n_1 + k - \tfrac{2}{3} n) \nabla_1 \,\mathrm{div}\, \mathbf{D} + (n + n_1) \nabla^2 \mathrm{D}_1 \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

is the $\mathbf{i}$ component of the translational force ; the complete force $\mathbf{F}$ is therefore

$$\mathbf{F} = (n + n_1)\, \nabla^2\mathbf{D} + (k + \tfrac{1}{3}n - n_1)\, \nabla \operatorname{div} \mathbf{D}\,; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (57)$$

or, in another form, if

$$\mathrm{P} = -k \operatorname{div} \mathbf{D},$$

P being the isotropic pressure,

$$\mathbf{F} = -\nabla\mathrm{P} + n\,(\nabla^2\mathbf{D} + \tfrac{1}{3}\, \nabla \operatorname{div} \mathbf{D}) - n_1 \operatorname{curl}^2 \mathbf{D}, \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (58)$$

remembering (23) and (53).

We see that in (57) the term involving div **D** may vanish in a compressible solid by the relation $n_1 = k + \frac{1}{3}n$; this makes

$$n + n_1 = k + \tfrac{4}{3}n, \qquad n_1 - n = k - \tfrac{2}{3}n, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (59)$$

which are the moduli, longitudinal and lateral, of a simple longitudinal strain; that is, multiplied by the extension, they give the longitudinal traction, and the lateral traction required to prevent lateral contraction.

The activity per unit volume, other than translational, is

$$\begin{aligned}\Sigma\mathbf{Q}\nabla q &= (n - n_1)\,\big(\nabla_1\mathbf{D}.\nabla q_1 + \nabla_2\mathbf{D}.\nabla q_2 + \nabla_3\mathbf{D}.\nabla q_3\big)\\ &\quad + (n + n_1)\,\big(\nabla\mathrm{D}_1.\nabla q_1 + \nabla\mathrm{D}_2.\nabla q_2 + \nabla\mathrm{D}_3.\nabla q_3\big)\\ &\quad + (k - \tfrac{2}{3}n) \operatorname{div} \mathbf{D} \operatorname{div} \mathbf{q}\\ &= n\,\big(\nabla_1\mathbf{D}.\nabla q_1 + \nabla_2\mathbf{D}.\nabla q_2 + \nabla_3\mathbf{D}.\nabla q_3 + \nabla\mathrm{D}_1\nabla q_1 + \nabla\mathrm{D}_2\nabla q_2 + \nabla\mathrm{D}_3\nabla q_3\big)\\ &\quad + (k - \tfrac{2}{3}n) \operatorname{div} \mathbf{D} \operatorname{div} \mathbf{q} + n_1 \operatorname{curl} \mathbf{D} \operatorname{curl} \mathbf{q}\,; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (60)\end{aligned}$$

or, which is the same,

$$\begin{aligned}\Sigma\mathbf{Q}\nabla q = \frac{d}{dt}\big[&\tfrac{1}{2}k\,(\operatorname{div}\mathbf{D})^2 + \tfrac{1}{2}n_1\,(\operatorname{curl}\mathbf{D})^2\\ &+ \tfrac{1}{2}n\{(\nabla\mathrm{D}_1)^2 + (\nabla\mathrm{D}_2)^2 + (\nabla\mathrm{D}_3)^2 + \nabla\mathrm{D}_1.\nabla_1\mathbf{D} + \nabla\mathrm{D}_2.\nabla_2\mathbf{D} + \nabla\mathrm{D}_3.\nabla_3\mathbf{D}\} - \tfrac{1}{3}n\,(\operatorname{div}\mathbf{D})^2\big], \quad . \qquad (61)\end{aligned}$$

where the quantity in square brackets is the potential energy of an *infinitesimal* distortion and rotation. The italicised reservation appears to be necessary, as we shall see from the equation of activity later, that the convection of the potential energy destroys the completeness of the statement

$$\Sigma\mathbf{Q}\nabla q = \dot{\mathrm{U}},$$

if U be the potential energy.

In an elastic solid of the ordinary kind, with $n_1 = 0$, we have

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= n\,(2 \operatorname{curl} \mathrm{V}\mathbf{D}\mathbf{N} + \mathrm{V}\mathbf{N} \operatorname{curl} \mathbf{D}),\\ \mathbf{F} &= -n \operatorname{curl}^2 \mathbf{D}.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (62)$$

In the case of a medium in which n is zero but n_1 finite (Sir W. THOMSON'S rotational ether),

$$\left.\begin{aligned} \mathbf{P}_{\mathrm{N}} &= n_1 \mathrm{V}\,\mathrm{curl}\,\mathbf{D}.\mathbf{N}, \\ \mathbf{F} &= -n_1\,\mathrm{curl}^2\,\mathbf{D}. \end{aligned}\right\} \qquad (63)$$

Thirdly, if we have both $k = -\frac{4}{3}n$ and $n = n_1$, then

$$\left.\begin{aligned} \mathbf{P}_{\mathrm{N}} &= 2n\,\mathrm{curl}\,\mathrm{V}\mathbf{D}\mathbf{N}, \\ \mathbf{F} &= -2n\,\mathrm{curl}^2\,\mathbf{D}, \end{aligned}\right\} (\mathbf{E} = n\,\mathrm{curl}\,\mathbf{D}), \qquad (64)$$

i.e., the sums of the previous two stresses and forces.

§ **13.** As already observed, the vector flux of energy, due to the stress, is

$$-\Sigma\mathbf{Q}q = -\mathbf{Q}_q q = -(\mathbf{Q}_1 q_1 + \mathbf{Q}_2 q_2 + \mathbf{Q}_3 q_3). \qquad (65)$$

Besides this, there is the flux of energy

$$\mathbf{q}\,(\mathrm{U} + \mathrm{T})$$

by convection, where U is potential and T kinetic energy. Therefore,

$$\mathbf{W} = \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q \qquad (66)$$

represents the complete energy flux, so far as the stress and motion are concerned. Its convergence increases the potential energy, the kinetic energy, or is dissipated. But if there be an impressed translational force **f** its activity is $\mathbf{fq}$. This supply of energy is independent of the convergence of **W**. Hence

$$\mathbf{fq} + \mathrm{Q} + \dot{\mathrm{U}} = \dot{\mathrm{T}} + \mathrm{div}\left[\mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q\right] \qquad (67)$$

is the equation of activity.

But this splits into two parts at least. For (67) is the same as

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} + \Sigma\mathbf{Q}\nabla q = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\,(\mathrm{U} + \mathrm{T}), \qquad (68)$$

and the translational portion may be removed altogether. That is,

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} = \mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0 + \mathrm{div}\,\mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0), \qquad (69)$$

if the quantities with the zero suffix are only translationally involved. For example, if

$$\mathbf{f} + \mathbf{F} = \rho\frac{\partial\mathbf{q}}{\partial t}, \qquad (70)$$

as in fluid motion, without frictional or elastic forces associated with the translation, then

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} = \rho\mathbf{q}\frac{\partial\mathbf{q}}{\partial t} = \dot{\mathrm{T}} + \operatorname{div}\mathbf{q}\mathrm{T}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

if $\mathrm{T} = \frac{1}{2}\rho q^2$, the kinetic energy per unit volume. The complete form (69) comes in by the addition of elastic and frictional resisting forces. So deducting (69) from (68) there is left

$$\Sigma\mathbf{Q}\nabla q = \mathrm{Q}_1 + \dot{\mathrm{U}}_1 + \dot{\mathrm{T}}_1 + \operatorname{div}\mathbf{q}\,(\mathrm{U}_1 + \mathrm{T}_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

where the quantities with suffix unity are connected with the distortion and the rotation, and there may plainly be two sets of dissipative terms, and of energy (stored) terms. Thus the relation

$$\boldsymbol{\epsilon} = \left(n_1 + n_2\frac{d}{dt} + n_3\frac{d^2}{dt^2}\right)\operatorname{curl}\mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

will bring in dissipation and kinetic energy, as well as the former potential energy of rotation associated with n_1.

That there can be dissipative terms associated with the distortion is also clear enough, remembering STOKES's theory of a viscous fluid. Thus, for simplicity, do away with the rotating stress, by putting $\boldsymbol{\epsilon} = 0$, making $\mathbf{P}_{\mathrm{N}}$ and $\mathbf{Q}_{\mathrm{N}}$ identical. Then take the stress on the **i** plane to be given by

$$\mathbf{P}_1 = \left(n + \mu\frac{d}{dt} + \nu\frac{d^2}{dt^2}\right)(\nabla\mathrm{D}_1 + \nabla_1\mathbf{D}) - \mathbf{i}\left\{\mathrm{P} + \frac{2}{3}\left(n + \mu\frac{d}{dt} + \nu\frac{d^2}{dt^2}\right)\operatorname{div}\mathbf{D}\right\}, \quad . \quad . \quad (74)$$

and similarly for any other plane; where $\mathrm{P} = -k\operatorname{div}\mathbf{D}$.

When $\mu = 0$, $\nu = 0$, we have the elastic solid with rigidity and compressibility. When $n = 0$, $\nu = 0$, we have the viscous fluid of STOKES. When $\nu = 0$ only, we have a viscous elastic solid, the viscous resistance being purely distortional, and proportional to the speed of distortion. But with n, μ, ν, all finite, we still further associate kinetic energy with the potential energy and dissipation introduced by n and μ.

We have

$$\Sigma\mathbf{P}\nabla q = \mathrm{Q}_2 + \dot{\mathrm{U}}_2 + \dot{\mathrm{T}}_2$$

for infinitesimal strains, omitting the effect of convection of energy; where

$$\mathrm{T}_2 = \tfrac{1}{2}\nu\left[-\tfrac{2}{3}(\operatorname{div}\mathbf{q})^2 + \nabla q_1(\nabla q_1 + \nabla_1\mathbf{q}) + \nabla q_2(\nabla q_2 + \nabla_2\mathbf{q}) + \nabla q_3(\nabla q_3 + \nabla_3\mathbf{q})\right], \quad . \quad . \quad . \quad . \quad (75)$$

$$\mathrm{Q}_2 = \mu\left[-\tfrac{2}{3}(\operatorname{div}\mathbf{q})^2 + \nabla q_1(\nabla q_1 + \nabla_1\mathbf{q}) + \nabla q_2(\nabla q_2 + \nabla_2\mathbf{q}) + \nabla q_3(\nabla q_3 + \nabla_3\mathbf{q})\right], \quad . \quad . \quad . \quad . \quad (76)$$

$$\mathrm{U}_2 = \tfrac{1}{2}n\left[\left(\frac{k}{n} - \tfrac{2}{3}\right)(\operatorname{div}\mathbf{D})^2 + \nabla\mathrm{D}_1(\nabla\mathrm{D}_1 + \nabla_1\mathbf{D}) + \nabla\mathrm{D}_2(\nabla\mathrm{D}_2 + \nabla_2\mathbf{D}) + \nabla\mathrm{D}_3(\nabla\mathrm{D}_3 + \nabla_3\mathbf{D})\right]. \quad (77)$$

Observe that T_2 and Q_2 only differ in the exchange of μ to $\frac{1}{2}\nu$; but U_2, the potential energy, is not the same function of n and **D** that T_2 is of ν and **q**. But if we take $k = 0$, we produce similarity. An elastic solid having no resistance to compression is also one of Sir W. THOMSON'S ethers.

When $n = 0$, $\mu = 0$, $\nu = 0$, we come down to the frictionless fluid, in which

$$\mathbf{f} - \nabla \mathrm{P} = \rho \frac{\partial \mathbf{q}}{\partial t}, \qquad (78)$$

and

$$\Sigma \mathrm{P} \nabla q = - \mathrm{P} \operatorname{div} \mathbf{q}, \qquad (79)$$

with the equation of activity

$$\mathbf{fq} = \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} (\mathrm{U} + \mathrm{T} + \mathrm{P})\, \mathbf{q}, \qquad (80)$$

the only parts of which are not always easy to interpret are the Pq term, and the proper measure of U. By analogy, and conformably with more general cases, we should take

$$\mathrm{P} = - k \operatorname{div} \mathbf{D}, \qquad \text{and} \qquad \mathrm{U} = \tfrac{1}{2} k (\operatorname{div} \mathbf{D})^2,$$

reckoning the expansion or compression from some mean condition.

The Electromagnetic Equations in a Moving Medium.

§ **14.** The study of the forms of the equation of activity in purely mechanical cases, and the interpretation of the same is useful, because in the electromagnetic problem of a moving medium we have still greater generality, and difficulty of safe and sure interpretation. To bring it as near to abstract dynamics as possible, all we need say regarding the two fluxes, electric displacement **D** and magnetic induction **B**, is that they are linear functions of the electric force **E** and magnetic force **H**, say

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = c\mathbf{E}, \qquad (81)$$

where c and μ are linear operators of the symmetrical kind, and that associated with them are the stored energies U and T, electric and magnetic respectively (per unit volume), given by

$$\mathrm{U} = \tfrac{1}{2}\mathbf{ED}, \qquad \mathrm{T} = \tfrac{1}{2}\mathbf{HB}, \qquad (82)$$

In isotropic media c is the permittivity, μ the inductivity. It is unnecessary to say more regarding the well-known variability of μ and hysteresis than that a magnet is here an ideal magnet of constant inductivity.

As there may be impressed forces, **E** is divisible into the force of the field and an impressed part; for distinctness, then, the complete **E** may be called the "force of the flux" **D**. Similarly as regards **H** and **B**.

There is also waste of energy (in conductors, namely) at the rates

$$\mathbf{Q}_1 = \mathbf{EC}; \qquad \mathbf{Q}_2 = \mathbf{HK}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

where the fluxes **C** and **K** are also linear functions of **E** and **H** respectively; thus

$$\mathbf{C} = k\mathbf{E}, \qquad \mathbf{K} = g\mathbf{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

where, when the force is parallel to the flux, and k is scalar, it is the electric conductivity. Its magnetic analogue is g, the magnetic conductivity. That is, a magnetic conductor is a (fictitious) body which cannot support magnetic force without continuously dissipating energy.

Electrification is the divergence of the displacement, and its analogue, magnetification, is the divergence of the induction; thus

$$\rho = \operatorname{div} \mathbf{D}, \qquad \sigma = \operatorname{div} \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

are their volume densities. The quantity σ is probably quite fictitious, like **K**.

According to MAXWELL'S doctrine, the true electric current is always circuital, and is the sum of the conduction current and the current of displacement, which is the time rate of increase of the displacement. But, to preserve circuitality, we must add the convection current when electrification is moving, so that the true current becomes

$$\mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

where **q** is the velocity of the electrification ρ. Similarly

$$\mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

should be the corresponding magnetic current.

§ **15.** MAXWELL'S equation of electric current in terms of magnetic force in a medium at rest, say,

$$\operatorname{curl} \mathbf{H}_1 = \mathbf{C} + \dot{\mathbf{D}},$$

where $\mathbf{H}_1$ is the force of the field, should be made, using **H** instead,

$$\operatorname{curl} (\mathbf{H} - \mathbf{h}_0) = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho,$$

and here $\mathbf{h}_0$ will be the force of intrinsic magnetisation, such that $\mu\mathbf{h}_0$ is the intensity of intrinsic magnetisation. But I have shown that when there is motion, another impressed term is required, viz., the motional magnetic force

$$\mathbf{h} = \mathrm{V}\mathbf{D}\mathbf{q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

making the first circuital law become

$$\operatorname{curl}(\mathbf{H}-\mathbf{h}_0-\mathbf{h})=\mathbf{J}=\mathbf{C}+\dot{\mathbf{D}}+\mathbf{q}\rho. \qquad (89)$$

MAXWELL's other connection to form the equations of propagation is made through his vector-potential **A** and scalar potential Ψ. Finding this method not practically workable, and also not sufficiently general, I have introduced instead a companion equation to (89) in the form

$$-\operatorname{curl}(\mathbf{E}-\mathbf{e}_0-\mathbf{e})=\mathbf{G}=\mathbf{K}+\dot{\mathbf{B}}+\mathbf{q}\sigma, \qquad (90)$$

where $\mathbf{e}_0$ expresses intrinsic force, and $\mathbf{e}$ is the motional electric force given by

$$\mathbf{e}=\mathrm{V}\mathbf{q}\mathbf{B}, \qquad (91)$$

which is one of the terms in MAXWELL's equation of electromotive force. As for $\mathbf{e}_0$, it includes not merely the force of intrinsic electrification, the analogue of intrinsic magnetisation, but also the sources of energy, voltaic force, thermoelectric force, &c.

(89) and (90) are thus the working equations, with (88) and (91) in case the medium moves; along with the linear relations before mentioned, and the definitions of energy and waste of energy per unit volume. The fictitious **K** and σ are useful in symmetrizing the equations, if for no other purpose.

Another way of writing the two equations of curl is by removing the **e** and **h** terms to the right side. Let

$$\left.\begin{aligned}\operatorname{curl}\mathbf{h}&=\mathbf{j}, & \mathbf{J}+\mathbf{j}&=\mathbf{J}_0,\\ -\operatorname{curl}\mathbf{e}&=\mathbf{g}, & \mathbf{G}+\mathbf{g}&=\mathbf{G}_0.\end{aligned}\right\} \qquad (92)$$

Then (89) and (90) may be written

$$\left.\begin{aligned}\operatorname{curl}(\mathbf{H}-\mathbf{h}_0)&=\mathbf{J}_0=\mathbf{C}+\dot{\mathbf{D}}+\mathbf{q}\rho+\mathbf{j},\\ -\operatorname{curl}(\mathbf{E}-\mathbf{e}_0)&=\mathbf{G}_0=\mathbf{K}+\dot{\mathbf{B}}+\mathbf{q}\sigma+\mathbf{g}.\end{aligned}\right\} \qquad (93)$$

So far as circuitality of the current goes, the change is needless, and still further complicates the make-up of the true current, supposed now to be $\mathbf{J}_0$. On the other hand, it is a simplification on the left side, deriving the current from the force of the flux or of the field more simply.

A question to be settled is whether **J** or $\mathbf{J}_0$ should be the true current. There seems only one crucial test, viz., to find whether $\mathbf{e}_0\mathbf{J}$ or $\mathbf{e}_0\mathbf{J}_0$ is the rate of supply of energy to the electromagnetic system by an intrinsic force $\mathbf{e}_0$. This requires, however, a full and rigorous examination of all the fluxes of energy concerned.

The Electromagnetic Flux of Energy in a stationary Medium.

§ **16.** First let the medium be at rest, giving us the equations

$$\text{curl}\,(\mathbf{H}-\mathbf{h}_0)=\mathbf{J}=\mathbf{C}+\dot{\mathbf{D}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (94)$$

$$-\,\text{curl}\,(\mathbf{E}-\mathbf{e}_0)=\mathbf{G}=\mathbf{H}+\dot{\mathbf{B}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (95)$$

Multiply (94) by $(\mathbf{E}-\mathbf{e}_0)$, and (95) by $(\mathbf{H}-\mathbf{h}_0)$, and add the results. Thus,

$$(\mathbf{E}-\mathbf{e}_0)\,\mathbf{J}+(\mathbf{H}-\mathbf{h}_0)\,\mathbf{G}=(\mathbf{E}-\mathbf{e}_0)\,\text{curl}\,(\mathbf{H}-\mathbf{h}_0)-(\mathbf{H}-\mathbf{h}_0)\,\text{curl}\,(\mathbf{E}-\mathbf{e}_0),$$

which, by the formula (25), becomes

$$\mathbf{e}_0\mathbf{J}+\mathbf{h}_0\mathbf{G}=\mathbf{EJ}+\mathbf{HG}+\text{div}\,\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0)\,;$$

or, by the use of (82), (83),

$$\mathbf{e}_0\mathbf{J}+\mathbf{h}_0\mathbf{G}=\mathrm{Q}+\dot{\mathrm{U}}+\dot{\mathrm{T}}+\text{div}\,\mathbf{W}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (96)$$

where the new vector **W** is given by

$$\mathbf{W}=\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (97)$$

The form of (96) is quite explicit, and the interpretation sufficiently clear. The left side indicates the rate of supply of energy from intrinsic sources. These $(\mathrm{Q}+\dot{\mathrm{U}}+\dot{\mathrm{T}})$ shows the rate of waste and of storage of energy in this unit volume. The remainder, therefore, indicates the rate at which energy is passed out from the unit volume; and the flux **W** represents the flux of energy necessitated by the postulated localisation of energy and its waste, when **E** and **H** are connected in the manner shown by (94) and (95).

There might also be an independent circuital flux of energy, but, being useless, it would be superfluous to bring it in.

The very important formula (97) was first discovered and interpreted by Professor POYNTING, and independently discovered and interpreted a little later by myself in an extended form. It will be observed that in my mode of proof above there is no limitation as to homogeneity or isotropy as regards the permittivity, inductivity, and conductivity. But c and μ should be symmetrical. On the other hand, k and g do not require this limitation in deducing (97).*

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* The method of treating MAXWELL'S electromagnetic scheme employed in the text (first introduced in "Electromagnetic Induction and its Propagation," 'The Electrician,' January 3, 1885, and later)

It is important to recognize that this flux of energy is not dependent upon the translational motion of the medium, for it is assumed explicitly to be at rest. The vector **W** cannot, therefore, be a flux of the kind $\mathbf{Q}_q q$ before discussed, unless possibly it be merely a rotating stress that is concerned.

The only dynamical analogy with which I am acquainted which seems at all satisfactory is that furnished by Sir W. THOMSON's theory of a rotational ether. Take the case of $\mathbf{e}_0 = 0$, $\mathbf{h}_0 = 0$, $k = 0$, $g = 0$, and c and μ constants, that is, pure ether uncontaminated by ordinary matter. Then

$$\text{curl}\,\mathbf{H} = c\dot{\mathbf{E}}, \quad \ldots \ldots \ldots \ldots \quad (98)$$

$$-\,\text{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}}. \quad \ldots \ldots \ldots \ldots \quad (99)$$

Now, let **H** be velocity, μ density; then, by (99), $-$ curl **E** is the translational force due to the stress, which is, therefore, a rotating stress; thus,

$$\mathbf{P}_N = \mathbf{VEN}, \qquad \mathbf{Q}_N = \mathbf{VNE}; \quad \ldots \ldots \ldots \quad (100)$$

and $2\mathbf{E}$ is the torque. The coefficient c represents the compliancy or reciprocal of the quasi-rigidity. The kinetic energy $\frac{1}{2}\mu H^2$ represents the magnetic energy, and the potential energy of the rotation represents the electric energy; whilst the flux of energy is **VEH**. For the activity of the torque is

$$2\mathbf{E}.\frac{\text{curl}\,\mathbf{H}}{2} = \mathbf{E}\,\text{curl}\,\mathbf{H},$$

and the translational activity is

$$-\,\mathbf{H}\,\text{curl}\,\mathbf{E}.$$

Their sum is

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may, perhaps, be appropriately termed the Duplex method, since its characteristics are the exhibition of the electric, magnetic, and electromagnetic relations in a duplex form, symmetrical with respect to the electric and magnetic sides. But it is not merely a method of exhibiting the relations in a manner suitable to the subject, bringing to light useful relations which were formerly hidden from view by the intervention of the vector-potential and its parasites, but constitutes a method of working as well. There are considerable difficulties in the way of the practical employment of MAXWELL's equations of propagation, even as they stand in his treatise. These difficulties are greatly magnified when we proceed to more general cases, involving heterogeneity and eolotropy and motion of the medium supporting the fluxes. The duplex method supplies what is wanted. Potentials do not appear, at least initially. They are regarded strictly as auxiliary functions which do not represent any physical state of the medium. In special problems they may be of great service for calculating purposes; but in general investigations their avoidance simplifies matters greatly. The state of the field is settled by **E** and **H**, and these are the primary objects of attention in the duplex system.

As the papers to which I have referred are not readily accessible, I may take this opportunity of mentioning that a Reprint of my 'Electrical Papers' is in the press (MACMILLAN and Co.), and that the first volume is nearly ready.

$$-\operatorname{div} \mathbf{VEH},$$

making **VEH** the flux of energy.*

All attempts to construct an elastic solid analogy with a distortional stress fail to give satisfactory results, because the energy is wrongly localised, and the flux of energy incorrect. Bearing this in mind, the above analogy is at first sight very enticing. But when we come to remember that the d/dt in (98) and (99) should be $\partial/\partial t$, and find extraordinary difficulty in extending the analogy to include the conduction current, and also remember that the electromagnetic stress has to be accounted for (in other words, the known mechanical forces), the perfection of the analogy, as far as it goes, becomes disheartening. It would further seem, from the explicit assumption that $\mathbf{q} = 0$ in obtaining **W** above, that no analogy of this kind can be sufficiently comprehensive to form the basis of a physical theory. We must go altogether beyond the elastic solid with the additional property of rotational elasticity. I should mention, to avoid misconception, that Sir W. THOMSON does not push the analogy even so far as is done above, or give to μ and c the same interpretation. The particular meaning here given to μ is that assumed by Professor LODGE in his "Modern Views of Electricity," on the ordinary elastic solid theory, however. I have found it very convenient from its making the curl of the electric force be a Newtonian force (per unit volume). When impressed electric force $\mathbf{e}_0$ produces disturbances, their real source is, as I have shown, not the seat of $\mathbf{e}_0$, but of curl $\mathbf{e}_0$. So we may with facility translate problems in electromagnetic waves into elastic solid problems by taking the electromagnetic source to represent the mechanical source of motion, impressed Newtonian force.

Examination of the Flux of Energy in a moving Medium, and Establishment of the Measure of "True" Current.

§ **17.** Now pass to the more general case of a moving medium with the equations

$$\operatorname{curl} \mathbf{H}_1 = \operatorname{curl} (\mathbf{H} - \mathbf{h}_0 - \mathbf{h}) = \mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

$$-\operatorname{curl} \mathbf{E}_1 = -\operatorname{curl} (\mathbf{E} - \mathbf{e}_0 - \mathbf{e}) = \mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

where $\mathbf{E}_1$ is, for brevity, what the force **E** of the flux becomes after deducting the intrinsic and motional forces; and similarly for $\mathbf{H}_1$.

From these, in the same way as before, we deduce

$$(\mathbf{e}_0 + \mathbf{e})\,\mathbf{J} + (\mathbf{h}_0 + \mathbf{h})\,\mathbf{G} = \mathbf{EJ} + \mathbf{HG} + \operatorname{div} \mathbf{VE}_1\mathbf{H}_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

and it would seem at first sight to be the same case again, but with impressed forces

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* This form of application of the rotating ether I gave in 'The Electrician,' January 23, 1891, p. 360.

$(\mathbf{e} + \mathbf{e}_0)$ and $(\mathbf{h} + \mathbf{h}_0)$ instead of $\mathbf{e}_0$ and $\mathbf{h}_0$, whilst the POYNTING flux requires us to reckon only $\mathbf{E}_1$ and $\mathbf{H}_1$ as the effective electric and magnetic forces concerned in it.*

But we must devolop $(\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}})$ plainly first. We have, by (86), (87), used in (103),

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = \mathbf{E}\,(\mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho) + \mathbf{H}\,(\mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma) - (\mathbf{eJ} + \mathbf{hG}) + \operatorname{div} \mathrm{V}\mathbf{E}_1\mathbf{H}_1. \quad . \quad . \quad (104)$$

Now here we have

$$\left.\begin{aligned} \dot{\mathrm{U}} = \frac{d}{dt}\tfrac{1}{2}\mathbf{ED} &= \tfrac{1}{2}\mathbf{E}\dot{\mathbf{D}} + \tfrac{1}{2}\mathbf{D}\dot{\mathbf{E}} \\ &= \mathbf{E}\dot{\mathbf{D}} + \tfrac{1}{2}(\mathbf{D}\dot{\mathbf{E}} - \mathbf{E}\dot{\mathbf{D}}) \\ &= \mathbf{E}\dot{\mathbf{D}} - \tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} \\ &= \mathbf{E}\dot{\mathbf{D}} - \dot{\mathrm{U}}_c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

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* It will be observed that the constant 4π, which usually appears in the electrical equations, is absent from the above investigations. This demands a few words of explanation. The units employed in the text are rational units, founded upon the principle of continuity in space of vector functions, and the corresponding appropriate measure of discontinuity, viz., by the amount of divergence. In popular language, the *unit* pole sends out *one* line of force, in the rational system, instead of 4π lines, as in the irrational system. The effect of the rationalisation is to introduce 4π into the formulæ of central forces and potentials, and to abolish the swarm of 4π's that appear in the practical formulæ of the practice of theory on FARADAY-MAXWELL lines, which receives its fullest and most appropriate expression in the rational method. The rational system was explained by me in 'The Electrician,' in 1882, and applied to the general theory of potentials and connected functions in 1883. (Reprint, vol. 1, p. 199, and later, especially p. 262.) I then returned to irrational formulæ because I did not think, then, that a reform of the units was practicable, partly on account of the labours of the B. A. Committee on Electrical Units, and partly on account of the ignorance of, and indifference to, theoretical matters which prevailed at that time. But the circumstances have greatly changed, and I do think a change is now practicable. There has been great advance in the knowledge of the meaning of MAXWELL's theory, and a diffusion of this knowledge, not merely amongst scientific men, but amongst a large body of practicians called into existence by the extension of the practical applications of electricity. Electricity is becoming, not only a master science, but also a very practical one. It is fitting, therefore, that learned traditions should not be allowed to control matters too greatly, and that the units should be rationalised. To make a beginning, I am employing rational units throughout in my work on "Electromagnetic Theory," commenced in 'The Electrician,' in January, 1891, and continued as fast as circumstances will permit; to be republished in book form. In Section XVII. (October 16, 1891, p. 655), will be found stated more fully the nature of the change proposed, and the reasons for it. I point out, in conclusion, that as regards theoretical treatises and investigations, there is no difficulty in the way, since the connection of the rational and irrational units may be explained separately; and I express the belief that when the merits of the rational system are fully recognised, there will arise a demand for the rationalisation of the practical units. We are, in the opinion of men qualified to judge, within a measurable distance of adopting the metric system in England. Surely the smaller reform I advocate should precede this. To put the matter plainly, the present system of units contains an absurdity running all through it of the same nature as would exist in the metric system of common units were we to define the unit area to be the area of a circle of unit diameter. The absurdity is only different in being less obvious in the electrical case. It would not matter much if it were not that electricity is a practical science.

Comparison of the third with the second form of (105) defines the generalised meaning of $\dot{c}$ when c is not a mere scalar. Or thus,

$$\begin{aligned}\dot{\mathrm{U}}_c &= \mathbf{E}^2\dot{c} = \tfrac{1}{2}\frac{d}{dt}(\mathbf{ED})_c \\ &= \tfrac{1}{2}\dot{c}_{11}\mathrm{E}_1{}^2 + \tfrac{1}{2}\dot{c}_{22}\mathrm{E}_2{}^2 + \tfrac{1}{2}\dot{c}_{33}\mathrm{E}_3{}^2 + \dot{c}_{12}\mathrm{E}_1\mathrm{E}_2 + \dot{c}_{23}\mathrm{E}_2\mathrm{E}_3 + \dot{c}_{31}\mathrm{E}_1\mathrm{E}_3, \quad \ldots\ldots \quad (106)\end{aligned}$$

representing the time-variation of U due to variation in the c's only.

Similarly

$$\dot{\mathrm{T}} = \mathbf{H}\dot{\mathbf{B}} - \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H} = \mathbf{H}\dot{\mathbf{B}} - \dot{\mathrm{T}}_\mu, \quad \ldots\ldots \quad (107)$$

with the equivalent meaning for $\dot{\mu}$ generalised.

Using these in (104) we have the result

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{q}\,(\mathbf{E}\rho + \mathbf{H}\sigma) + (\tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} + \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H}) - (\mathbf{eJ} + \mathbf{hG}) + \mathrm{div}\ \mathrm{V}\mathbf{E}_1\mathbf{H}_1. \quad . \quad (108)$$

Here we have, besides $(\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}})$, terms indicating the activity of a translational force. Thus $\mathbf{E}\rho$ is the force on electrification ρ, and $\mathbf{Eq}\rho$ its activity. Again,

$$\frac{\partial c}{\partial t} = \dot{c} + \mathbf{q}\nabla.c;$$

so that we have

$$\dot{c} = \frac{\partial c}{\partial t} - \mathbf{q}\nabla.c,$$

and, similarly,

$$\left.\begin{aligned}\dot{c} &= \frac{\partial c}{\partial t} - \mathbf{q}\nabla.c, \\ \dot{\mu} &= \frac{\partial \mu}{\partial t} - \mathbf{q}\nabla.\mu,\end{aligned}\right\} \quad \ldots\ldots \quad (109)$$

the generalised meaning of which is indicated by

$$-\frac{\partial \mathrm{U}_c}{\partial t} + \tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} = -\tfrac{1}{2}\mathbf{E}\,(\mathbf{q}\nabla\,.\,c)\,\mathbf{E} = -\mathbf{q}\nabla\mathrm{U}_c; \quad \ldots\ldots \quad (110)$$

where, in terms of scalar products involving $\mathbf{E}$ and $\mathbf{D}$,

$$-\mathbf{q}\nabla\mathrm{U}_c = -\tfrac{1}{2}(\mathbf{E}.\mathbf{q}\nabla.\mathbf{D} - \mathbf{D}.\mathbf{q}\nabla.\mathbf{E}).. \quad \ldots\ldots \quad (111)$$

This is also the activity of a translational force. Similarly,

$$-\frac{\partial \mathrm{T}_\mu}{\partial t} + \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H} = -\mathbf{q}\nabla\mathrm{T}_\mu \quad \ldots\ldots \quad (112)$$

is the activity of a translational force. Then again

$$-(\mathbf{eJ} + \mathbf{hG}) = -\mathbf{J}\mathrm{V}\mathbf{qB} - \mathbf{G}\mathrm{V}\mathbf{Dq} = \mathbf{q}\,(\mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG}) \quad \ldots\ldots \quad (113)$$

expresses a translational activity. Using them all in (108) it becomes

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{q}\left(\mathbf{E}\rho + \mathbf{H}\sigma - \nabla\mathrm{U}_c - \nabla\mathrm{T}_\mu + \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG}\right) + \operatorname{div}\mathrm{V}\mathbf{E}_1\mathbf{H}_1 + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu). \quad . \quad . \quad . \quad . \quad (114)$$

It is clear that we should make the factor of $\mathbf{q}$ be the complete translational force. But that has to be found; and it is equally clear that, although we appear to have exhausted all the terms at disposal, the factor of $\mathbf{q}$ in (114) is not the complete force, because there is no term by which the force on intrinsically magnetised or electrized matter could be exhibited. These involve $\mathbf{e}_0$ and $\mathbf{h}_0$. But as we have

$$\mathbf{q}\,(\mathrm{V}\mathbf{j}_0\mathbf{B} + \mathrm{V}\mathbf{D}\mathbf{g}_0) = -(\mathbf{e}\mathbf{j}_0 + \mathbf{h}\mathbf{g}_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

a possible way of bringing them in is to add the left member and subtract the right member of (115) from the right member of (114); bringing the translational force to $\mathbf{f}$, say, where

$$\mathbf{f} = \mathbf{E}\rho + \mathbf{H}\sigma - \nabla\mathrm{U}_c - \nabla\mathrm{T} + \mathrm{V}(\mathbf{J} + \mathbf{j}_0)\,\mathbf{B} + \mathrm{V}\,(\mathbf{G} + \mathbf{g}_0)\,\mathbf{D}. \quad . \quad . \quad . \quad . \quad . \quad (116)$$

But there is still the right number of (115) to be accounted for. We have

$$-\operatorname{div}(\mathrm{V}\mathbf{e}\mathbf{h}_0 + \mathrm{V}\mathbf{e}_0\mathbf{h}) = \mathbf{e}\mathbf{j}_0 + \mathbf{h}\mathbf{g}_0 + \mathbf{e}_0\mathbf{j} + \mathbf{h}_0\mathbf{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117)$$

and, by using this in (114), through (115), (116), (117), we bring it to

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{f}\mathbf{q} - (\mathbf{e}_0\mathbf{j} + \mathbf{h}_0\mathbf{g}) + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu); \quad (118)$$

or, transferring the $\mathbf{e}_0$, $\mathbf{h}_0$ terms from the right to the left side,

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{f}\mathbf{q} + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \quad . \quad . \quad (119)$$

Here we see that we have *a* correct form of activity equation, though it may not be *the* correct form. Another form, equally probable, is to be obtained by bringing in $\mathrm{V}\mathbf{e}\mathbf{h}$; thus

$$\operatorname{div}\mathrm{V}\mathbf{e}\mathbf{h} = \mathbf{h}\operatorname{curl}\mathbf{e} - \mathbf{e}\operatorname{curl}\mathbf{h} = -(\mathbf{e}\mathbf{j} + \mathbf{h}\mathbf{g}) = \mathbf{q}\,(\mathrm{V}\mathbf{j}\mathbf{B} + \mathrm{V}\mathbf{D}\mathbf{g}), \quad . \quad . \quad . \quad . \quad (120)$$

which converts (119) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{F}\mathbf{q} + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h} - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \quad . \quad (121)$$

where $\mathbf{F}$ is the translational force

$$\mathbf{F} = \mathbf{E}\rho + \mathbf{H}\sigma - \nabla\mathrm{U}_c - \nabla\mathrm{T}_\mu + \mathrm{V}\operatorname{curl}\mathbf{H}.\mathbf{B} + \mathrm{V}\operatorname{curl}\mathbf{E}.\mathbf{D}, \quad . \quad . \quad . \quad . \quad (122)$$

which is perfectly symmetrical as regards $\mathbf{E}$ and $\mathbf{H}$, and in the vector products utilises the fluxes and their complete forces, whereas former forms did this only partially. Observe, too, that we have only been able to bring the activity equation to a correct form (either (119) or (122)) by making $\mathbf{e}_0\mathbf{J}_0$ be the activity of intrinsic force $\mathbf{e}_0$, which requires that $\mathbf{J}_0$ should be the true electric current, according to the energy criterion, not $\mathbf{J}$.

§ **18.** Now, to test (119) and (121), we must interpret the flux in (121), or say

$$\mathbf{Y} = \mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h} - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (123)$$

which has replaced the Poynting flux $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$ when $\mathbf{q} = 0$, along with the other changes. Since $\mathbf{Y}$ reduces to $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$ when $\mathbf{q} = 0$, there must still be a Poynting flux when $\mathbf{q}$ is finite, though we do not know its precise form of expression. There is also the stress flux of energy and the flux of energy by convection, making a total flux

$$\mathbf{X} = \mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q + \mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (124)$$

where $\mathbf{W}$ is the Poynting flux, and $-\Sigma\mathbf{Q}q$ that of the stress, whilst $\mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0)$ means convection of energy connected with the translational force. We should therefore have

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + (\mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0) + \operatorname{div} \mathbf{X} \quad . \; . \; . \; . \; . \quad (125)$$

to express the continuity of energy. More explicitly

$$\begin{aligned}\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div}\left[\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right] \\ + \mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0 + \operatorname{div}\left[-\Sigma\mathbf{Q}q + \mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0)\right] \quad . \; . \; . \; . \; . \; . \; . \; . \quad (126)\end{aligned}$$

But here we may simplify by using the result (69) (with, however, $\mathbf{f}$ put $= 0$), making (126) become

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{F}\mathbf{q} + \mathbf{S}\mathbf{a} + \operatorname{div}\left[\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\,\mathbf{Q}q\right], \quad . \; . \; . \quad (127)$$

where $\mathbf{S}$ is the torque, and $\mathbf{a}$ the spin.

Comparing this with (121), we see that we require

$$\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q = \mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h} - \mathrm{V}\mathbf{e}_0\mathbf{h} - \mathrm{V}\mathbf{e}\mathbf{h}_0, \quad . \; . \; . \; . \; . \quad (128)$$

with a similar equation when (119) is used instead; and we have now to separate the right member into two parts, one for the Poynting flux, the other for the stress flux, in such a way that the force due to the stress is the force $\mathbf{F}$ in (121), (122), or the force $\mathbf{f}$ in (119), (116); or similarly in other cases. It is unnecessary to give the failures; the only one that stands the test is (121), which satisfies it completely.

I argued that

$$\mathbf{W} = \mathrm{V}(\mathbf{E} - \mathbf{e}_0)(\mathbf{H} - \mathbf{h}_0) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (129)$$

was the probable form of the POYNTING flux in the case of a moving medium, not $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$, because when a medium is endowed with a *uniform* translational motion, the transmission of disturbances through it takes place just as if it were at rest. With this expression (129) for $\mathbf{W}$, we have, identically,

$$\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{e}_0\mathbf{h} - \mathrm{V}\mathbf{eh}_0 = \mathbf{W} - \mathrm{V}\mathbf{eH} - \mathrm{V}\mathbf{Eh}. \quad . \; . \; . \; . \; . \; . \; . \quad (130)$$

Therefore, by (128) and (130), we get

$$\Sigma \mathbf{Q}q = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}(\mathrm{U} + \mathrm{T}), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (131)$$

to represent the negative of the stress flux of energy, so that, finally, the fully significant equation of activity is

$$\begin{aligned} \mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{Fq} + \mathbf{Sa} + \mathrm{div}\left[\mathrm{V}(\mathbf{E} - \mathbf{e}_0)(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}(\mathrm{U} + \mathrm{T})\right] \\ - \mathrm{div}\left[\mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}(\mathrm{U} + \mathrm{T})\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (132) \end{aligned}$$

This is, of course, an identity, subject to the electromagnetic equations we started from, and is only one of the multitude of forms which may be given to it, many being far simpler. But the particular importance of this form arises from its being the only form apparently possible which shall exhibit the principle of continuity of energy without outstanding terms, and without loss of generality; and this is only possible by taking $\mathbf{J}_0$ as the proper flux for $\mathbf{e}_0$ to work upon.*

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* In the original an erroneous estimate of the value of $(\partial/\partial t)(\mathrm{U}_e + \mathrm{T}_\mu)$ was used in some of the above equations. This is corrected. The following contains full details of the calculation. We require the value of $(\partial/\partial t)\mathrm{U}_e$, or of $\frac{1}{2}\mathbf{E}\,(\partial c/\partial t)\mathbf{E}$, where $\partial c/\partial t$ is the linear operator whose components are the time-variations (for the same matter), of those of c. The calculation is very lengthy in terms of these six components. But vectorially it is not difficult. In (27), (28), we have

$$\left.\begin{aligned} \mathbf{D} = c\mathbf{E} &= \mathbf{i}.\mathbf{c}_1\mathbf{E} + \mathbf{j}.\mathbf{c}_2\mathbf{E} + \mathbf{k}.\mathbf{c}_3\mathbf{E} \\ &= (\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathbf{E}, \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (132a)$$

if the vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, are given by

$$\mathbf{c}_1 = \mathbf{i}c_{11} + \mathbf{j}c_{12} + \mathbf{k}c_{13}, \qquad \mathbf{c}_2 = \mathbf{i}c_{21} + \mathbf{j}c_{22} + \mathbf{k}c_{23}, \qquad \mathbf{c}_3 = \mathbf{i}c_{31} + \mathbf{j}c_{32} + \mathbf{k}c_{33}.$$

We, therefore, have

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}\left(\frac{\partial \mathbf{i}}{\partial t}.\mathbf{c}_1 + \frac{\partial \mathbf{j}}{\partial t}.\mathbf{c}_2 + \frac{\partial \mathbf{k}}{\partial t}.\mathbf{c}_3\right)\mathbf{E} + \mathbf{E}\left(\mathbf{i}.\frac{\partial \mathbf{c}_1}{\partial t} + \mathbf{j}.\frac{\partial \mathbf{c}_2}{\partial t} + \mathbf{k}.\frac{\partial \mathbf{c}_3}{\partial t}\right)\mathbf{E} \quad . \; . \; . \; . \quad (132b)$$

The part played by the dots is to clearly separate the scalar products.

Now suppose that the eolotropic property symbolised by c is intrinsically unchanged by the shift of the matter. The mere translation does not, therefore, affect it, nor does the distortion; but the rotation

Derivation of the Electric and Magnetic Stresses and Forces from the Flux of Energy.

§ 19. It will be observed that the convection of energy disappears by occurring twice oppositely signed; but as it comes necessarily into the expression for the stress flux of energy, I have preserved the cancelling terms in (132). A comparison of the stress flux with the POYNTING flux is interesting. Both are of the same form, viz., vector products of the electric and magnetic forces with convection terms; but whereas in the latter the forces in the vector product are those of the field (*i.e.*, only intrinsic forces deducted from **E** and **H**), in the former we have the motional forces **e** and **h** combined with the complete **E** and **H** of the fluxes. Thus the stress depends

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does. For if we turn round an eolotropic portion of matter, keeping **E** unchanged, the value of U is altered by the rotation of the principal axes of c along with the matter, so that a torque is required.

In equation (132*a*), then, to produce (132*b*), we keep **E** constant, and let the six vectors, **i**, **j**, **k**, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ rotate as a rigid body with the spin $\mathbf{a} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{q}$. But when a vector magnitude **i** is turned round in this way, its rate of time-change $\partial\mathbf{i}/\partial t$ is Vai. Thus, for $\partial/\partial t$, we may put Va throughout. Therefore, by (132*b*),

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}(\mathrm{V}\mathbf{ai}.\mathbf{c}_1 + \mathrm{V}\mathbf{aj}.\mathbf{c}_2 + \mathrm{V}\mathbf{ak}.\mathbf{c}_3)\mathbf{E} + \mathbf{E}(\mathbf{i}.\mathrm{V}\mathbf{ac}_1 + \mathbf{j}.\mathrm{V}\mathbf{ac}_2 + \mathbf{k}.\mathrm{V}\mathbf{ac}_3)\mathbf{E}. \quad . \quad . \quad (132c)$$

In this use the parallelepipedal transformation (12), and it becomes

$$\begin{aligned}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} &= \mathrm{V}\mathbf{Ea}(\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_3 + \mathbf{k}.\mathbf{c}_3)\mathbf{E} + \mathbf{E}(\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathrm{V}\mathbf{Ea} \\ &= (\mathrm{V}\mathbf{Ea})c\mathbf{E} + \mathbf{E}c(\mathrm{V}\mathbf{Ea}) = (\mathbf{D} + \mathbf{D}')\mathrm{V}\mathbf{Ea}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132d)\end{aligned}$$

by (132*a*), if $\mathbf{D}'$ is conjugate to **D**; that is, $\mathbf{D}' = c'\mathbf{E} = \mathbf{E}c$. So, when $c = c'$, as in the electrical case, we have

$$\left.\begin{aligned}\frac{\partial \mathrm{U}_c}{\partial t} &= \tfrac{1}{2}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{D}\mathrm{V}\mathbf{Ea} = \mathbf{a}\mathrm{V}\mathbf{DE}, \\ \frac{\partial \mathrm{T}_\mu}{\partial t} &= \tfrac{1}{2}\mathbf{H}\frac{\partial \mu}{\partial t}\mathbf{H} = \mathbf{B}\mathrm{V}\mathbf{Ha} = \mathbf{a}\mathrm{V}\mathbf{BH}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (132e)$$

and similarly

Now the torque arising from the stress is (see (139))

$$\mathbf{S} = \mathrm{V}\mathbf{DE} + \mathrm{V}\mathbf{BH},$$

so we have

$$\frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) = \mathbf{Sa} = \text{torque} \times \text{spin}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132f)$$

The variation allowed to **i**, **j**, **k** may seem to conflict with their constancy (as reference vectors) in general. But they merely vary for a temporary purpose, being fixed in the matter instead of in space. But we may, perhaps better, discard **i**, **j**, **k** altogether, and use any independent vectors, **l**, **m**, **n** instead, making

$$\mathbf{D} = (\mathbf{l}.\mathbf{c}_1 + \mathbf{m}.\mathbf{c}_2 + \mathbf{n}.\mathbf{c}_3)\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132g)$$

wherein the **c**'s are properly chosen to suit the new axes. The calculation then proceeds as before, half entirely on the fluxes, however they be produced, in this respect resembling the electric and magnetic energies.

To exhibit the stress, we have (131), or

$$\mathbf{Q}_1 q_1 + \mathbf{Q}_2 q_2 + \mathbf{Q}_3 q_3 = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (133)$$

In this use the expressions for $\mathbf{e}$ and $\mathbf{h}$, giving

$$\begin{aligned} \Sigma \mathbf{Q}q &= \mathrm{V}\mathbf{H}\mathrm{V}\mathbf{Bq} + \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{Dq} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) \\ &= \mathbf{B}.\mathbf{Hq} - \mathbf{q}.\mathbf{HB} + \mathbf{D}.\mathbf{Eq} - \mathbf{q}.\mathbf{ED} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) \\ &= (\mathbf{B}.\mathbf{Hq} - \mathbf{q}\mathrm{T}) + (\mathbf{D}.\mathbf{Eq} - \mathbf{q}\mathrm{U})\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (134) \end{aligned}$$

where observe the singularity that $\mathbf{q}\,(\mathrm{U} + \mathrm{T})$ has changed its sign. The first set belongs to the magnetic, the second to the electric stress, since we see that the complete stress is thus divisible.

The divergence of $\Sigma \mathbf{Q}q$ being the activity of the stress-variation per unit volume, its $\mathbf{N}$ component is the activity of the stress per unit surface, that is,

$$(\mathbf{NB}.\mathbf{Hq} - \mathbf{Nq}.\mathrm{T}) + (\mathbf{ND}.\mathbf{Eq} - \mathbf{Nq}.\mathrm{U}) = \mathbf{q}\,(\mathbf{H}.\mathbf{BN} + \mathbf{E}.\mathbf{DN} - \mathbf{N}\mathrm{U} - \mathbf{N}\mathrm{T}) = \mathbf{P}_{\mathrm{N}}\mathbf{q}. \quad . \qquad (135)$$

The stress itself is therefore

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the value of $\partial \mathrm{U}_c/\partial t$ arising from the variation of $\mathbf{l}, \mathbf{m}, \mathbf{n}$, and the other half from the $\mathbf{c}$'s, provided c is irrotational.

Or we may choose the three principal axes of c in the body, when $\mathbf{l}, \mathbf{m}, \mathbf{n}$ will coincide with, and therefore move with them.

Lastly, we may proceed thus :—

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} - \mathbf{D}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{E}\mathrm{V}\mathbf{aD} - \mathbf{D}\mathrm{V}\mathbf{aE} = 2\mathbf{a}\mathrm{V}\mathbf{DE}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (132h)$$

That is, replace $\partial/\partial t$ by $\mathrm{V}\mathbf{a}$ when the operands are $\mathbf{E}$ and $\mathbf{D}$. This is the correct result, but it is not easy to justify the process directly and plainly; although the clue is given by observing that what we do is to take a difference, from which the time-variation of $\mathbf{E}$ disappears.

If it is $\mathbf{D}$ that is kept constant, the result is $2\mathbf{a}\mathrm{V}\mathbf{ED}$, the negative of the above.

It is also worth noticing that if we split up $\mathbf{E}$ into $\mathbf{E}_1 + \mathbf{E}_2$ we shall have

$$\left.\begin{aligned} \mathbf{E}_1\frac{\partial c}{\partial t}\mathbf{E}_2 &= \mathbf{a}\,[\mathrm{V}(\mathbf{E}_1 c)\mathbf{E}_2 - \mathrm{V}\mathbf{E}_1(c\mathbf{E}_2)], \\ \mathbf{E}_2\frac{\partial c}{\partial t}\mathbf{E}_1 &= \mathbf{a}\,[\mathrm{V}(\mathbf{E}_2 c)\mathbf{E}_1 - \mathrm{V}_2(c\mathbf{E}_1)]. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (132i)$$

These are only equal when $c = c'$, or $\mathbf{E}c = c\mathbf{E}$; so that, in the expansion of the torque,

$$\mathrm{V}\mathbf{DE} = \mathrm{V}\mathbf{D}_1\mathbf{E}_1 + \mathrm{V}\mathbf{D}_2\mathbf{E}_2 + \mathrm{V}\mathbf{D}_2\mathbf{E}_1 + \mathrm{V}\mathbf{D}_1\mathbf{E}_2,$$

the cross-torques are not $\mathrm{V}\mathbf{D}_2\mathbf{E}_1$ and $\mathrm{V}\mathbf{D}_1\mathbf{E}_2$, which are unequal, but are each equal to half the sum of these vector-products.

$$\mathbf{P}_{\mathrm{N}} = (\mathbf{E}.\mathbf{DN} - \mathbf{N}\mathrm{U}) + (\mathbf{H}.\mathbf{BN} - \mathbf{N}\mathrm{T}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

divided into electric and magnetic portions. This is with restriction to symmetrical μ and c, and with persistence of their forms as a particle moves, but is otherwise unrestricted.

Neither stress is of the symmetrical or irrotational type in case of eolotropy, and there appears to be no getting an irrotational stress save by arbitrary assumptions which destroy the validity of the stress as a correct deduction from the electromagnetic equations. But, in case of isotropy, with consequent directional identity of **E** and **D**, and of **H** and **B**, we see, by taking **N** in turns parallel to, or perpendicular to **E** in the electric case, and to **H** in the magnetic case, that the electric stress consists of a tension U parallel to **E** combined with an equal lateral pressure, whilst the magnetic stress consists of a tension T parallel to **H** combined with an equal lateral pressure. There are, in fact, MAXWELL'S stresses in an isotropic medium homogeneous as regards μ and c. The difference from MAXWELL arises when μ and c are variable (including abrupt changes from one value to another of μ and c), and when there is intrinsic magnetisation, MAXWELL'S stresses and forces being then different.

The stress on the plane whose normal is V**EH**, is

$$\frac{\mathbf{E}.\mathbf{D}\mathrm{V}\mathbf{EH} + \mathbf{H}.\mathbf{B}\mathrm{V}\mathbf{EH} - (\mathrm{U} + \mathrm{T})\,\mathrm{V}\mathbf{EH}}{\mathrm{V}_0\mathbf{EH}},$$

$$= \frac{\mathbf{E}.\mathbf{H}\mathrm{V}\mathbf{DE} + \mathbf{H}.\mathbf{E}\mathrm{V}\mathbf{HB} - (\mathrm{U} + \mathrm{T})\,\mathrm{V}\mathbf{EH}}{\mathrm{V}_0\mathbf{EH}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

reducing simply to a pressure (U + T) in lines parallel to V**EH** in case of isotropy.

§ **20.** To find the force **F**, we have

$$\begin{aligned} \mathbf{FN} &= \mathrm{div}\,\mathbf{Q}_{\mathrm{N}} = \mathrm{div}\,(\mathbf{D}.\mathbf{EN} - \mathbf{N}\mathrm{U} + \mathbf{B}.\mathbf{HN} - \mathbf{N}\mathrm{T}) \\ &= \mathbf{EN}.\rho + \mathbf{D}\nabla.\mathbf{EN} - \tfrac{1}{2}\mathbf{E}.\mathbf{N}\nabla.\mathbf{D} - \tfrac{1}{2}\mathbf{D}.\mathbf{N}\nabla.\mathbf{E} + \&c. \\ &= \mathbf{EN}.\rho + \mathbf{D}\,(\nabla.\mathbf{EN} - \mathbf{N}\nabla.\mathbf{E}) + \tfrac{1}{2}\,(\mathbf{D}.\mathbf{N}\nabla.\mathbf{E} - \mathbf{E}.\mathbf{N}\nabla.\mathbf{D}) + \&c. \\ &= \mathbf{N}\left[\mathbf{E}\rho + \mathrm{V}\ \mathrm{curl}\ \mathbf{E}.\mathbf{D} - \nabla\mathrm{U}_c + \&c.\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138) \end{aligned}$$

where the unwritten terms are the similar magnetic terms. This being the **N** component of **F**, the force itself is given by (122), as is necessary.

It is V curl $\mathbf{h}_0 . \mathbf{B}$ that expresses the translational force on intrinsically magnetised matter, and this harmonises with the fact that the flux **B** due to any impressed force $\mathbf{h}_0$ depends solely upon curl $\mathbf{h}_0$.

Also, it is $-\nabla\mathrm{T}_\mu$ that explains the forcive on elastically magnetised matter, *e.g.*, FARADAY'S motion of matter to or away from the places of greatest intensity of the field, independent of its direction.

If **S** be the torque, it is given by

$$\mathbf{VSN} = \mathbf{P}_{\mathrm{N}} - \mathbf{Q}_{\mathrm{N}} = \mathbf{E.DN} - \mathbf{D.EN} + \&c.$$

$$\mathbf{VN}\,(\mathbf{VED} + \mathbf{VHB});$$

therefore

$$\mathbf{S} = \mathbf{VDE} + \mathbf{VBH} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

But the matter is put more plainly by considering the convergence of the stress flux of energy and dividing it into translational and other parts. Thus

$$\mathrm{div}\,\Sigma\mathbf{Q}q = \mathbf{F}\mathrm{q} + \big(\mathbf{E.D}\nabla.\mathrm{q} - \mathrm{U}\ \mathrm{div}\ \mathrm{q}\big) + \big(\mathbf{H.B}\nabla.\mathrm{q} - \mathrm{T}\ \mathrm{div}\ \mathrm{q}\big), \quad . \quad . \quad . \quad . \quad . \quad (140)$$

where the terms following **Fq** express the sum of the distortional and rotational activities.

Shorter Way of going from the Circuital Equations to the Flux of Energy, Stresses, and Forces.

§ **21.** I have given the investigation in §§ 17 to 19 in the form in which it occurred to me before I knew the precise nature of the results, being uncertain as regards the true measure of current, the proper form of the Poynting flux, and how it worked in harmony with the stress flux of energy. But knowing the results, a short demonstration may be easily drawn up, though the former course is the most instructive. Thus, start now from

$$\left.\begin{aligned} \mathrm{curl}\,(\mathbf{H} - \mathbf{h}_0) &= \mathbf{J}_0, \\ -\,\mathrm{curl}\,(\mathbf{E} - \mathbf{e}_0) &= \mathbf{G}_0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

on the understanding that $\mathbf{J}_0$ and $\mathbf{G}_0$ are the currents which make $\mathbf{e}_0\mathbf{J}_0$ and $\mathbf{h}_0\mathbf{G}_0$ the activities of $\mathbf{e}_0$ and $\mathbf{h}_0$ the intrinsic forces. Then

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathbf{EJ}_0 + \mathbf{HG}_0 + \mathrm{div}\ \mathbf{W}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

where

$$\mathbf{W} = \mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

and we now take this to be the proper form of the Poynting flux. Now develop $\mathbf{EJ}_0$ and $\mathbf{HG}_0$ thus :—

$$\begin{aligned} \mathbf{EJ}_0 + \mathbf{HG}_0 &= \mathbf{E}\big(\mathbf{C} + \dot{\mathbf{D}} + \mathrm{q}\rho + \mathrm{curl}\ \mathbf{h}\big) + \mathbf{H}\big(\mathbf{K} + \dot{\mathbf{B}} + \mathrm{q}\sigma - \mathrm{curl}\ \mathbf{e}\big), \quad \text{by (93)}; \\ &= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + \mathbf{E}\mathrm{q}\rho + \mathbf{E}\ \mathrm{curl}\ \mathrm{V}\mathbf{D}\mathrm{q} \\ &\quad + \mathrm{Q}_2 + \dot{\mathrm{T}} + \dot{\mathrm{T}}_\mu + \mathbf{H}\mathrm{q}\sigma + \mathbf{H}\ \mathrm{curl}\ \mathrm{V}\mathbf{B}\mathrm{q}, \quad \text{by (88) and (91)}; \\ &= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + \mathbf{E}\mathrm{q}\rho + \mathbf{E}\,(\mathbf{D}\ \mathrm{div}\ \mathrm{q} + \mathrm{q}\nabla.\mathbf{D} - \mathrm{q}\ \mathrm{div}\ \mathbf{D} - \mathbf{D}\nabla.\mathrm{q}) \\ &\quad + \mathrm{Q}_2 + \dot{\mathrm{T}} + \dot{\mathrm{T}}_\mu + \mathbf{H}\mathrm{q}\sigma + \mathbf{H}\,(\mathbf{B}\ \mathrm{div}\ \mathrm{q} + \mathrm{q}\nabla.\mathbf{B} - \mathrm{q}\ \mathrm{div}\ \mathbf{B} - \mathbf{B}\nabla.\mathrm{q}), \quad \text{by (26)}, \\ &= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + 2\mathrm{U}\ \mathrm{div}\ \mathrm{q} + \mathbf{E}.\mathrm{q}\nabla.\mathbf{D} - \mathbf{E.D}\nabla.\mathrm{q} \\ &\quad + \text{magnetic terms}, \\ &= \big(\mathrm{Q}_1 + \dot{\mathrm{U}} + \mathrm{div}\ \mathrm{qU}\big) + \big(\mathrm{U}\ \mathrm{div}\,\mathrm{q} - \mathbf{E.D}\nabla.\mathrm{q}\big) + \big(\dot{\mathrm{U}}_c - \mathrm{q}\nabla.\mathrm{U} + \mathbf{E}.\mathrm{q}\nabla.\mathbf{D}\big) \\ &\quad + \text{magnetic terms.} \quad . \quad (144) \end{aligned}$$

Now here

$$\mathrm{q}\nabla.\mathrm{U} = \tfrac{1}{2}\mathbf{E}.\mathrm{q}\nabla.\mathbf{D} + \tfrac{1}{2}\mathbf{D}.\mathrm{q}\nabla.\mathbf{E},$$

so that the terms in the third pair of brackets in (144) represent

$$\dot{\mathrm{U}}_c + \mathrm{q}\nabla.\mathrm{U}_c = \frac{\partial \mathrm{U}_c}{\partial t} = \tfrac{1}{2}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E},$$

with the generalised meaning before explained. So finally

$$\begin{aligned}\mathbf{EJ}_0 + \mathbf{HG}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} \mathbf{q}(\mathrm{U} + \mathrm{T}) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \\ + (\mathrm{U} \operatorname{div} \mathbf{q} - \mathbf{E.D}\nabla.\mathbf{q}) + (\mathrm{T} \operatorname{div} \mathbf{q} - \mathbf{H.B}\nabla.\mathbf{q}), \quad . \ . \ . \ . \quad (145)\end{aligned}$$

which brings (142) to

$$\begin{aligned}\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} \{\mathbf{W} + \mathbf{q}(\mathrm{U} + \mathrm{T})\} \\ + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) + (\mathrm{U} \operatorname{div} \mathbf{q} - \mathbf{E.D}\nabla.\mathbf{q}) + (\mathrm{T} \operatorname{div} \mathbf{q} - \mathbf{H.B}\nabla.\mathbf{q}), \quad . \ . \ . \ . \quad (146)\end{aligned}$$

which has to be interpreted in accordance with the principle of continuity of energy.

Use the form (127), first, however, eliminating $\mathbf{Fq}$ by means of

$$\operatorname{div} \Sigma \mathbf{Q}q = \mathbf{Fq} + \Sigma \mathbf{Q}\nabla\mathbf{q},$$

which brings (127) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathbf{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} \{\mathbf{W} + \mathbf{q}(\mathrm{U} + \mathrm{T})\} - \Sigma\mathbf{Q}\nabla q + \mathbf{Sa}\,; \quad . \ . \ . \ . \ . \ . \quad (147)$$

and now, by comparison of (147) with (146) we see that

$$\begin{aligned}- \mathbf{Sa} + \Sigma\mathbf{Q}\nabla q = (\mathbf{E.D}\nabla.\mathbf{q} - \mathrm{U} \operatorname{div} \mathbf{q}) - \frac{\partial \mathrm{U}_c}{\partial t} \\ + (\mathbf{H.B}\nabla.\mathbf{q} - \mathrm{T} \operatorname{div} \mathbf{q}) - \frac{\partial \mathrm{T}_\mu}{\partial t}\,; \quad . \ . \ . \ . \ . \ . \ . \ . \quad (148)\end{aligned}$$

from which, when μ and c do not change intrinsically, we conclude that

$$\left.\begin{aligned}\mathbf{Q}_{\mathrm{N}} = \mathbf{B.HN} - \mathbf{N}\mathrm{T} + \mathbf{D.EN} - \mathbf{N}\mathrm{U}, \\ \mathbf{P}_{\mathrm{N}} = \mathbf{H.BN} - \mathbf{N}\mathrm{T} + \mathbf{E.DN} - \mathbf{N}\mathrm{U},\end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (149)$$

as before. In this method we lose sight altogether of the translational force which formed so prominent an object in the former method as a guide.

Some Remarks on HERTZ'S *investigation relating to the Stresses.*

§ **22.** Variations of c and μ in the same portion of matter may occur in different ways, and altogether independently of the strain variations. Equation (146) shows how their influence affects the energy transformations; but if we consider only such changes as depend on the strain, *i.e.*, the small changes of value which μ and c undergo as the strain changes, we may express them by thirty-six new coefficients each (there being six distortion elements, and six elements in μ, and six in c), and so reduce the expressions for $\partial \mathrm{U}_c/\partial t$ and $\partial \mathrm{T}_\mu/\partial t$ in (148) to the form suitable for exhibiting the corresponding change in $\mathbf{Q}_{\mathrm{N}}$ and in the stress function $\mathbf{P}_{\mathrm{N}}$. As is usual in such cases of secondary corrections, the magnitude of the resulting formula is out of all proportion to the importance of the correction terms in relation to the primary formula to which they are added.

Professor H. HERTZ[*] has considered this question, and also refers to VON HELMOLTZ's previous investigation relating to a fluid. The c and μ can then only depend on the density, or on the compression, so that a single coefficient takes the place of the thirty-six. But I cannot quite follow HERTZ's stress investigation. First, I would remark that in developing the expression for the distortional (*plus* rotational) activity, he assumes that all the coefficients of the spin vanish identically; this is done in order to make the stress be of the irrotational type. But it may easily be seen that the assumption is inadmissible by examining its consequence, for which we need only take the case of c and μ intrinsically constant. By (139) we see that it makes $\mathbf{S} = 0$, and therefore (since the electric and magnetic stress are separable), $\mathbf{VHB} = 0$, and $\mathbf{VED} = 0$; that is, it produces directional identity of the force $\mathbf{E}$ and the flux $\mathbf{D}$, and of the force $\mathbf{H}$ and the flux $\mathbf{B}$. This means isotropy, and, therefore, breaks down the investigation so far as the eolotropic application, with six μ and six c coefficients, goes. Abolish the assumption made, and the stress will become that used by me above.

Another point deserving of close attention in HERTZ's investigation, relates to the principle to be followed in deducing the stress from the electromagnetic equations. Translating into my notation it would appear to amount to this, the *à priori* assumption that the quantity

$$\frac{1}{v}\frac{\partial}{\partial t}\left(\mathrm{T}v\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

where v indicates the volume of a moving unit element undergoing distortion, may be taken to represent the distortional (*plus* rotational) activity of the magnetic stress. Similarly as regards the electric stress.

Expanding (150) we obtain

$$\frac{\partial \mathrm{T}}{\partial t} + \frac{\mathrm{T}}{v}\frac{\partial v}{\partial t} = \mathbf{H}\frac{\partial \mathbf{B}}{\partial t} + \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

Now the second circuital law (90) may be written

$$-\,\mathrm{curl}\,(\mathbf{E} - \mathbf{e}_0) = \mathbf{K} + \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B}\,\mathrm{div}\,\mathbf{q} - \mathbf{B}\nabla.\mathbf{q}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

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[*] 'WIEDEMANN'S Annalen,' v. 41, p. 369.

Here ignore $\mathbf{e}_0$, $\mathbf{K}$, *and ignore the curl of the electric force,* and we obtain, by using (152) in (151),

$$\mathbf{H.B}\nabla.\mathbf{q} - \mathbf{HB}\,\mathrm{div}\,\mathbf{q} + \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t} = \mathbf{H.B}\nabla.\mathbf{q} - \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t}, \quad . \; . \; . \; . \quad (153)$$

which represents the distortional activity (my form, not equating to zero the coefficients of curl $\mathbf{q}$ in its development). We *can*, therefore, derive the magnetic stress in the manner indicated, that is, from (150), with the special meaning of $\partial\mathbf{B}/\partial t$ later stated, and the ignorations or nullifications.

In a similar manner, from the first circuital law (89), which may be written

$$\mathrm{curl}\,(\mathbf{H} - \mathbf{h}_0) = \mathbf{C} + \frac{\partial \mathbf{D}}{\partial t} + (\mathbf{D}\,\mathrm{div}\,\mathbf{q} - \mathbf{D}\nabla.\mathbf{q}), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (154)$$

we can, by ignoring the conduction current and the curl of the magnetic force, obtain

$$\frac{1}{v}\frac{\partial}{\partial t}(v\mathrm{U}) = \mathbf{E.D}\nabla.\mathbf{q} - \mathrm{U}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{U}_c}{\partial t}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (155)$$

which represents the distortional activity of the electric stress.

The difficulty here seems to me to make it evident *à priori* that (150), with the special reckoning of $\partial\mathbf{B}/\partial t$ *should* represent the distortional activity (*plus* rotational understood); this interesting property should, perhaps, rather be derived from the magnetic stress when obtained by a safe method. The same remark applies to the electric stress. Also, in (150) to (155) we overlook the POYNTING flux. I am not sure how far this is intentional on Professor HERTZ's part, but its neglect does not seem to give a sufficiently comprehensive view of the subject.

The complete expansion of the magnetic distortional activity is, in fact,

$$\mathbf{H.B}\nabla.\mathbf{q} - \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t} = \mathrm{Q}_2 + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\mathrm{T} - \mathbf{HG}_0; \quad . \; . \; . \; . \; . \; . \quad (156)$$

and similarly, that of the electric stress is

$$\mathbf{E.D}\nabla.\mathbf{q} - \mathrm{U}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{U}_c}{\partial t} = \mathrm{Q}_1 + \dot{\mathrm{U}} + \mathrm{div}\,\mathbf{q}\mathrm{U} - \mathbf{EJ}_0. \quad . \; . \; . \; . \; . \; . \; . \quad (157)$$

It is the last term of (156) and the last term of (157), together, which bring in the POYNTING flux. Thus, adding these equations,

$$\Sigma\mathbf{Q}\nabla q - \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\,(\mathrm{U} + \mathrm{T}) - (\mathbf{EJ}_0 + \mathbf{HG}_0), \quad . \; . \; . \quad (158)$$

where

$$(\mathbf{EJ}_0 + \mathbf{HG}_0) = (\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0) - \mathrm{div}\,\mathbf{W}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (159)$$

and so we come round to the equation of activity again, in the form (146), by using (159) in (158).

Modified Form of Stress-vector, and Application to the Surface separating two Regions.

§ **23.** The electromagnetic stress, $\mathbf{P}_N$ of (149) and (136) may be put into another interesting form. We may write it

$$\mathbf{P}_N = \tfrac{1}{2}(\mathbf{E}.\mathbf{ND} + \mathrm{V}.\mathrm{V}\mathbf{NE}.\mathbf{D}) + \tfrac{1}{2}(\mathbf{H}.\mathbf{NB} + \mathrm{V}.\mathrm{V}\mathbf{NH}.\mathbf{B}). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (160)$$

Now, **ND** is the surface equivalent of div **D** and **NB** of div **B**; whilst V**NE** and V**NH** are the surface equivalents of curl **E** and curl **H**. We may, therefore, write

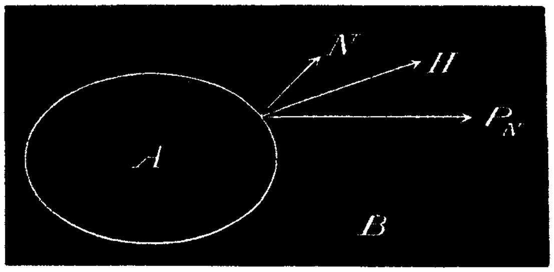
$$\mathbf{P}_N = \tfrac{1}{2}(\mathbf{E}\rho' + \mathrm{V}\mathbf{DG}') + \tfrac{1}{2}(\mathbf{H}\sigma' + \mathrm{V}\mathbf{J}'\mathbf{B}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (161)$$

and this is the force, reckoned as a pull, on unit area of the surface whose normal is **N**. Here the accented letters are the surface equivalents of the same quantities unaccented, which have reference to unit volume.

Comparing with (122) we see that the type is preserved, except as regards the terms in **F** due to variation of c and μ in space. That is, the stress is represented in (101) as the translational force, due to **E** and **H**, on the fictitious electrification, magnetification, electric current, and magnetic current produced by imagining **E** and **H** to terminate at the surface across which $\mathbf{P}_N$ is the stress.

The coefficient $\frac{1}{2}$ which occurs in (161) is understandable by supposing the fictitious quantities ("matter" and "current") to be distributed uniformly within a very thin layer, so that the forces **E** and **H** which act upon them do not then terminate quite abruptly, but fall off gradually through the layer from their full values on one side to zero on the other. The mean values of **E** and **H** through the layer, that is, $\frac{1}{2}\mathbf{E}$ and $\frac{1}{2}\mathbf{H}$ are thus the effective electric and magnetic forces on the layer as a whole, per unit volume density of matter or current; or $\frac{1}{2}\mathbf{E}$ and $\frac{1}{2}\mathbf{H}$ per unit surface density when the layer is indefinitely reduced in thickness.

Considering the electric field only, the quantities concerned are electrification and magnetic current. In the magnetic field only they are magnetification and electric current. Imagine the medium divided into two regions A and B, of which A is internal, B external, and let **N** be the unit normal from the surface into the external region. The mechanical action between the two regions is fully represented by the stress $\mathbf{P}_N$ over their interface, and the forcive of B upon A is fully represented by the **E** and **H** in B acting upon the fictitious matter and current produced on the boundary of B, on the assumption that **E** and **H** terminate there. If the normal and $\mathbf{P}_N$ be drawn the other way, thus negativing them both, as well as the fictitious matter and current on the interface, then it is the forcive of A on B that is represented by the action of **E** and **H** in A on the new interfacial matter and current. That is, the **E** and **H** in the region A may be done away with altogether, because their abolition will immediately introduce the fictitious matter and current equivalent, so far as B is concerned, to the influence of the region A. Similarly **E** and **H** in B may be abolished without altering them in A. And, generally, any portion of the medium may be taken by itself and regarded as being subjected to an equilibrating system of forces, when treated as a rigid body.



§ **24.** When c and μ do not vary in space, we do away with the forces $-\frac{1}{2}E^2\nabla c$ and $-\frac{1}{2}H^2\nabla\mu$, and make the form of the surface and volume translational forces agree. We may then regard every element of ρ or of σ as a source sending out from itself displacement and induction isotropically, and every element of **J** or **G** as causing induction or displacement according to AMPÈRE's rule for electric current and its analogue for magnetic current. Thus

$$\mathbf{E} = \Sigma \frac{\rho/c + \mathrm{V}\mathbf{r}_1\mathbf{G}}{4\pi r^2}, \quad \dots \dots \dots \dots \quad (162)$$

$$\mathbf{H} = \Sigma \frac{\sigma/\mu + \mathrm{V}\mathbf{J}\mathbf{r}_1}{4\pi r^2}, \quad \dots \dots \dots \dots \quad (163)$$

where $\mathbf{r}_1$ is a unit vector drawn from the infinitesimal unit volume in the summation to the point at distance r where **E** or **H** is reckoned. Or, introducing potentials,

$$\mathbf{E} = -\nabla\Sigma\frac{\rho/c}{4\pi r} - \mathrm{curl}\,\Sigma\frac{\mathbf{G}}{4\pi r}, \quad \dots \dots \dots \dots \quad (164)$$

$$\mathbf{H} = -\nabla\Sigma\frac{\sigma/\mu}{4\pi r} + \mathrm{curl}\,\Sigma\frac{\mathbf{J}}{4\pi r}. \quad \dots \dots \dots \dots \quad (165)$$

These apply to the whole medium, or to any portion of the same, with, in the latter case, the surface matter and current included, there being no **E** or **H** outside the region, whilst within it **E** and **H** are the same as due to the matter and current in the whole region ("matter," ρ and σ; "current," **J** and **G**). But there is no known general method of finding the potentials when c and μ vary.

We may also divide **E** and **H** into two parts each, say $\mathbf{E}_1$ and $\mathbf{H}_1$ due to matter and current in the region A, and $\mathbf{E}_2$, $\mathbf{H}_2$ due to matter and current in the region B surrounding it, determinable in the isotropic homogeneous case by the above formulæ. Then we may ignore $\mathbf{E}_1$ and $\mathbf{H}_1$ in estimating the forcive on the matter and current in the region A; thus,

$$\Sigma(\mathbf{H}_2\sigma_1 + \mathrm{V}\mathbf{J}_1\mathbf{B}_2) + \Sigma(\mathbf{E}_2\rho_1 + \mathrm{V}\mathbf{D}_2\mathbf{G}_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (166)$$

where $\sigma_1 = \operatorname{div} \mathbf{B}_1 = \operatorname{div} \mathbf{B}$, and $\mathbf{J}_1 = \operatorname{curl} \mathbf{H}_1 = \operatorname{curl} \mathbf{H}$ in region A, is the resultant force on the region A, and

$$\Sigma(\mathbf{H}_1\sigma_2 + \mathrm{V}\mathbf{J}_2\mathbf{B}_1) + \Sigma(\mathbf{E}_1\rho_2 + \mathrm{V}\mathbf{D}_1\mathbf{G}_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (167)$$

is the resultant force on the region B; the resultant force on A due to its own **E** and **H** being zero, and similarly for B. These resultant forces are equal and opposite, and so are the equivalent surface-integrals

$$\Sigma(\mathbf{H}_2\sigma_1' + \mathrm{V}\mathbf{J}_1'\mathbf{B}_2) + \Sigma(\mathbf{E}_2\rho_1' + \mathrm{V}\mathbf{D}_2\mathbf{G}_1'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (168)$$

and

$$\Sigma(\mathbf{H}_1\sigma_2' + \mathrm{V}\mathbf{J}_2'\mathbf{B}_1) + \Sigma(\mathbf{E}_1\rho_2' + \mathrm{V}\mathbf{D}_1\mathbf{G}_2'), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (169)$$

taken over the interface. The quantity summed is that part of the stress-vector, $\mathbf{P}_\mathrm{N}$, which depends upon products of the **H** of one region and the **B** of the other, &c. Thus, for the magnetic stress only,

$$\begin{aligned}\mathbf{H}.\mathbf{B}\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}\mathbf{B} = {} & \left(\mathbf{H}_1.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_1\mathbf{B}_1\right) + \left(\mathbf{H}_1.\mathbf{B}_2\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_1\mathbf{B}_2\right) \\ & + \left(\mathbf{H}_2.\mathbf{B}_2\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_2\mathbf{B}_2\right) + \left(\mathbf{H}_2.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_2\mathbf{B}_1\right), \quad . \quad . \quad . \qquad (170)\end{aligned}$$

and it is the terms in the second and fourth brackets (which, be it observed, are not equal) which together make up the magnetic part of (168) and (169) or their negatives, according to the direction taken for the normal; that is, since $\mathbf{H}_1\mathbf{B}_2 = \mathbf{H}_2\mathbf{B}_1$,

$$\begin{aligned}\Sigma\mathbf{P}_\mathrm{N} &= \Sigma\left(\mathbf{H}_1.\mathbf{B}_2\mathbf{N} + \mathbf{H}_2.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\mathbf{H}_1\mathbf{B}_2\right) = \Sigma\left(\mathbf{H}.\mathbf{B}\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}\mathbf{B}\right) \\ &= \Sigma\left(\mathbf{H}_1\sigma_2' + \mathrm{V}\mathbf{J}_2'\mathbf{B}_1\right) = \Sigma\left(\mathbf{H}_2\sigma_1' + \mathrm{V}\mathbf{J}_1'\mathbf{B}_2\right) = \Sigma\left(\mathbf{H}\sigma' + \mathrm{V}\mathbf{J}'\mathbf{B}\right) \\ &= \Sigma\mathbf{F} = \Sigma\left(\mathbf{H}_1\sigma_2 + \mathrm{V}\mathbf{J}_2\mathbf{B}_1\right) = \Sigma\left(\mathbf{H}_2\sigma_1 + \mathrm{V}\mathbf{J}_1\mathbf{B}_2\right) = \Sigma\left(\mathbf{H}\sigma + \mathrm{V}\mathbf{J}\mathbf{B}\right), \quad . \quad . \quad . \qquad (171)\end{aligned}$$

where the first six expressions are interfacial summations, and the four last summations throughout one or the other region, the last summation applying to either region. No special reckoning of the sign to be prefixed has been made. The notation is such that $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, $\sigma = \sigma_1 + \sigma_2$, &c., &c.

The comparison of the two aspects of electromagnetic theory is exceedingly curious; namely, the precise mathematical equivalence of "explanation" by means of instantaneous action at a distance between the different elements of matter and current, each according to its kind, and by propagation through a medium in time at a finite velocity. But the day has gone by for any serious consideration of the former view other than as a mathematical curiosity.

Quaternionic Form of Stress-Vector.

§ **25.** We may also notice the Quaternion form for the stress function, which is so vital a part of the mathematics of forces varying as the inverse square of the distance, and of potential theory. Isotropy being understood, the electric stress may be written

$$\mathbf{P}_{\mathrm{N}} = \tfrac{1}{2} c\,[\mathbf{E}\mathbf{N}^{-1}\mathbf{E}], \quad \ldots\ldots\ldots\ldots \quad (172)$$

where the quantity in the square brackets is to be understood quaternionically. It is, however, a pure vector. Or,

$$\left[\frac{\mathbf{P}_{\mathrm{N}}}{\mathbf{E}}\right] = \frac{c}{2}\left[\frac{\mathbf{E}}{\mathbf{N}}\right], \quad \ldots\ldots\ldots\ldots \quad (173)$$

that is, not counting the factor $\tfrac{1}{2} c$, the quaternion $\left[\frac{\mathbf{P}_{\mathrm{N}}}{\mathbf{E}}\right]$ is the same as the quaternion $\left[\frac{\mathbf{N}}{\mathbf{E}}\right]$; or the same operation which turns **N** to **E** also turns **E** to $\mathbf{P}_{\mathrm{N}}$. Thus, **N**, **E**, and $\mathbf{P}_{\mathrm{N}}$ are in the same plane, and the angle between **N** and **E** equals that between **E** and $\mathbf{P}_{\mathrm{N}}$; and **E** and $\mathbf{P}_{\mathrm{N}}$ are on the same side of **N** when **E** makes an acute angle with **N**. Also, the tensor of $\mathbf{P}_{\mathrm{N}}$ is U, so that its normal and tangential components are $\mathrm{U} \cos 2\theta$ and $\mathrm{U} \sin 2\theta$, if $\theta = \widehat{\mathbf{NE}}$.

Otherwise

$$\mathbf{P}_{\mathrm{N}} = -\tfrac{1}{2} c\,[\mathbf{E}\mathbf{N}\mathbf{E}], \quad \ldots\ldots\ldots\ldots \quad (174)$$

since the quaternionic reciprocal of a vector has the reverse direction. The corresponding volume translational force is

$$\mathbf{F} = -c\mathrm{V}\,[\mathbf{E}\nabla\mathbf{E}], \quad \ldots\ldots\ldots\ldots \quad (175)$$

which is also to be understood quaternionically, and expanded, and separated into parts to become physically significant. I only use the square brackets in this paragraph to emphasise the difference in notation. It rarely occurs that any advantage is gained by the use of the quaternion, in saying which, I merely repeat what Professor WILLARD GIBBS has been lately telling us; and I further believe the disadvantages usually far outweigh the advantages. Nevertheless, apart from practical application, and looking at it from the purely quaternionic point of view, I ought to also add that the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics; but to find out quaternions required a genius.

Remarks on the Translational Force in Free Ether.

§ **26.** The little vector **Veh**, which has an important influence in the activity equation, where **e** and **h** are the motional forces

$$\mathbf{e} = \mathbf{VqB}, \qquad \mathbf{h} = \mathbf{VDq},$$

has an interesting form, viz., by expansion,

$$\mathbf{Veh} = \mathbf{q.qVDB} = \frac{\mathbf{q}}{v^2} \cdot \mathbf{qVEH}, \quad \ldots \ldots \ldots \ldots \quad (176)$$

if v be the speed of propagation of disturbances. We also have, in connection therewith, the equivalence

$$\mathbf{eD} = \mathbf{hB}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (177)$$

always.

The translational force in a non-conducting dielectric, free from electrification and intrinsic force, is

$$\mathbf{F} = \mathbf{VJB} + \mathbf{VDG} + \mathbf{VjB} + \mathbf{VDg},$$

or, approximately,

$$= \mathbf{V\dot{D}B} + \mathbf{VD\dot{B}} = \frac{d}{dt}\mathbf{VDB} = \frac{1}{v^2}\frac{d}{dt}\mathbf{VEH} = \frac{\mathbf{\dot{W}}}{v^2}. \quad \ldots \ldots \ldots \quad (178)$$

The vector **VDB**, or the flux of energy divided by the square of the speed of propagation, is, therefore, the momentum (translational, not magnetic, which is quite a different thing), provided the force **F** *is* the complete force from all causes acting, and we neglect the small terms **VjB** and **VDg**.

But have we any right to safely write

$$\mathbf{F} = m\frac{\partial \mathbf{q}}{\partial t}, \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (179)$$

where m is the density of the ether? To do so is to assume that **F** is the only force acting, and, therefore, equivalent to the time-variation of the momentum of a moving particle.*

Now, if we say that there is a certain forcive upon a conductor supporting electric current; or, equivalently, that there is a certain distribution of stress, the magnetic stress, acting upon the same, we do not at all mean that the accelerations of momentum of the different parts are represented by the translational force, the "electromagnetic force." It is, on the other hand, a dynamical problem in which the electromagnetic force plays the part of an impressed force, and similarly as regards the magnetic

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* Professor J. J. THOMSON has endeavoured to make practical use of the idea, 'Phil. Mag.,' March, 1891. See also my article, 'The Electrician,' January 15, 1886.

stress; the actual forces and stresses being only determinable from a knowledge of the mechanical conditions of the conductor, as its density, elastic constants, and the way it is constrained. Now, if there is any dynamical meaning at all in the electromagnetic equations, we must treat the ether in precisely the same way. But we do not know, and have not formularised, the equations of motion of the ether, but only the way it propagates disturbance through itself, with due allowance made for the effect thereon of given motions, and with formularisation of the reaction between the electromagnetic effects and the motion. Thus the theory of the stresses and forces in the ether and its motions is an unsolved problem, only a portion of it being known so far, *i.e.*, assuming that the Maxwellian equations do express the known part.

When we assume the ether to be motionless, there is a partial similarity to the theory of the propagation of vibrations of infinitely small range in elastic bodies, when the effect thereon of the actual translation of the matter is neglected.

But in ordinary electromagnetic phenomena, it does not seem that the ignoration of $\mathbf{q}$ can make any sensible difference, because the speed of propagation of disturbances through the ether is so enormous, that if the ether were stirred about round a magnet, for example, there would be an almost instantaneous adjustment of the magnetic induction to what it would be were the ether at rest.

Static Consideration of the Stresses.—Indeterminateness.

§ **27.** In the following the stresses are considered from the static point of view, principally to examine the results produced by changing the form of the stress function. Either the electric or the magnetic stress alone may be taken in hand. Start then, from a knowledge that the force on a magnetic pole of strength m is $\mathbf{R}m$, where $\mathbf{R}$ is the polar force of any distribution of intrinsic magnetisation in a medium, the whole of which has unit inductivity, so that

$$\text{div}\,\mathbf{R} = m = \text{conv}\,\mathbf{h}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

measures the density of the fictitious "magnetic" matter; $\mathbf{h}_0$ being the intrinsic force, or, since here $\mu = 1$, the intensity of magnetisation. The induction is $\mathbf{B} = \mathbf{h} + \mathbf{R}$. This rudimentary theory locates the force on a magnet at its poles, superficial or internal, by

$$\mathbf{F} = \mathbf{R}\,\text{div}\,\mathbf{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

The $\mathbf{N}$ component of $\mathbf{F}$ is

$$\mathbf{FN} = \mathbf{RN}.\,\text{div}\,\mathbf{R} = \text{div}\left\{\mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

because curl $\mathbf{R} = 0$. Therefore

$$\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

is the appropriate stress, of irrotational type. Now, however uncertain we may be about the stress in the interior of a magnet, there can be no question as to the possible validity of this stress in the air outside our magnet, for we know that the force **R** is then a polar force, and that is all that is wanted, *m* and **h** being merely auxiliaries, derived from **R**.

Now consider a region A, containing magnets of this kind, enclosed in B, the rest of space, also containing magnets. The mutual force between the two regions is expressed by $\Sigma \mathbf{P}_{\mathbf{N}}$ over the interface, which we may exchange for $\Sigma \mathbf{R}m$ through either region A or B, still on the assumption that **R** remains polar.

But if we remove this restriction upon the nature of **R**, and allow it to be arbitrary, say in region B or in any portion thereof, we find

$$\mathbf{NF} = \text{div}\ \mathbf{P}_{\mathbf{N}} = \mathbf{RN}\ \text{div}\ \mathbf{R} + \mathbf{NV}\ \text{curl}\ \mathbf{R}\ .\ \mathbf{R};$$

or

$$\mathbf{F} = \mathbf{R}m + \mathbf{VJR},$$

if $\mathbf{J} = \text{curl}\ \mathbf{R}$. This gives us, from a knowledge of the external magnetic field of polar magnets only, the mechanical force exerted by a magnet on a region containing **J**, whatever that may be, provided it be measurable as above; and without any experimental knowledge of electric currents, we could now predict their mechanical effects in every respect by the principle of the equality of action and reaction, not merely as regards the mutual influence of a magnet and a closed current, but as regards the mutual influence of the closed currents themselves; the magnetic force of a closed current, for instance, being the force on unit of *m*, is equivalently the force exerted by *m* on the closed current, which, by the above, we know. Also, we see that according to this magnetic notion of electric current, it is necessarily circuital.

At the same time, it is to be remarked that our real knowledge must cease at the boundary of the region containing electric current, a metallic conductor for instance; the surface over which $\mathbf{P}_{\mathbf{N}}$ is reckoned, on one side of which is the magnet, on the other side electric current, can only be pushed up as far as the conductor. The stress $\mathbf{P}_{\mathbf{N}}$ may therefore cease altogether on reaching the conductor, where it forms a distribution of surface force fully representing the action of the magnet on the conductor. Similarly, we need not continue the stress into the interior of the magnet. Then, so far as the resultant force on the magnet as a whole, in translating or rotating it, and, similarly, so far as the action on the conductor, is concerned, the simple stress $\mathbf{P}_{\mathbf{N}}$ of constant tensor $\frac{1}{2}\mathbf{R}^2$, varying from a tension parallel to **R** to an equal pressure laterally, acting in the medium between the magnet and conductor, accounts, by its terminal pulls or pushes, for the mechanical forces on them. The lateral pressure is especially prominent in the case of conductors, whilst the tension goes more or less out of sight, as the immediate cause of motion. Thus, when parallel currents appear to attract one another, the conductors are really pushed together by the lateral pressure on each conductor being greater on the side remote from the other than on the near side: whilst if the currents are oppositely directed, the pressure on the near sides is greater than on the remote sides, and they appear to repel one another.

The effect of continuing the stress into the interior of a conductor of unit inductivity, according to the same law, instead of stopping it on its boundary, is to distribute the translational force bodily, according to the formula $\Sigma \mathbf{VJB}$, instead of superficially, according to $\Sigma \mathbf{P}_N$. In either case, of course, the conductor must be strained by the magnetic stress, with the consequent production of a mechanical stress. But the strain (and associated stress) will be different in the two cases, the applied forces being differently localised. The effect of the stress on a straight portion of a wire supporting current, due to its own field only, is to compress it laterally, and to lengthen it. Besides this, there will be resultant force on it arising from the different pressures on its opposite sides due to the proximity of the return conductor or rest of the circuit, tending to move it so as to increase the induction through the circuit per unit current, that is, the inductance of the circuit.

§ **28.** If now, we bring an elastically magnetisable body into a magnetic field, it modifies the field by its presence, causing more or less induction to go through it than passed previously in the air it replaces, according as its inductivity exceeds or is less than that of the air. The forcive on it, considered as a rigid body, is completely accounted for by the simple stress $\mathbf{P}_N$ in the air outside it, reckoned according to the changed field, and supposed to terminate on the surface of the disturbing body. This is true whether the body be isotropic or heterotropic in its inductivity; nor need the induction be a linear function of the magnetic force. It is also true when the body is intrinsically magnetised; or is the seat of electric current. In short, since the external stress depends upon the magnetic force outside the body, when we take the external field as we may find it, that is, as modified by any known or unknown causes within the body, the corresponding stress, terminated upon its boundary, fully represents the forcive on the body, as a whole, due to magnetic causes. This follows from the equality of action and reaction; the force on the body due to a unit pole is the opposite of that of the body on the pole.

If we wish to continue the stress into the interior of the body, surrounded on all sides by the unmagnetised medium of unit inductivity, as we must do if we wish to arrive ultimately at the mutual actions of its different parts, and how they are modified by variations of inductivity, by intrinsic magnetisation, and by electric current in the body, we may, so far as the resultant force and torque on it are concerned, do it in any way we please, provided we do not interfere with the stress outside. For the internal stress, of any type, will have no resultant force or torque on the body, and there is merely left the real external stress.

Practically, however, we should be guided by the known relations of magnetic force, induction, magnetisation, and current, and not go to work in a fanciful manner; furthermore, we should always choose the stress in such a way that if, in its expression, we take the inductivity to be unity, and the intrinsic magnetisation zero, it must reduce to the simple Maxwellian stress in air (assumed to represent ether here).

But as we do not know definitely the forcive arising from the magnetic stress in the interior of a magnet, there are several formulæ that suggest themselves as possible.

Special Kinds of Stress Formulæ statically suggested.

§ **29.** Thus, first we have the stress (183); let this be quite general, then

$$(1)\ \begin{cases} \mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2, & \qquad (184) \\ \mathbf{F} \ = \mathbf{R}\ \mathrm{div}\ \mathbf{R} + \mathrm{V}\mathbf{JR}. & \qquad (185) \end{cases}$$

Here **R** is the magnetic force of the field, not of the flux **B**. If $\mu = 1$, div **R** is the density of magnetic matter, the convergence of the intrinsic magnetisation, but not otherwise. In general, it is the density of the matter of the magnetic potential, calculated on the assumption $\mu = 1$. The force on a magnet is located in this system at its poles, whether the magnetisation be intrinsic or induced. The second term in (185) represents the force on matter bearing electric current ($\mathbf{J} = \mathrm{curl}\ \mathbf{R}$), but has to be supplemented by the first term, unless div $\mathbf{R} = 0$ at the place.

§ **30.** Next, let the stress be μ times as great for the same magnetic force, but be still of the same simple type, μ being the inductivity, which is unity outside the body, but having any positive value, which may be variable, within it. Then we shall have

$$(2)\ \begin{cases} \mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{N}\mu\mathbf{R} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}\mu\mathbf{R}, & \qquad (186) \\ \mathbf{F} \ = \mathbf{R}m + \mathrm{V}\mathbf{J}\mu\mathbf{R} - \tfrac{1}{2}\mathbf{R}^2\nabla\mu, & \qquad (187) \end{cases}$$

where $m = \mathrm{conv}\ \mu\mathbf{h}_0 = \mathrm{div}\ \mu\mathbf{R}$ is the density of magnetic matter, $\mu\mathbf{h}_0$ being the intensity of intrinsic magnetisation.

The electromagnetic force is made μ times as great for the same magnetic force; the force on an intrinsic magnet is at its poles; and there is, in addition, a force wherever μ varies, including the intrinsic magnet, and not forgetting that a sudden change in μ, as at the boundary of a magnet, has to count. This force, the third term in (187), explains the force on inductively magnetised matter. It is in the direction of most rapid decrease of μ.

§ **31.** Thirdly, let the stress be of the same simple type, but taking **H** instead of **R**, **H** being the force of the flux $\mathbf{B} = \mu\mathbf{H} = \mu\,(\mathbf{R} + \mathbf{h}_0)$, where $\mathbf{h}_0$ is as before. We now have

$$(3)\ \begin{cases} \mathbf{P}_{\mathrm{N}} = \mathbf{H}.\mathbf{NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB}, & \qquad (188) \\ \mathbf{F} \ = \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{j}_0\mathbf{B} - \tfrac{1}{2}\mathbf{H}^2\nabla\mu, & \qquad (189) \end{cases}$$

where $\mathbf{j}_0 = \mathrm{curl}\ \mathbf{h}_0$ is the distribution of fictitious electric current which produces the same induction as the intrinsic magnetisation $\mu\mathbf{h}_0$, and **J** is, as before, the real current.

It is now *quasi*-electromagnetic force that acts on an intrinsic magnet, with, however, the force due to $\nabla\mu$, since a magnet has usually large μ compared with air.

The above three stresses are all of the simple type (equal tension and perpendicular pressure), and are irrotational, unless μ be the eolotropic operator. No change is, in the latter case, needed in (186), (188), whilst in the force formulæ (187), (189), the only change needed is to give the generalised meaning to $\nabla\mu$. Thus, in (189), instead of $\mathbf{H}^2\nabla\mu$, use $2\nabla\mathrm{T}_\mu$,

or

$$\nabla_\mu\,(\mathbf{H}\mu\mathbf{H}),$$

or

$$\mathbf{i}\left(\mathbf{H}\frac{d\mu}{dx}\mathbf{H}\right)+\mathbf{j}\left(\mathbf{H}\frac{d\mu}{dy}\mathbf{H}\right)+\mathbf{k}\left(\mathbf{H}\frac{d\mu}{dz}\mathbf{H}\right),$$

or

$$(\nabla_{\mathrm{B}}-\nabla_{\mathrm{H}})\,\mathbf{HB},$$

or

$$\mathbf{i}\,(\mathbf{H}\nabla_1\mathbf{B}-\mathbf{B}\nabla_1\mathbf{H})+\mathbf{j}\,(\mathbf{H}\nabla_2\mathbf{B}-\mathbf{B}\nabla_2\mathbf{H})+\mathbf{k}\,(\mathbf{H}\nabla_3\mathbf{B}-\mathbf{B}\nabla_3\mathbf{H}),$$

showing the **i**, **j**, **k** components.

Similarly in the other cases occurring later.

The following stresses are not of the simple type, though all consist of a tension parallel to **R** or **H** combined with an isotropic pressure.

§ **32.** Alter the stress so as to locate the force on an intrinsic magnet bodily upon its magnetised elements. Add $\mathbf{R}.\mu\mathbf{h}_0\mathbf{N}$ to the stress (186), and therefore $\mu\mathbf{h}_0.\mathbf{RN}$ to its conjugate; then the divergence of the latter must be added to the **N**-component of the force (187). Thus we get, if $\mathbf{I}=\mu\mathbf{h}_0$,

$$(4)\quad\begin{cases}\mathbf{P}_{\mathrm{N}}=\mathbf{R}.\mathbf{BN}-\mathbf{N}.\tfrac{1}{2}\,\mathbf{R}\mu\mathbf{R}, & \qquad (190)\\ \mathbf{F}\;=\mathbf{I}\nabla.\mathbf{R}+\mathrm{V}\mathbf{J}\mu\mathbf{R}-\tfrac{1}{2}\mathbf{R}^2\nabla\mu. & \qquad (191)\end{cases}$$

But here the sum of the first two terms in **F** may be put in a different form. Thus,

$$\begin{aligned}\mathbf{I}\nabla.\mathbf{R}&=\mathrm{I}_1\nabla_1\mathbf{R}+\mathrm{I}_2\nabla_2\mathbf{R}+\mathrm{I}_3\nabla_3\mathbf{R}\\&=\mathbf{i}.\mathbf{I}\nabla\mathrm{R}_1+\mathbf{j}.\mathbf{I}\nabla\mathrm{R}_2+\mathbf{k}.\mathbf{I}\nabla\mathrm{R}_3.\end{aligned}$$

Also

$$\mathbf{I}\nabla\mathrm{R}_1=\mathbf{I}\nabla_1\mathbf{R}+\mathbf{I}\,(\nabla\mathrm{R}_1-\nabla_1\mathbf{R})=\mathbf{I}\nabla_1\mathbf{R}+\mathbf{i}\mathrm{V}\mathbf{JI}.$$

These bring (191) to

$$\mathbf{F}=(\mathbf{i}.\mathbf{I}\nabla_1\mathbf{R}+\mathbf{j}.\mathbf{I}\nabla_2\mathbf{R}+\mathbf{k}.\mathbf{I}\nabla_3\mathbf{R})+\mathrm{V}\mathbf{JB}-\tfrac{1}{2}\,\mathbf{R}^2\nabla\mu, \qquad (192)$$

where the first component (the bracketted part) is MAXWELL's force on intrinsic magnetisation, and the second his electromagnetic force. The third, as before, is required where μ varies.

§ **33.** To the stress (190) add $-\,\mathbf{N}.\tfrac{1}{2}\,\mathbf{RI}$, without altering the conjugate stress making

$$(5)\begin{cases}\mathbf{P}_{\mathrm{N}} = \mathbf{R.BN} - \mathbf{N}.\tfrac{1}{2}\,\mathbf{RB}, & (193)\\ \mathbf{F} = \mathbf{VJB} - \tfrac{1}{2}\,\{\mathbf{i}\,(\mathbf{R}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{R}) + \mathbf{j}\,(\mathbf{R}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{R}) + \mathbf{k}\,(\mathbf{R}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{R})\}. & (194)\end{cases}$$

$$= \mathbf{VJB} - (\nabla_{\mathrm{B}} - \nabla_{\mathrm{R}})\,\tfrac{1}{2}\mathbf{RB}.$$

This we need not discuss, as it is merely a transition to the next form.

§ **34.** To the stress (193) add $\mathbf{h}_0.\mathbf{NB}$; we then get

$$(6)\begin{cases}\mathbf{P}_{\mathrm{N}} = \mathbf{H.NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{RB}, & (195)\\ \mathbf{F} = \mathbf{VJB} + \{\mathbf{i.B}\nabla h_1 + \mathbf{j.B}\nabla h_2 + \mathbf{k.B}\nabla h_3\} & \\ \quad - \tfrac{1}{2}\,\{\mathbf{i}.\,(\mathbf{R}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{R}) + \mathbf{j}.\,(\mathbf{R}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{R}) + \mathbf{k}.\,(\mathbf{R}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{R})\}, & (196)\end{cases}$$

$$= \mathbf{VJB} + \mathbf{B}\nabla.\mathbf{h}_0 - (\nabla_{\mathrm{B}} - \nabla_{\mathrm{R}})\,\tfrac{1}{2}\mathbf{RB},$$

where h_1, h_2, h_3 are the components of $\mathbf{h}_0$.

Now if to this last stress (195) we add $-\ \mathbf{N}.\tfrac{1}{2}\mathbf{h}_0\mathbf{B}$, we shall come back to the third stress, (188), of the simple type.

Perhaps the most instructive order in which to take the six stresses is (1), (2), (4), (5), (6), and (3); merely adding on to the force, in passing from one stress to the next, the new part which the alteration in the stress necessitates.

To the above we should add MAXWELL's general stress, which is

$$(7)\begin{cases}\mathrm{P}_{\mathrm{N}} = \mathbf{R.NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2, & (197)\\ \mathbf{F} = \mathbf{VJB} + \{\mathbf{i.I}\nabla_1\mathbf{R} + \mathbf{j.I}\nabla_2\mathbf{R} + \mathbf{k.I}\nabla_3\mathbf{R}\} & \\ \qquad + \{\mathbf{i.M}\nabla_1\mathbf{R} + \mathbf{j.M}\nabla_2\mathbf{R} + \mathbf{k.M}\nabla_2\mathbf{R}\}, & (198)\end{cases}$$

$$= \mathbf{VJB} + \nabla_{\mathrm{R}}[\mathbf{R}\,(\mathbf{I} + \mathbf{M})],$$

where $\mathbf{M} = (\mu - 1)\,\mathbf{R} =$ intensity of induced magnetisation. There is a good deal to be said against this stress; some of which later.

Remarks on MAXWELL's *General Stress.*

§ **35.** All the above force formulæ refer to the unit volume; whenever, therefore, a discontinuity in the stress occurs at a surface, the corresponding expression per unit surface is needed; *i.e.*, in making a special application, for it is wasted labour else. It might be thought that as MAXWELL gives the force (198), and in his treatise usually gives surface expressions separately, so none is required in the case of this his force system (198). But this formula will give entirely erroneous results if carried out literally. It forms no exception to the rule that all the expressions require surface additions.

MAXWELL's general stress has the apparent advantage of simplicity. It merely requires an alteration in the tension parallel to **R**, from $\mathbf{R}^2$ to **RB**, whilst the lateral pressure remains $\tfrac{1}{2}\mathbf{R}^2$, when we pass from unmagnetised to magnetised matter. The force to which it gives rise is also apparently simple, being merely the sum of two forces, one the electromagnetic, $\mathrm{V}\mathbf{J}\mathbf{B}$, the other a force on magnetised matter whose i component is $(\mathbf{I} + \mathbf{M})\nabla_1\mathbf{R}$, both per unit volume, the latter being accompanied (in case of eolotropy) by a torque. Now $\mathbf{I}$ is the intrinsic and $\mathbf{M}$ the induced magnetisation, so the force is made irrespective of the proportion in which the magnetisation exists as intrinsic or induced. In fact, MAXWELL'S "magnetisation" is the sum of the two without reservation or distinction. But to unite them is against the whole behaviour of induced and intrinsic magnetisation in the electromagnetic scheme of MAXWELL, as I interpret it. Intrinsic magnetisation (using Sir W. THOMSON'S term) should be regarded as impressed ($\mathbf{I} = \mu\mathbf{h}_0$, where $\mathbf{h}_0$ is the equivalent impressed magnetic force); on the other hand, "induced" magnetisation depends on the force of the field $\{\mathbf{M} = (\mu - 1)\,\mathbf{R}\}$. Intrinsic magnetisation keeps up a field of force. Induced magnetisation is kept up by the field. In the circuital law $\mathbf{I}$ and $\mathbf{M}$ therefore behave differently. There may be absolutely no difference whatever between the magnetisation of a molecule of iron in the two cases of being in a permanent or a temporary magnet. That, however, is not in question. We have no concern with molecules in a theory which ignores molecules, and whose element of volume must be large enough to contain so many molecules as to swamp the characteristics of individuals. It is the resultant magnetisation of the whole assembly that is in question, and there is a great difference between its nature according as it disappears on removal of an external cause, or is intrinsic. The complete amalgamation of the two in MAXWELL'S formula must certainly, I think, be regarded as a false step.

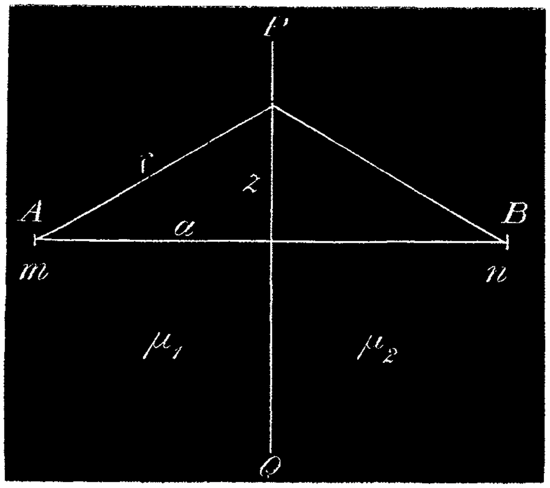
We may also argue thus against the probability of the formula. If we have a system of electric current in an unmagnetisable ($\mu = 1$) medium, and then change μ everywhere in the same ratio, we do not change the magnetic force at all, the induction is made μ times as great, and the magnetic energy μ times as great, and is similarly distributed. The mechanical forces are, therefore, μ times as great, and are similarly distributed. That is, the translational force in the $\mu = 1$ medium, or $\mathrm{V}\mathbf{J}\mathbf{R}$, becomes $\mathrm{V}\mathbf{J}\mu\mathbf{R}$ in the second case in which the inductivity is μ, without other change. But there is no force brought in on magnetised matter *per se*.

Similarly, if in the $\mu = 1$ medium we have intrinsic magnetisation $\mathbf{I}$, and then alter μ in any ratio everywhere alike, keeping $\mathbf{I}$ unchanged, it is now the induction that remains unaltered, the magnetic force becoming μ^{-1} times, and the energy μ^{-1} times the former values, without alteration in distribution (referring to permanent states, of course). Again, therefore, we see that there is no translational force brought in on magnetised matter merely because it is magnetised.

Whatever formula, therefore, we should select for the stress function, it would certainly not be MAXWELL'S, for cumulative reasons. When, some six years ago, I had occasion to examine the subject of the stresses, I was unable to arrive at any very definite results, except outside of magnets or conductors. It was a perfectly indeterminate problem to find the magnetic stress inside a body from the existence of a known, or highly probable, stress outside it. All one could do was to examine the consequences of assuming certain stresses, and to reject those which did not work well. After going into considerable detail, the only two which seemed possible were the second and third above (those of equations (186) and (188) above). As regards the seventh (MAXWELL's stress equation (198) above), the apparent simplicity produced by the union of intrinsic and induced magnetisation, turned out, when examined into its consequences, to lead to great complication and unnaturalness. This will be illustrated in the following example, a simple case in which we can readily and fully calculate all details by different methods, so as to be quite sure of the results we ought to obtain.

A worked-out Example to Exhibit the Forcives contained in Different Stresses.

§ **36**. Given a fluid medium of inductivity μ_1, in which is an intrinsic magnet of the same inductivity. Calculate the attraction between the magnet and a large solid mass of different inductivity μ_2. Here it is only needful to calculate the force on a single pole, so let the magnet be infinitely thin and long, with one pole of strength m at distance a from the medium μ_2, which may have an infinitely extended plane boundary. By placing a fictitious pole of suitable strength at the optical image in the second medium of the real pole in the first, we may readily obtain the solution.



Let PQ be the interface and the real pole be at A and its image at B. We have first to calculate the distribution of **R**, magnetic force, in both media due to the pole m, as disturbed by the change of inductivity. We have div $\mu_1\mathbf{R}_1 = m$ in the first medium, and div $\mu_2\mathbf{R}_2 = 0$ in the second, therefore **R** has divergence only on the interface. Let σ be the surface density of the fictitious interfacial matter to correspond; its force goes symmetrically both ways; the continuity of the normal induction therefore gives, at distance r from A, the condition

$$\mu_1\left(\frac{ma}{4\pi\mu_1 r^3} - \tfrac{1}{2}\sigma\right) = \mu_2\left(\frac{ma}{4\pi\mu_1 r^3} + \tfrac{1}{2}\sigma\right), \qquad (199)$$

because $m/4\pi\mu_1 r^2$ is the tensor of the magnetic force due to m in the μ_1 medium when of infinite extent. Therefore

$$\sigma = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{ma}{2\pi\mu_1 r^3}. \qquad (200)$$

The magnetic potential Ω, such that $\mathbf{B} = -\nabla\Omega$ is the polar force in either region, is therefore the potential of m/μ_1 at A and of σ over the interface.

But if we put matter n at the image B, of amount

$$n = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m}{\mu_1}, \qquad (201)$$

the normal component of $\mathbf{B}_1$ on the μ_1 side due to n and the pole m will be

$$\frac{ma}{4\pi\mu_1 r^3} - \frac{na}{4\pi r^3} = \frac{ma}{4\pi\mu_1 r^3} - \tfrac{1}{2}\sigma, \qquad (202)$$

the same value as before; the force $\mathbf{B}_1$ on the μ_1 side is, therefore, the same as that due to matter m/μ_1 at A and matter n at B; whilst on the μ_2 side the force $\mathbf{B}_2$ is that due to matter m/μ_1 at A and matter n also at A, that is, to matter $\dfrac{2m}{\mu_1 + \mu_2}$ at A. Thus in the μ_2 medium the force $\mathbf{B}_2$ is radial from A as if there were no change of inductivity, though altered in intensity.

The repulsion between the pole m and the solid mass is not the repulsion between the matters m/μ_1 and n of the potential, but is

$$\begin{aligned} &= m \times \text{magnetic force at A due to matter } n \text{ at B,} \\ &= n \times \text{magnetic force at B due to matter } m/\mu_1 \text{ at A,} \\ &= \frac{mn}{4\pi(2a)^2} = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m^2}{4\pi\mu_1(2a)^2}, \qquad (203) \end{aligned}$$

becoming an attraction when $\mu_2 > \mu_1$, making n negative. When $\mu_2 = 0$, the repulsion is

$$\frac{m^2}{4\pi\mu_1(2a)^2};$$

when $\mu_2 = \infty$, it is turned into an attraction of equal amount.

Similarly, if we consider the attraction to be the resultant force between m and the interfacial matter σ, we shall get the same result by

$$\Sigma \frac{\sigma ma}{4\pi r^3}, \qquad (204)$$

the quantity summed (over the interface) being $\sigma \times$ normal component of magnetic force due to matter m in a medium of unit inductivity, or the normal component of induction due to m in its own medium. For this is

$$\int \frac{ma}{4\pi r^3} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{ma}{2\pi\mu_1 r^3} 2\pi r\, dr = \frac{m^2 a^2}{4\pi\mu_1} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \int \frac{dr}{r^5} = (203) \text{ again.}$$

Another way is to calculate the variation of energy made by displacing either the pole m or the μ_2 mass. The potential energy is expressed by

$$\tfrac{1}{2}(\mathrm{P} + p)\, m = \tfrac{1}{2}\mathrm{P}m + \tfrac{1}{2}\Sigma \mathrm{P}\sigma\mu, \quad \ldots \ldots \ldots \quad (205)$$

where $\mathrm{P} = m/4\pi\mu_1 r$ and $p = \Sigma\, \sigma/4\pi r$, the potentials of matter m/μ_1 and σ, where r is the distance from m or from σ to the point where P and p are reckoned.

The value of the second part in (205), depending upon σ, comes to

$$\tfrac{1}{2} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m^2}{4\pi\mu_1 . 2a}, \quad \ldots \ldots \ldots \quad (206)$$

and its rate of decrease with respect to a expresses the repulsion between the pole and the μ_2 region. This gives (203) again.

A fourth way is by means of the *quasi*-electromagnetic force on fictitious interfacial electric current, instead of matter, the current being circular about the axis of symmetry AB. The formula for the attraction is

$$\Sigma \mathrm{V}\ \mathrm{curl}\ \mathbf{B}.\mathbf{R}_0, \quad \ldots \ldots \ldots \quad (207)$$

if $\mathbf{R}_0$ be the radial magnetic force from m in its own medium, tensor $m/4\pi\mu_1 r^2$. Here the curl of $\mathbf{B}$ is represented by the interfacial discontinuity in the tangential induction, or

$$\frac{2zm}{4\pi r^3} \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}.$$

Also the tangential component of $\mathbf{R}_0$ is $mz/4\pi\mu_1 r^3$. Therefore the repulsion is

$$\int \frac{2mz}{4\pi r^3} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{mz}{4\pi\mu_1 r^3} 2\pi r\, dr = \frac{m^2}{4\pi\mu_1} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \int_a^\infty \frac{r^2 - a^2}{r^5}\, dr = \frac{m^2}{4\pi\mu_1} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{1}{4a^2}, \quad . \quad (208)$$

as before, equation (203). This method (207) is analogous to (204).

§ **37.** There are several other ways of representing the attraction, employing fictitious matter and current; but now let us change the method, and observe how the attraction between the magnetic pole and the iron mass is accounted for by a stress distribution, and its space-variation. The best stress is the third, equation (188) § 31. Applying this, we have simply a tension of magnitude $\tfrac{1}{2}\mu_1 \mathrm{R}_1^2 = \mathrm{T}_1$ in the first medium and $\frac{1}{2}\mu_2 R_2{}^2 = T_2$ in the second, parallel to $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively, each combined with an equal lateral pressure, so that the tensor of the stress vector is constant.

But, so far as the attraction is concerned, we may ignore the stress in the second medium altogether, and consider it as the $\Sigma \mathbf{P}_N$ of the stress-vector in the first medium over the surface of the second medium. The tangential component summed has zero resultant; the attraction is therefore the sum of the normal components, or $\Sigma T_1 \cos 2\theta_1$, where θ_1 is the angle between $\mathbf{R}_1$ and the normal. This is the same as $\Sigma \frac{1}{2}\mu_1 (R_N{}^2 - R_T{}^2)$, if R_N and R_T are the normal and tangential components of $\mathbf{R}_1$; or

$$\int_a^\infty 2\pi r\, dr\, \tfrac{1}{2}\mu_1 \left[\left(\frac{ma}{4\pi\mu_1 r^3}\, \frac{2\mu_2}{\mu_1 + \mu_2}\right)^2 - \left(\frac{mz}{4\pi\mu_1 r^3}\, \frac{2\mu_1}{(\mu_1 + \mu_2)}\right)\right]; \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

which on evaluation gives the required result (203).

But this method does not give the true distribution of translational force due to the stresses. In the first medium there is no translational force, except on the magnet. Nor is there any translational force in the second μ_2 medium. But at the interface, where μ changes, there is the force $- \frac{1}{2}R^2\nabla\mu$ per unit volume, and this is represented by the stress-difference at the interface. It is easily seen that the tangential stress-difference is zero, because

$$T \sin 2\theta = \mu R_N R_T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

and both the normal induction and the tangential magnetic force are continuous. The real force is, therefore, the difference of the normal components of the stress-vectors, and is, therefore, normal to the interface. This we could conclude from the expression $- \frac{1}{2}R^2\nabla\mu$. But since the resultant of the interfacial stress in the second medium is zero, we need not reckon it, so far as the attraction of the pole is concerned. The normal traction on the interface, due to both stresses, is of amount

$$\frac{m^2}{8\pi^2 r^6}\, \frac{\mu_2 - \mu_1}{(\mu_1 + \mu_2)^2}\left(r^2 + a^2\,\frac{\mu_2 - \mu_1}{\mu_1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

per unit area. Summed up, it gives (203) again.

That (211) properly represents the force $- \frac{1}{2}R^2\nabla\mu$ when μ is discontinuous, we may also verify by supposing μ to vary continuously in a very thin layer, and then proceed to the limit.

The change from an attraction to a repulsion as μ_2 changes from being greater to being less than μ_1, depends upon the relative importance of the tensions parallel to the magnetic force and the lateral pressures operative at different parts of the interface. In the extreme case of $\mu_2 = 0$, we have $\mathbf{R}_1$ tangential, with, therefore, a pressure everywhere. For the other extreme, $\mathbf{R}_1$ is normal, and there is a pull on the second medium everywhere. When μ_2 is finite there is a certain circular area on the interface within which the translational force due to the stress in the medium containing the pole m is towards that medium, whilst outside it the force is the other way. But when both stresses are allowed for, we see that when $\mu_2 > \mu_1$ the pull is towards the first medium in all parts of the interface, and that this becomes a push in all parts when $\mu_1 > \mu_2$.

A definite Stress only obtainable by Kinetic Consideration of the Circuital Equations and Storage and Flux of Energy.

§ **38**. We see that the stress considered in the last paragraph gives a rationally intelligible interpretation of the attraction or repulsion. The same may be said of other stresses than that chosen. But the use of MAXWELL's stress, or any stress leading to a force on inductively magnetised matter as this stress does, leads us into great difficulties. By (198) we see that there is first a bodily force on the whole of the μ_2 medium, because it is magnetised, unless $\mu_2 = 1$. When summed up, the resultant does not give the required attraction. For, secondly, the μ_1 medium is also magnetised, unless $\mu_1 = 1$, and there is a bodily force throughout the whole of it. When this is summed up (not counting the force on the magnet), its resultant added on to the former resultant still does not make up the attraction (*i.e.*, equivalently, the force on the magnet). For, thirdly, the stress is discontinuous at the interface (though not in the same manner as in the last paragraph). The resultant of this stress-discontinuity, added on to the former resultants, makes up the required attraction. It is unnecessary to give the details relating to so improbable a system of force.

Our preference must naturally be for a more simple system, such as the previously considered stress. But there is really no decisive settlement possible from the theoretical statical standpoint, and nothing short of actual experimental determination of the strains produced and their exhaustive analysis would be sufficient to determine the proper stress-function. But when the subject is attacked from the dynamical standpoint, the indeterminateness disappears. From the two circuital laws of variable states of electric and magnetic force in a moving medium, combined with certain distributions of stored energy, we are led to just one stress-vector, viz. (136). It is, in the magnetic case, the same as (188); that is, it reduces to the latter when the medium is kept at rest, so that $\mathbf{J}_0$ and $\mathbf{G}_0$ become $\mathbf{J}$ and $\mathbf{G}$.

It is of the simple type in case of isotropy (constant tensor), but is a rotational stress in general, as indeed are all the statically probable stresses that suggest themselves. The translational force due to it being divisible conveniently into (*a*) the electromagnetic force on electric current, (*b*) the ditto on the fictitious electric current taking the place of intrinsic magnetisation, (*c*) force depending upon space-variation of μ; we see that the really striking part is (*b*). Of all the various ways of representing the forcive on an intrinsic magnet it is the most extreme. The magnetic "matter" does not enter into it, nor does the distribution of magnetisation; it is where the intrinsic force $\mathbf{h}_0$ has curl that the translational force operates, usually on the sides of a magnet. From actual experiments with bar magnets, needles, &c., one would naturally prefer to regard the polar règions as the seat of translational force. But the equivalent forcive $\Sigma \mathbf{j}_0 \mathbf{B}$ has one striking recommendation (apart from the dynamical method of deducing it), viz., that the induction of an intrinsic magnet is determined by curl $\mathbf{h}_0$, not by $\mathbf{h}_0$ itself; and this, I have shown, is true when $\mathbf{h}_0$ is imagined to vary, the whole varying states of the fluxes **B**, **D**, **C** due to impressed force being determined by the curls of $\mathbf{e}_0$ and $\mathbf{h}_0$. which are the sources of the disturbances (though not of the energy).

The rotational peculiarity in eolotropic substances does not seem to be a very formidable objection. Are they not solid?

As regards the assumed constancy of μ, a more complete theory must, to be correct, reduce to one assuming constancy of μ, because, as Lord RAYLEIGH* has shown, the assumed law has a limited range of validity, and is therefore justifiable as a preparation for more complete views. Theoretical requirements are not identical with those of the practical engineer.

But, for quite other reasons, the dynamically determined stress might be entirely wrong. Electric and magnetic "force" and their energies are facts. But it is the total of the energies in concrete cases that should be regarded as the facts, rather then their distribution; for example, that, as Sir W. THOMSON proved, the "mechanical value" of a simple closed current C is $\frac{1}{2}LC^2$, where L is the inductance of the circuit (coefficient of electromagnetic capacity), rather than that its distribution in space is given by $\frac{1}{2}\mathbf{HB}$ per unit volume. Other distributions may give the same total amount of energy. For example, the energy of distortion of an elastic solid may be expressed in terms of the square of the rotation and the square of the expansion, if its boundary be held at rest; but this does not correctly localise the energy. If, then, we choose some other distribution of the energy for the same displacement and induction, we should find quite a different flux of energy. But I have not succeeded in making any other arrangement than MAXWELL's work practically, or without an immediate introduction of great obscurities. Perhaps the least certain part of MAXWELL's scheme, as modified by myself, is the estimation of magnetic energy as $\frac{1}{2}\mathbf{HB}$ in intrinsic magnets, as well as outside them, that is, by $\frac{1}{2}\mathbf{B}\mu^{-1}\mathbf{B}$, however **B** may be caused. Yet, only in this way are thoroughly consistent results apparently obtainable when the electromagnetic field is considered comprehensively and dynamically.

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APPENDIX.

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Extension of the Kinetic Method of arriving at the Stresses to cases of Non-linear Connection between the Electric and Magnetic Forces and the Fluxes. Preservation of Type of the Flux of Energy Formula.

§ **39**. It may be worth while to give the results to which we are led regarding the stress and flux of energy when the restriction of simple proportionality between "forces" and "fluxes," electric and magnetic respectively, is removed. The course to be followed, to obtain an interpretable form of the equation of activity, is sufficiently clear in the light of the experience gained in the case of proportionality.

First assume that the two circuital laws (89) and (90), or the two in (93), hold good generally, without any initially stated relation between the electric force **E** and its associated fluxes **C** and **D**, or between the magnetic force **H** and its associated fluxes **K** and **B**. When written in the form most convenient for the present application, these laws are

$$\mathrm{curl}\,(\mathbf{H}-\mathbf{h}_0)=\mathbf{J}_0=\mathbf{C}+\frac{\partial\mathbf{D}}{\partial t}+\left(\mathbf{D}\,\mathrm{div}\,\mathbf{q}-\mathbf{D}\nabla.\mathbf{q}\right), \quad . \ . \ . \ . \ . \ . \ . \quad (212)$$

$$-\,\mathrm{curl}\,(\mathbf{E}-\mathbf{e}_0)=\mathbf{G}_0=\mathbf{K}+\frac{\partial\mathbf{B}}{\partial t}+\left(\mathbf{B}\,\mathrm{div}\,\mathbf{q}-\mathbf{B}\nabla.\mathbf{q}\right). \quad . \ . \ . \ . \ . \ . \ . \quad (213)$$

Now derive the equation of activity in the manner previously followed, and arrange it in the particular form

$$\begin{aligned}&\mathbf{e}_0\mathbf{J}_0+\mathbf{h}_0\mathbf{G}_0+\mathrm{conv}\ \mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0)\\&\qquad=(\mathbf{EC}+\mathbf{HK})+\left(\mathbf{E}\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\frac{\partial\mathbf{B}}{\partial t}\right)+\left(\mathbf{E.D}\nabla.\mathbf{q}-\mathbf{ED}\,\mathrm{div}\,\mathbf{q}\right)+\left(\mathbf{H.B}\nabla.\mathbf{q}-\mathbf{HB}\,\mathrm{div}\,\mathbf{q}\right), \quad . \quad (214)\end{aligned}$$

which will best facilitate interpretation.

Although independent of the relation between **E** and **D**, &c., of course the dimensions must be suitably chosen so that this equation may really represent activity per unit volume in every term.

Now, guided by the previous investigation, we can assume that $(\mathbf{e}_0\mathbf{J}_0+\mathbf{h}_0\mathbf{G}_0)$ represents the rate of supply of energy from intrinsic sources, and also that $\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0)$, which is a flux of energy independent of $\mathbf{q}$, is the correct form in general. Also, if there be no other intrinsic sources of energy than $\mathbf{e}_0$, $\mathbf{h}_0$, and no other fluxes of energy besides that just mentioned except the convective flux and that due to the stress, the equation of activity should be representable by

$$
\begin{aligned}
&(\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0) + \text{conv}\left[\mathrm{V}(\mathbf{E} - \mathbf{e}_0)(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}(\mathrm{U} + \mathrm{T})\right] \\
&= (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{Fq} + \text{conv}\,\mathbf{Q}_q q \\
&= (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \Sigma\mathbf{Q}\nabla q, \qquad (215)
\end{aligned}
$$

where $\mathbf{Q}$ is the conjugate of the stress vector, $\mathbf{F}$ the translational force, and Q, U, and T the rate of waste and the stored energies, whatever they may be.

Comparing with the preceding equation (214), we see that we require

$$
\begin{aligned}
\Sigma\mathbf{Q}\nabla q = (\mathrm{Q} - \mathbf{EC} - \mathbf{HK}) + \left(\frac{\partial \mathrm{U}}{\partial t} - \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}\right) + \left(\frac{\partial \mathrm{T}}{\partial t} - \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right) \\
+ \left[\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - (\mathbf{ED} - \mathrm{U})\,\text{div}\,\mathbf{q}\right] + \left[\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - (\mathbf{HB} - \mathrm{T})\,\text{div}\,\mathbf{q}\right]. \qquad (216)
\end{aligned}
$$

Now assume that there is no waste of energy except by conduction; then

$$\mathrm{Q} = \mathbf{EC} + \mathbf{HK}. \qquad (217a)$$

Also assume that

$$\frac{\partial \mathrm{U}}{\partial t} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}, \qquad \frac{\partial \mathrm{T}}{\partial t} = \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}. \qquad (217b)$$

These imply that the relation between $\mathbf{E}$ and $\mathbf{D}$ is, for the same particle of matter, an invariable one, and that the stored electric energy is

$$\mathrm{U} = \int_0^{\mathbf{D}} \mathbf{E}\,d\mathbf{D}, \qquad (218)$$

where $\mathbf{E}$ is a function of $\mathbf{D}$. Similarly,

$$\mathrm{T} = \int_0^{\mathbf{B}} \mathbf{H}\,d\mathbf{B}, \qquad (219)$$

expresses the stored magnetic energy, and $\mathbf{H}$ must be a definite function of $\mathbf{B}$.

On these assumptions, (216) reduces to

$$\Sigma\mathbf{Q}\nabla q = \left[\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - (\mathbf{ED} - \mathrm{U})\,\text{div}\,\mathbf{q}\right] + \left[\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - (\mathbf{HB} - \mathrm{T})\,\text{div}\,\mathbf{q}\right], \qquad (220)$$

from which the stress-vector follows, namely,

$$\mathbf{P}_{\mathrm{N}} = \left[\mathbf{E}.\mathbf{DN} - \mathbf{N}(\mathbf{ED} - \mathrm{U})\right] + \left[\mathbf{H}.\mathbf{BN} - \mathbf{N}(\mathbf{HB} - \mathrm{T})\right]. \qquad (221)$$

Or,

$$\mathbf{P}_{\mathrm{N}} = (\mathrm{V}\mathbf{D}\mathrm{V}\mathbf{EN} + \mathbf{N}\mathrm{U}) + (\mathrm{V}\mathbf{B}\mathrm{V}\mathbf{HN} + \mathbf{N}\mathrm{T}). \qquad (222)$$

Thus, in case of isotropy, the stress is a tension U parallel to $\mathbf{E}$ combined with a lateral pressure ($\mathbf{ED}$ — U); and a tension T parallel to $\mathbf{H}$ combined with a lateral pressure ($\mathbf{HB}$ — T).

The corresponding translational force is

$$
\begin{aligned}
\mathbf{F} &= \mathbf{E}\,\text{div}\,\mathbf{D} + \mathbf{D}\nabla.\mathbf{E} - \nabla(\mathbf{ED} - \mathrm{U}) \\
&\quad + \mathbf{H}\,\text{div}\,\mathbf{B} + \mathbf{B}\nabla.\mathbf{H} - \nabla(\mathbf{HB} - \mathrm{T}), \qquad (223)
\end{aligned}
$$

which it is unnecessary to put in terms of the currents.

Exchange **E** and **D**, and **H** and **B**, in (221) or (222) to obtain the conjugate vector $\mathbf{Q}_{\mathrm{N}}$; from which we obtain the flux of energy due to the stress,

$$\begin{aligned} -q\mathbf{Q}_q &= \mathbf{D}.\mathbf{E}\mathbf{q} - \mathbf{q}\,(\mathbf{E}\mathbf{D} - \mathrm{U}) + \mathbf{B}.\mathbf{H}\mathbf{q} - \mathbf{q}\,(\mathbf{H}\mathbf{B} - \mathrm{T}) \\ &= \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{D}\mathbf{q} + \mathrm{V}\mathbf{H}\mathrm{V}\mathbf{B}\mathbf{q} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}), \quad \ldots\ldots\ldots \quad (224) \end{aligned}$$

or

$$-q\mathbf{Q}_q = \mathrm{V}\mathbf{e}\mathbf{H} + \mathrm{V}\mathbf{E}\mathbf{h} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}), \quad \ldots\ldots\ldots\ldots \quad (225)$$

where **e** and **h** are the motional electric and magnetic forces, of the same form as before (88) and (91); so that the complete form of the equation of activity, showing the fluxes of energy and their convergence, is

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 + \mathrm{conv}\left[\mathrm{V}(\mathbf{E}-\mathbf{e}_0)(\mathbf{H}-\mathbf{h}_0) + \mathbf{q}(\mathrm{U}+\mathrm{T})\right] - \mathrm{conv}\left[\mathrm{V}\mathbf{e}\mathbf{H} + \mathrm{V}\mathbf{E}\mathbf{h} + \mathbf{q}(\mathrm{U}+\mathrm{T})\right] = \mathbf{F}\mathbf{q} + (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}), \quad (226)$$

where **F** has the above meaning.

There is thus a remarkable preservation of form as compared with the corresponding formulæ when there is proportionality between force and flux. For we produce harmony by means of a POYNTING flux of identical expression and a flux due to the stress, which is also of identical expression, although U and T now have a more general meaning, of course.*

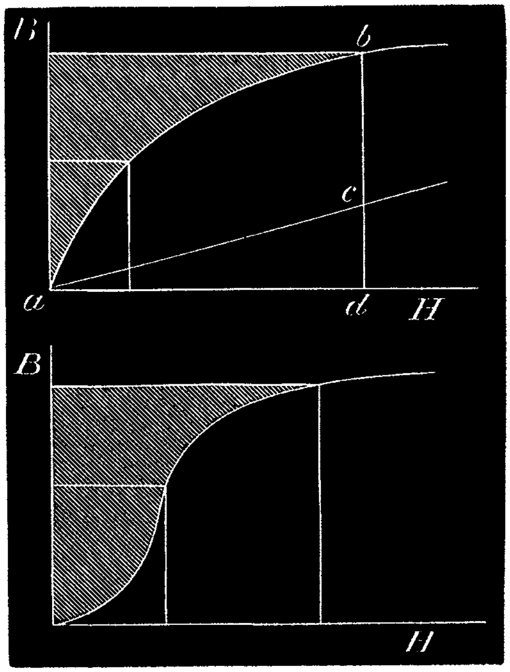
Example of the above, and Remarks on Intrinsic Magnetisation when there is Hysteresis.

§ **40.** In the stress-vector itself (for either the electric or the magnetic stress) the relative magnitude of the tension and the lateral pressure varies unless the curve

---

* As the investigation in this Appendix has some pretensions to generality, we should try to settle the amount it is fairly entitled to. No objection is likely to be raised to the use of the circuital equations (212), (213), with the restriction of strict proportionality between **E** and **H** and the fluxes **D** and **B**, or **C** and **K** entirely removed; nor to the estimation of $\mathbf{J}_0$ and $\mathbf{G}_0$ as the "true" currents; nor to the use of the same form of flux of electromagnetic energy when the medium is stationary. For these things are obviously suggested by the preceding investigations, and their justification is in their being found to continue to work, which is the case. But the use in the text of language appropriate to linear functions, which arose from the notation, &c., being the same as before, is unjustifiable. We may, however, remove this misuse of language, and make the equation (226), showing the flux of energy, rest entirely upon the two circuital equations. In fact, if we substitute in (226) the relations (217*a*), (217*b*), it becomes merely a particular way of writing (214).

It is, therefore, to (217*a*), (217*b*) that we should look for limitations. As regards (217*a*), there does not seem to be any limitation necessary. That is, there is no kind of relation imposed between **E** and **C**, and **H** and **K**. This seems to arise merely from Q meaning energy wasted for good, and having no further entry into the system. But as regards (217*b*), the case is different. For it seems necessary, in order to exclude terms corresponding to $\mathbf{E}(\partial c/\partial t)\mathbf{E}$ and $\mathbf{H}(\partial \mu/\partial t)\mathbf{H}$ in the linear theory, when there is connecting the force and the induction be a straight line. Thus, if the curve be of the type shown in the first figure, the shaded area will represent the stored energy and the tension, and the remainder of the rectangle will represent the lateral pressure. They are equal when **H** is small; later on the pressure preponderates, and more and more so the bigger **H** becomes.



But if the curve be of the type shown in the second figure, then, after initial equality the tension preponderates; though, later on, when **H** is very big the pressure preponderates.

To obtain an idea of the effect, take the concrete example of an infinitely long rod, uniformly axially inductized by a steady current in an overlapping solenoid, and consider the forcive on the rod. Here both **H** and **B** are axial or longitudinal; and so, by equation (223), the translational force would be a normal force on the surface of the rod, acting outwards, of amount

$$(\mathbf{HB} - \mathrm{T}) - \tfrac{1}{2}\mathbf{H}_0\mathbf{B}_0$$

per unit area; this being the excess of the lateral pressure in the rod over $\frac{1}{2}\mathrm{H}_0\mathrm{B}_0$, the lateral pressure just outside it.

In case of proportionality of force to flux, the first pressure is $\frac{1}{2}$HB, and if there is no intrinsic magnetisation H and H_0 are equal. The outward force is therefore positive for paramagnetic, and negative for diamagnetic substances, and the result would be lateral expansion or contraction, since the infinite length would prevent elongation.

rotation, that **E** and **D** should be parallel, and likewise **H** and **B**. At any rate, if such terms be allowed, some modification may be required in the subsequent reckoning of the mechanical force. In other respects, it is merely implied by (217*b*) that **E** and **D** are definitely connected, likewise **H** and **B**, so that there is no waste of energy other than that expressed by Q.

But if the curve in the rod be of the type of the first figure, and the straight line *ac* be the air curve to correspond, it is the area *abc* that now represents the outward force per unit area when the magnetic force has the value *ad*. If the straight line can cross the curve *ab*, we see that by sufficiently increasing H we can make the external air pressure preponderate, so that the rod, after initially expanding, would end by contracting.

If the rod be a ring of large diameter compared with its thickness, the forcive would be approximately the same, viz., an outward surface force equal to the difference of the lateral pressures in the rod and air. The result would then be elongation, with final retraction when the external pressure came to exceed the internal.

BIDWELL found a phenomenon of this kind in iron, but it does not seem possible that the above supposititious case is capable of explaining it, though of course the true explanation may be in some respects of a similar nature. But the circumstances are not the same as those supposed. The assumption of a definite connexion between H and B, and elastic storage of the energy T, is very inadequate to represent the facts of magnetisation of iron, save within a small range.

Magneticians usually plot the curve connecting $H - h_0$ and B, not between H and B, or which would be the same, between $H - h_0$ and $B - B_0$, where B_0 is the intrinsic magnetisation. Now when an iron ring is subjected to a given gaussage (or magneto motive force), going through a sequence of values, there is no definite curve connecting $H - h_0$ and B, on account of the intrinsic magnetisation. But, with proper allowance for h_0, it might be that the resulting curve connecting H and B in a given specimen, would be approximately definite, at any rate, far more so than that connecting $H - h_0$ and B. Granting perfect definiteness, however, there is still insufficient information to make a theory. The energy put into iron is not wholly stored; that is, in increasing the coil current we increase B_0 as well as B, and in doing so dissipate energy; and although we know, by EWING's experiments, the amount of waste in cyclical changes, it is not so clear what the rate of waste is at a given moment. There is also the further peculiarity that the energy of the intrinsic magnetisation at a given moment, though apparently locked up, and really locked up temporarily, however loosely it may be secured, is not wholly irrecoverable, but comes into play again when H is reversed. Now it may be that the energy of the intrinsic magnetisation plays, in relation to the stress, an entirely different part from that of the elastic magnetisation. It is easy to make up formulæ to express special phenomena, but very difficult to make a comprehensive theory.

But in any case, apart from the obscurities connected with iron, it is desirable to be apologetic in making any application of MAXWELL's stresses or similar ones to practice when the actual strains produced are in question, bearing in mind the difficulty of interpreting and harmonising with MAXWELL's theory the results of KERR, QUINCKE, and others.