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Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$.

a) $R_1 = \{(1, 2), (2, 4), (3, 4), (4, 5)\}$ A function from $A$ to $B$

b) $R_2 = \{(1, 2), (2, 4), (2, 5), (4, 5)\}$ Not a function

c) $R_3 = \{(1, 2), (2, 4), (4, 5)\}$

d) $R_4 = A \times B$ Not a function

## Notation

We write $f(a) = b$ when $(a, b) \in f$ where $f$ is a function.
We say that $b$ is the *image* of $a$ under $f$, and $a$ is a *preimage* of $b$.

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Let $A = \{1, 2, 3, 4\}$ and $B = \{v, w, x, y, z\}$.
Let $f : A \rightarrow B$ be $f = \{(1, w), (2, y), (3, y), (4, z)\}$.

a) The domain of $f$ is $\{1, 2, 3, 4\}$.

b) The codomain of $f$ is $\{v, w, x, y, z\}$.

c) The range of $f$ is

$$\begin{aligned} f(A) &= f(\{1, 2, 3, 4\}) \\ &= \{f(1), f(2), f(3), f(4)\} \\ &= \{w, y, y, z\} = \{w, y, z\} \end{aligned}$$

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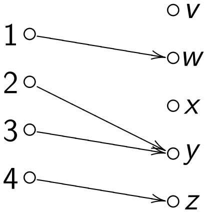

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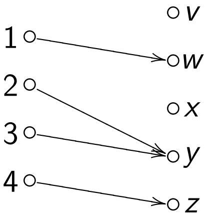

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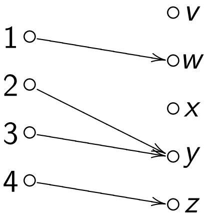

Let $P = \{Justin, Britney, Joey, Christina, Jessica, Kevin, Kelly, Nick, Sarah\}$ and let $S = \{man, woman\}$.

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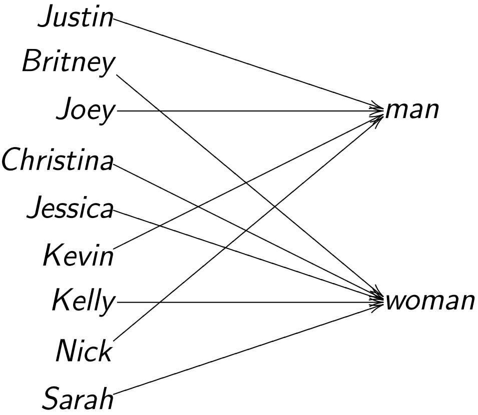

## Theorem 2

*For any sets $A$ and $B$, the number of functions from $A$ to $B$ is $|B|^{|A|}$*

**Proof.** Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. A function $f$ assigns each element $a_i$ of $A$ an element $b_j = f(a_i)$ of $B$; there are $m$ possibilities for each element of $A$, hence by the rule of product, we have $\underbrace{m \cdot m \cdot \ldots \cdot m}_{n} = m^n = |B|^{|A|}$ possible functions. □

Let $A = \{1, 2, 3, 4\}$ and $B = \{v, w, x, y, z\}$.

a) There are ? relations from $A$ to $B$.

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Let $A = \{1, 2, 3, 4\}$ and $B = \{v, w, x, y, z\}$.

a) There are $2^{|A||B|} = 2^{20} = 1,048,576$ relations from $A$ to $B$.

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a) There are $2^{|A||B|} = 2^{20} = 1,048,576$ relations from $A$ to $B$.

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## Definition 3

A function $f : A \to B$ is *one-to-one* or *injective* if each element of $B$ appears at most once as the image of an element of $A$.
A function $f : A \to B$ is *onto* or *surjective* if $f(A) = B$, that is, each element of $B$ appears at least once as the image of an element of $A$.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(x) = 3x + 7$.

$f = \{\ldots, (-3, -2), (-2, 1), (-1, 4), (0, 7), (1, 10), (2, 13), \ldots\}$

a) $f$ is injective

Suppose otherwise, i.e., $f(x) = f(y)$ for $x \neq y$
$f(x) = f(y) \implies 3x + 7 = 3y + 7 \implies 3x = 3y \implies x = y$ □

b) $f$ is not surjective

For $b = 2$ there is no $a$ such that $f(a) = b$, that is, $2 = 3a + 7$ holds for $a = -\frac{5}{3}$ which is not in $\mathbb{Z}$.

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A function $f : A \to B$ is *one-to-one* or *injective* if each element of $B$ appears at most once as the image of an element of $A$.
A function $f : A \to B$ is *onto* or *surjective* if $f(A) = B$, that is, each element of $B$ appears at least once as the image of an element of $A$.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(x) = 3x + 7$.

$f = \{\dots, (-3, -2), (-2, 1), (-1, 4), (0, 7), (1, 10), (2, 13), \dots\}$

a) $f$ is injective

Suppose otherwise, i.e., $f(x) = f(y)$ for $x \neq y$
$f(x) = f(y) \implies 3x + 7 = 3y + 7 \implies 3x = 3y \implies x = y$ $\square$

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## Theorem 4

*For any finite sets A and B, the number of one-to-one functions from A to B is* $\frac{|B|!}{(|B|-|A|)!} = P(|B|, |A|)$

**Proof.** Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. A one-to-one function $f$ assigns each element $a_i$ of $A$ a distinct element $b_j = f(a_i)$ of $B$; for $a_1$ there are $m$ choices, for $a_2$ there are $m-1$ choices,..., for $a_n$ there are $(m-(n-1))$ choices.

Hence by the rule of product, we have

$$\underbrace{m(m-1)\ldots(m-(n-1))}_{n} = \frac{m!}{(m-n)!} = \frac{|B|!}{(|B|-|A|)!} = P(|B|, |A|)$$

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## Formal notation (optional - good for proofs)

A relation $f$ from $A$ to $B$ is a function if

$$\forall x \in A \; \exists y \in B \; [ \; (x, y) \in f \; ]$$

$$\forall x \in A \; \forall y, z \in B \; [ \; (x, y) \in f \; \wedge \; (x, z) \in f \Longrightarrow y = z \; ]$$

A function $f : A \rightarrow B$ is injective if

$$\forall x, y \in A \; [ \; f(x) = f(y) \Longrightarrow x = y \; ]$$

A function $f : A \rightarrow B$ is surjective if

$$\forall y \in B \; \exists x \in A \; [ \; f(x) = y \; ]$$