

# REVISITING THE WILKIE INVESTMENT MODEL

June 12, 2008

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## ABSTRACT

In this paper we consider the Wilkie model for the retail prices index both without and with an ARCH model, share dividend yields, share dividends and share prices, long-term interest rates and short-term interest rates, in each case by updating the parameters to 2007. We discuss the validation of the models by applying some statistical tests on the residuals and estimate the parameters and their confidence intervals recursively to study their stability. When we update the parameters and examine the validation of the models we observe that for some of the models the performance gets worse, such as for retail prices including the ARCH model and especially long-term interest rates. We had to modify the long-term interest rate model slightly in order to update the parameters to avoid negative real interest rates. There is no significant change in the performance of the models for share dividends, share dividend yields and short-term interest rates and the models are still satisfactory. We also examine the parameter stability and conclude that most of the parameters are not stable.

## KEYWORDS

Stochastic Asset Models, Wilkie Investment Model, Price Inflation, Share Dividend Yields, Share Dividends, Share Prices, Long-Term Interest Rates, Short-Term Interest Rates

# 1 INTRODUCTION

The Wilkie stochastic investment model, developed by A. D. Wilkie, is described fully in two papers: the original version is described in "A Stochastic Investment Model For Actuarial Use" (1986) and the model is reviewed, updated and extended in "More On A Stochastic Asset Model For Actuarial Use" (1995).

The original Wilkie model (1986) was developed from U.K. data over the period 1919-1982, and was made up of four interconnected models for price inflation, share dividend yields, share dividends and long-term interest rates. Wilkie (1995) updated the original model and extended it to include an alternative autoregressive conditional heteroscedastic (ARCH) model for price inflation, and models for wage inflation, short-term interest rates, property yields and income and index-linked yields. Furthermore, these models were fitted to data from numerous developed countries and an exchange rate model was proposed.

Hardy (2003) describes the Wilkie model as a multivariate model, meaning that several related economic series are projected together. This is very useful for applications that require consistent projections of, for example, stock prices and inflation rates or fixed interest yields. It is designed for long-term actuarial applications such as simulating assets of financial institutions (life insurance companies, pension funds) over many years in the future to study the risk of insolvency. Since the model is designed to be applied to annual data it is not suitable in that form for assessing short-term hedging strategies.

There is an enormous number of papers such as Kitts (1990), Clarkson (1991), Geohagan et al. (1992), Ludvic (1993), Harris (1995), Huber (1995, 1997), Rambaruth (2003), Hardy (2004), Nam (2004), Lee and Wilkie (2000) and books such as Daykin et al. (1994), Booth (1999), Hardy (2003) which describe, compare or criticise the Wilkie Model. Moreover, the discussions attached to Wilkie's 1986 and 1995 papers may be counted as the most important references for comments on the Wilkie model. Especially in the 'Abstract of Discussion' part of the 1995 paper there are various comments and criticisms about the model from twenty academics and practitioners who examined and applied the model or developed new models which followed in the footsteps of Wilkie (1986, 1995).

## Methodology

The Wilkie investment model is based originally on Box-Jenkins (1976) time series models.

- Most of the parameters are estimated by using Least Square Estimates calculated by a non-linear optimization method (in practice the Nelder-Mead simplex method).
- Almost all models are stationary or integrated of autoregressive order one (AR(1) or ARIMA(1,1,0)).
- Some items are treated as if co-integrated. For example, Wilkie (1986, 1995) defined the share dividend yield as the dividend index divided by the price index where logarithm of the yield is equal to the difference between the logarithms of these two series; this means that these three series are co-integrated.

## Structure

The series in the Wilkie model are correlated and could be modelled simultaneously by multivariate analysis, using 'vector autoregressive models' (VAR). The model in fact started as a straightforward VAR model but after crossing out a great many non-significant values, it was simplified to a cascade model. Figure 1 illustrates the cascade structure of the model where the arrows indicate the direction of influence.

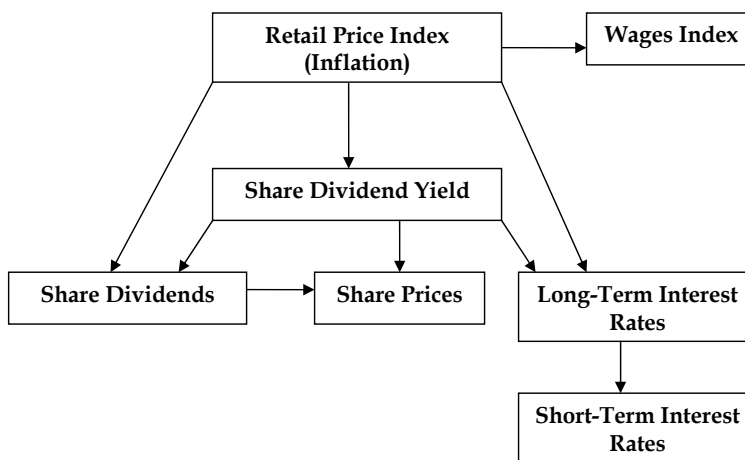


Figure 1: Structure of the Wilkie model

It can be seen that the complete model is wholly self-contained. The only inputs are the separate white noise series, and no exogenous variables are included. According to Wilkie (1986), whatever may be the case for short-term forecasting, such a self-contained model is better for long-term simulations.

This paper aims to review the Wilkie investment model only for UK data from a statistical viewpoint. Sections 2 to 7 describe the models for retail prices index, the ARCH modelling of that index, and the models for share dividend yields, share dividends, long-term interest rates and short term interest rates respectively, introducing the new parameters and examining the validation and parameter stability of the models. Section 8 concludes.

## 2 RETAIL PRICES INDEX

### Original Model

The most recent series used for the Retail Prices Index is the one called RPI, and not any of the other alternative series produced for the UK in recent years. The model for the U.K. Retail Prices Index (RPI) where  $Q(t)$  is the value of a retail price index at time  $t$ , is:

$$Q(t) = Q(t - 1) \cdot \exp(I(t))$$

so that  $I(t) = \ln Q(t) - \ln Q(t - 1)$  is the force of inflation over the year  $(t - 1, t)$ .

The force of inflation  $I(t)$ , which is defined as the difference in the logarithms of the RPI each year, is modelled as a first order autoregressive series. An AR(1) model is a statistically stationary series for suitable parameters, which means that in the long run the mean and variance are constant.

$$\begin{aligned}
 I(t) &= QMU + QA \cdot (I(t - 1) - QMU) + QE(t) \\
 QE(t) &= QSD \cdot QZ(t) \\
 QZ(t) &\sim iidN(0, 1)
 \end{aligned}$$

that is  $QZ(t)$  is a series of independent, identically distributed unit normal variates.

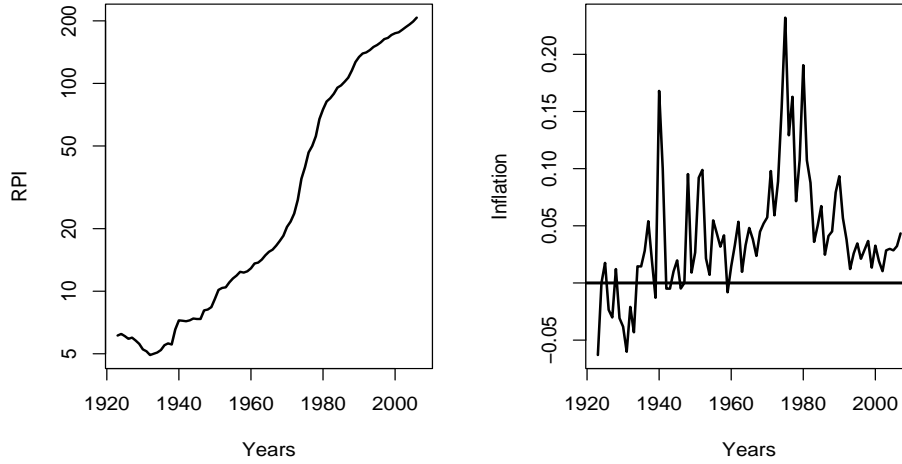


Figure 2: Retail price index and annual force of inflation, 1923-2007

$I(t)$	1923-1994	1923-2007
$QMU$	0.0473 (0.0119)	0.0446 (0.0102)
$QA$	0.5773 (0.0799)	0.5794 (0.0738)
$QSD$	0.0427 (0.0036)	0.0396 (0.0030)
$r_z(1)$	-0.006	-0.007
$r_{z^2}(1)$	0.041	0.071
skewness $\sqrt{\beta_1}$	1.10	1.25
kurtosis $\beta_2$	4.94	5.86
Jarque-Bera $\chi^2$	28.04	52.83
$p(\chi^2)$	0.000	0.000

Table 1: Estimates of parameters and standard errors of AR(1) model for inflation over 1923-1994 and 1923-2007

The model states that each year the force of inflation is equal to its mean rate,  $QMU$ , plus some proportion,  $QA$ , of last year's deviation from the mean, plus a random innovation which has zero mean and a constant standard deviation,  $QSD$ .

The retail price index,  $Q(t)$ , from 1923 to 2007 on a vertical logarithmic scale and the annual differences in the logarithms  $I(t)$ , are shown in Figure 2.

The graph of the RPI illustrates that prices fell after the First World War, until the mid 1930s and have risen fairly steadily upwards since then. The graph of the annual force of inflation shows that there was a fall in prices after the First World War, and big rises during the Second World War and the 1970s. Since the 1960s inflation has been positive and especially in the last 10-15 years it seems to have been low and stable.

### Updating and Rebasing to 1923-2007

We updated the data and re-estimated the parameters of the price inflation model and applied some statistical tests to explore the validation of the model. Table 1 shows the new parameters, their standard errors and some diagnostic tests results.

The estimated parameters over the two periods have not changed significantly.  $QMU$

and  $QSD$  have slightly decreased and  $QA$  has slightly increased. Standard errors (in brackets) show that all the parameters are significantly different from zero. When we compare these two periods by examining the diagnostic tests, it can be concluded that there is no significant improvement on the model based on the updated data. The residuals, the observed values of  $QE$ , are calculated for both periods. The autocorrelation coefficients of the residuals and squared residuals show nothing unusual, i.e. residuals can be considered to be independent and there is no simple ARCH effect. However, the skewness and kurtosis coefficients, based on the third and fourth moments of the residuals, are rather large:  $\sqrt{\beta_1} = 1.10$  and  $1.25$ , demonstrating substantial positive skewness; and  $\beta_2 = 4.94$  and  $5.86$ , implying quite heavy tails in the distribution.

A composite test of the skewness and kurtosis coefficients has been devised by Jarque & Bera, and this also shows significant non-normality. The test statistics are 28.04 and 52.83 for the two periods, which should be compared with a  $\chi^2$  variate with two degrees of freedom. The p-values are zero and therefore, the probability that such a result would occur at random is negligible.

### Parameter Stability

The parameter constancy of the models can be examined by recursively estimating the parameters on incrementally larger data sets. Figure 3 and Figure 4 present these recursive estimates and 95% confidence intervals of  $QMU$ ,  $QA$  and  $QSD$ , respectively, for earlier sub-periods (data sets starting in 1923) and later sub-periods (data sets ending in 2007). In the figures, solid lines show the parameter estimates and upper and lower confidence limits for the earlier sub-periods, which are calculated assuming normally distributed errors, whereas the dotted lines show the parameter estimates for the later sub-periods and the upper and lower confidence limits for these. Sub-periods with fewer than 10 observations are omitted.

When we look at Figure 3 we see recursive estimates of  $QMU$  (the mean value of inflation). The middle bold solid line, which is labelled  $QMU1$ , is obtained by calculating the parameter value for the earlier sub-periods and the middle bold dotted one which is labelled  $QMU2$  represents the estimated parameter value for the later sub-periods. Since we use the same data periods for the right hand end of  $QMU1$  and the left hand end of  $QMU2$  (i.e. data over the period 1923 to 2007), we have exactly the same parameter values and confidence limits at these points. The overall mean level of inflation is about 0.045. It is useful to explain the graph by giving an example: at year 1960,  $QMU1$  represents the parameter estimate for 1923 to 1960 inclusive, which is equal to 0.020, and  $QMU2$  represents the parameter estimate for 1960 to 2007 inclusive, which is equal to 0.061.

By examining the graph of  $QMU1$  we can see that for the first 10 to 17 years (1923-1939),  $QMU1$  is negative (between -0.02 and -0.004) which reflects the negative inflation of that post-war period. We can observe that  $QMU1$  tends to increase over most of the period, including two jumps in the early 1940s and the mid 1970s due to the effects of the Second World War and the oil crisis. The confidence limits for the early sub-periods are almost the same width over most of the period. On the other hand, the right hand end of  $QMU2$  gives the mean of the last ten years' inflation (1998-2007), which is about 0.027 and the confidence interval is very small. This may suggest that during a period of low inflation the uncertainty of prices is lower and inflation is therefore stable. Moreover, as Huber (1997) indicates, the recursive estimates of  $QMU$  tend to increase over most of the period. The graphs suggest that  $QMU$  may not be constant over time which may indicate different 'regimes'.

Figure 3 shows the recursive estimates for the autoregressive parameter,  $QA$ , of the

inflation model. Again, the bold solid line represents the recursive estimates for earlier sub-periods,  $QA1$  and the bold dotted line is for the later sub-periods,  $QA2$ . The graph illustrates that  $QA1$  jumps in the mid-1970s, from a value of approximately 0.37 to a value of approximately 0.58. Before and after this period it seems stable. When we look at the right hand end of  $QA2$ , the autocorrelation coefficient for the last ten years' inflation is very small. This means that during a low and stable inflation period, the rate of inflation does not depend on its previous values. If the rate of inflation is high, the dependence on the previous years' inflation increases. Furthermore, the confidence intervals are large when the data period is small and shrink as the period extends.

Figure 4 shows the recursive estimates of the standard deviation,  $QSD$ , of the inflation model. The graph of  $QSD1$  indicates that there are two jumps: one is in the early 1940s and the other is in the mid 1970s. The steadily increasing structure of  $QMU1$  and the two jumps due to the Second World War and oil crises cause bigger jumps in the volatility of the inflation. The confidence intervals are wide around these jumps and they get smaller as we increase the data period. When we look at the right hand end of  $QSD2$ , in which we used the latest years' values to estimate the parameter, we see that the standard deviation is very small (about 0.01) and the confidence limits are very close to each other. This result supports our comment on the right hand end of  $QMU2$  which is that during a low inflation period it is easier to predict the rate and there is decreased uncertainty.

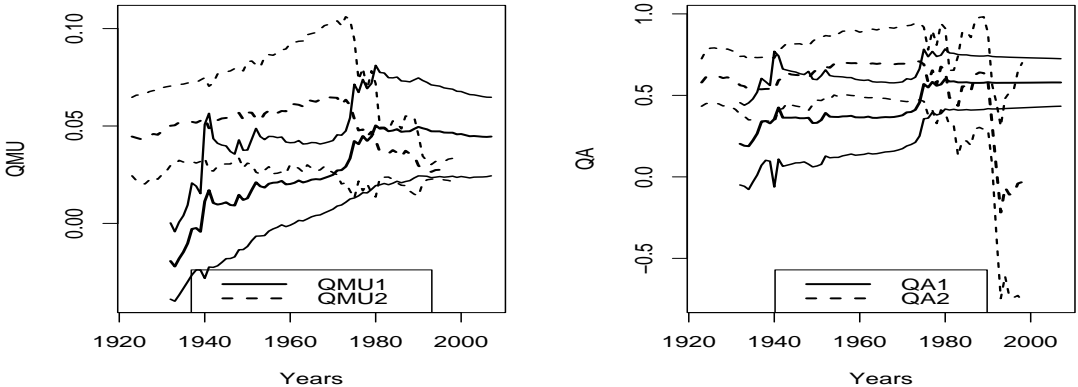


Figure 3: Recursive estimates of  $QMU$  and  $QA$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

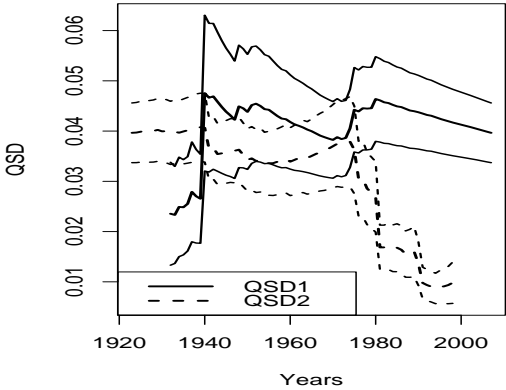


Figure 4: Recursive estimates of  $QSD$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

### 3 ARCH MODELLING

Although Wilkie initially assumed that the residuals of the inflation model were normally distributed, he observed in 1995 that they are much fatter tailed than normal distribution. One of the ways to model these fat tailed distributions is using an Autoregressive Conditional Heteroscedastic (ARCH) model (Engle, 1982). Therefore, Wilkie (1995) proposed a rather unusual, but straightforward, ARCH model for the standard deviation of the inflation model which is:

$$QSD(t)^2 = QSA^2 + QSB.(I(t - 1) - QSC)^2$$

$$QSB \geq 0$$

where  $QSA$ ,  $QSB$  and  $QSC$  are parameters to be determined.

We might also put  $QSC=QMU$ .

Note that Wilkie in 1995 started the formula with  $QSA$  rather than  $QSA^2$ , but he quoted the value as  $0.0256^2$ ; we have changed the notation to follow this. This model states that the variance each year depends on the square of the deviation of last year's observation of the force of inflation  $I(t)$ , from some middle value  $QSC$ , which might or might not equal the mean value  $QMU$ .

According to the model, if the previous year's inflation is unusually high or unusually low, then the uncertainty about the rate of inflation in the following year is increased. It might turn out to be again high or it might be much lower. If, on the other hand, the rate of inflation happens to have been near the value of  $QSC$ , then the standard deviation in the succeeding year is smaller.

#### Updating and Rebasing to 1923-2007

Wilkie fitted a first order ARCH model to the yearly data for 1923 to 1994, first allowing  $QSC$  to be chosen freely and then fixing  $QSC$  to equal  $QMU$ . Since fixing  $QSC$  to equal  $QMU$  fits the data practically as well as allowing it to vary, we applied this model to the updated data. Table 2 shows the estimates and the diagnostic test results of the models for the different periods.

I(t)	1923-1994	1923-2007
$QMU$	0.0404 (0.0058)	0.0368 (0.0044)
$QA$	0.6179 (0.1296)	0.6124 (0.1286)
$QSA^2$	0.0256 <sup>2</sup> (0.000019)	0.0212 <sup>2</sup> (0.00000971)
$QSB$	0.5524 (0.2148)	0.6579 (0.2129)
$r_z(1)$	-0.013	-0.027
$r_{z^2}(1)$	-0.042	-0.031
skewness $\sqrt{\beta_1}$	0.57	0.57
kurtosis $\beta_2$	3.63	4.07
Jarque-Bera $\chi^2$	5.76	9.68
$p(\chi^2)$	0.0562	0.0079

Table 2: Estimates of parameters of AR(1) ARCH models for inflation over 1923-1994 and 1923-2007

Except for  $QSA$  all the parameters for the ARCH model based on the different periods are significantly different from zero. The autocorrelation coefficients for the first lags suggest that the residuals are independent. When we compare the skewness and the kurtosis of the models, it is seen that values are closer to the theoretical values of the normal distribution, comparing with the values in Table 1, for the AR(1) inflation model. Besides, the Jarque-Bera statistic is considerably reduced in the ARCH model based on the period 1923-1994. When we examine the fit on the updated data, although the log likelihood improved, the skewness remains the same and the kurtosis gets worse and the Jarque-Bera test statistic indicates non-normality of the residuals. Therefore, it can be concluded that, although the ARCH model does not fit the updated data as well as the one Wilkie proposed in Wilkie (1995), it is still superior to the AR(1) inflation model.

Wilkie stated in 1995 that an ARCH model of this type is easy to use for simulation, but it produces larger variances in the long run. The long-run or unconditional variance of  $QE$  is given by:

$$QSD^2 = \{QSA + QSB(QMU - QSC)^2\} / \{1 - QSB / (1 - QA^2)\}$$

provided that  $QSB < (1 - QA^2)$  Wilkie (1995). This condition is satisfied by the parameters based on the period 1923-1994. However, the estimated  $QSB$  on the updated data is greater than  $(1 - QA^2)$ . Since this is the case, the variance would increase without limit as  $t$  increases; which is an undesirable feature of such a model.

### Parameter Stability

The parameter constancy of the models can be examined by recursively estimating the parameters on incrementally larger data sets. Figure 5 and Figure 6 present the recursive estimates and 95% confidence intervals of  $QMU$ ,  $QA$ ,  $QSA$  and  $QSB$  parameters for earlier and later sub-periods for the ARCH model. In the figures, solid lines show the parameter estimates and upper and lower confidence limits for the earlier sub-periods whereas dotted lines show the parameter estimates for the later sub-periods and the upper and lower confidence limits for these later sub-periods. Sub-periods with fewer than 10 observations are omitted but in this model and next several models we could omit the sub-periods with fewer than 20 observations because as the number of parameters of the model increased, we need more information (i.e. more data) to obtain reasonable estimates.

The autoregressive parameter,  $QA_{arch}$ , in Figure 5 also has a shape similar to that of the autoregressive parameter,  $QA$ , of the AR(1) model. For small sub-periods and specific years (Second World War, 1952 and the oil crisis) it has jumps and large confidence intervals. There is a continuous increase in  $QA_{arch1}$  between 1970 and 1980, which indicates that the increase in the rate of inflation increased the dependence on the previous years' inflation. After the 1980s  $QA_{arch1}$  seems quite stationary.

When we look at Figure 5, we see that mean parameter of the ARCH model for the earlier sub-periods, denoted as  $QMU_{arch1}$ , has almost the same shape with  $QMU1$  of the AR(1) inflation model. There is a big jump in 1952 which reflects the effect of relatively high inflation (about 10%) in the two years 1951 and 1952. As we mentioned above, if the previous year's inflation is unusually high or unusually low, then the uncertainty about the rate of inflation in the following year is increased. This increased uncertainty is represented by a very large confidence interval in that year. There is another significant jump in the early 1970s due to the effect of the oil crises. After the 1970s the parameter seems quite stable and as sub-periods are increased the confidence intervals shrink. When



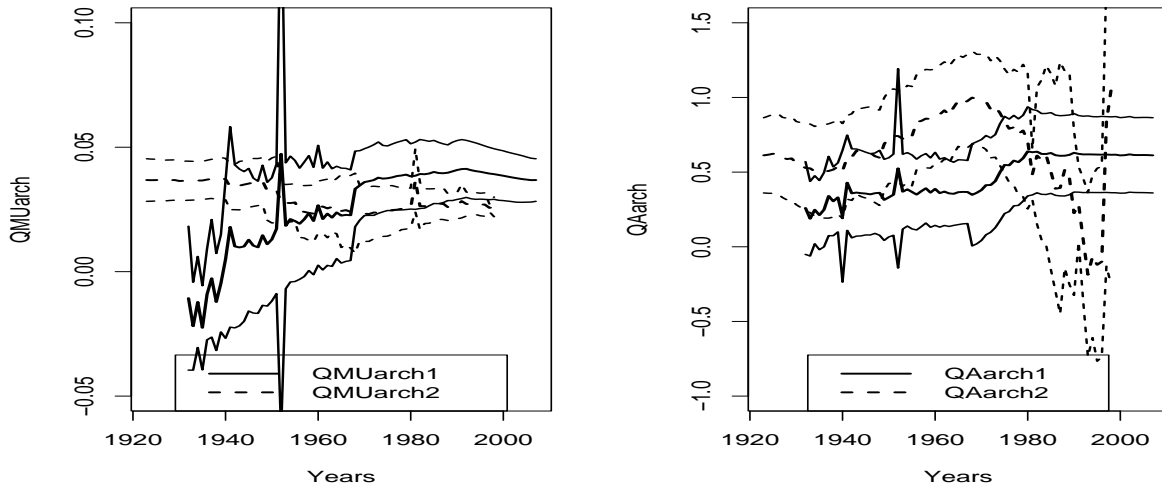


Figure 5: Recursive estimates of  $QMUarch$  and  $QAarch$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

we look at  $QMUarch2$ , we see that for the first 20 years the confidence intervals are small because of the stable inflation, which indicates that we might not need an ARCH model for these periods. The overall mean level for the ARCH model is about 0.037.

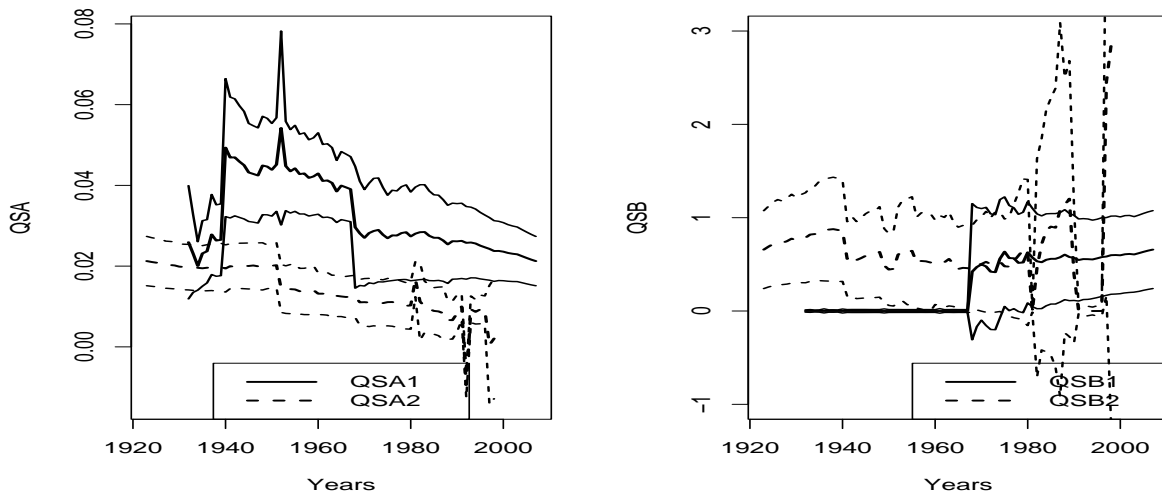


Figure 6: Recursive estimates of  $QSA$  and  $QSB$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

Figure 6 shows the recursive estimates of the  $QSA$  and  $QSB$  parameters. It is useful to interpret these two graphs together because  $QSB$  is the parameter which makes a difference between the  $AR(1)$  and ARCH inflation models. When  $QSB$  is set to 0, these two models become identical. The graph of  $QSA1$  is similar to its equivalent in the  $AR(1)$  model,  $QSD$ , until the 1970s. There are several jumps and two of them are significant: one is in the late 1930s and the other is in the early 1970s. On the other hand,  $QSB1$  is almost zero until the early 1970s which indicates that the  $AR(1)$  model is enough to

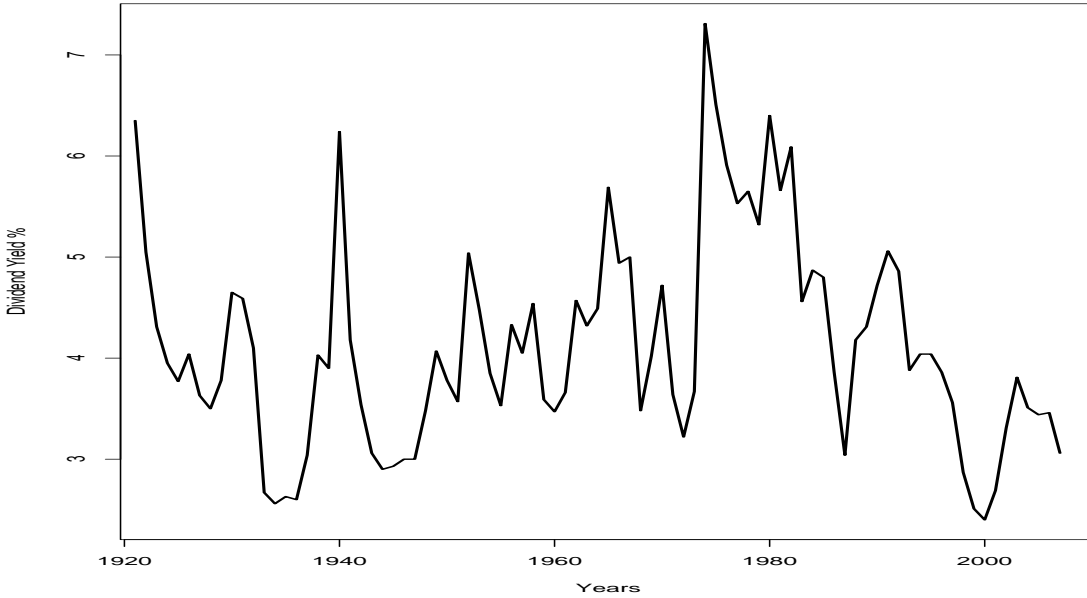


Figure 7: Dividend Yield, 1921-2007

model the rate of inflation until this year. The sharp decrease in  $QSA1$  coincides with the sharp increase in  $QSB1$  in these years, because after the first oil crises the  $AR(1)$  model is not sufficient and through the  $QSB$  parameter the ARCH effect comes into the model. Taking this into account decreases  $QSA1$  and stabilises it for the rest of the sub-periods. Besides,  $QSA2$  is quite stable except for two specific jumps and  $QSB2$  is informative just after the sub-periods including 20 or more years.

## 4 SHARE DIVIDEND YIELDS

### Original Model

The share dividend yield is based on a number of past indices, since 1962 on the FTSE-Actuaries All-Share Index. The yield for most of the period has been based on the gross dividend index, i.e. gross of income tax. However, since 1997 the quoted dividend yield is based on 'actual' dividends, and we have grossed these up by dividing by 0.9 since that date. The original model for share dividend yields, based on annual data from June 1923 to June 1994, where  $Y(t)$  is the dividend yield on ordinary shares at time  $t$ , is:

$$\begin{aligned}
 Y(t) &= \exp(YW.I(t) + \ln YMU + YN(t)) \\
 \ln Y(t) &= YW.I(t) + \ln YMU + YN(t) \\
 YN(t) &= YA.YN(t-1) + YE(t) \\
 YE(t) &= YSD.YZ(t) \\
 YZ(t) &\sim iidN(0,1)
 \end{aligned}$$

Thus, the logarithm of the dividend yield is equal to its mean value, plus an influence of its deviation a year ago from the mean, plus an additional influence from inflation over the previous year, plus a random innovation.

Values of the dividend yield from 1921 to 2007 are shown in Figure 7.

$\ln Y(t)$	1923-1994	1923-2007
$YW$	1.7940 (0.4704)	1.6473 (0.4566)
$YA$	0.5492 (0.0978)	0.6354 (0.0851)
$YMU$	3.77% (0.17%)	3.64% (0.18%)
$YSD$	0.1552 (0.0129)	0.1529 (0.0117)
$r_z(1)$	0.078	0.091
$r_{z^2}(1)$	-0.102	-0.100
skewness $\sqrt{\beta_1}$	0.22	0.39
kurtosis $\beta_2$	3.01	3.43
Jarque-Bera $\chi^2$	0.63	3.10
$p(\chi^2)$	0.7301	0.2118

Table 3: Parameter estimates for model for the  $\ln Y$  over 1923-1994 and 1923-2007

### Updating and Rebasing to 1923-2007

Table 3 shows the parameter estimates and diagnostic tests for the dividend yield model. Comparing the parameters of the model based on the period 1923-1994 with those in the updated data, we see that  $YW$  and  $YMU$  are smaller and  $YA$  is bigger.  $YSD$  is also smaller than it was before. However, all the new parameter estimates are within, or not much above, one standard deviation away from the original estimates (based on 1923-1994), so there is no strong evidence of a change in the parameters of the model on the updated data.

Diagnostic tests for both models show that the residuals appear to be independent; the autocorrelation function has no high values. The residuals appear to be normally distributed, too. The skewness and kurtosis coefficients are increased a bit but are still not far from their expected values. The Jarque-Bera statistic increased to 3.10, giving  $p(\chi^2) = 0.21$ . The model is still satisfactory.

### Parameter Stability

We examine the stability of the parameters by calculating the recursive estimates on incrementally larger data sets as we did in the previous sections. Figure 8 and Figure 9 present the recursive estimates and 95% confidence intervals of the inflation effect,  $YW$ , the autoregressive parameter of the yield,  $YA$ , the mean yield,  $YMU$ , and the standard deviation,  $YSD$ , respectively, for the earlier sub-periods (data sets starting in 1923) and the later sub-periods (data sets ending in 2007). The construction of the graphs is similar to the ones in the previous sections.

In Figure 8,  $YW$  is the parameter which reflects the effect of inflation on dividend yields. The graph for  $YW1$  shows two jumps in the years 1940 and 1974. These are the years in which the greatest increases in prices and in yield occurred. The graph suggests that when inflation is high, its effect on yield is also high. If last years' inflation is low, it does not have a significant effect on the dividend yield.

Figure 8 shows the autoregressive parameter,  $YA$  too. This parameter is quite stationary compared with the other parameters. We should note that when inflation is high (1940 and 1974),  $YA$  decreases which means that during these years the increasing inflation effect on yields ( $YW$  increases in these years) explains most of the variability in yield and yield does not depend so much on its previous value.

When we look at the  $YMU$  graph in Figure 9 we can see that, as we extend the period, the confidence intervals become smaller.  $YMU1$  has a similar path to  $QMU1$  which

justifies the proposition that high inflation, when it occurs, leads to a fall in share prices and hence to high dividend yields.

The *YSD* graph in Figure 9 shows that during low, stable inflation the standard deviation of the yields is small. Then right hand end of *YSD2* represents parameter estimates over more recent years and these values are significantly lower than the overall standard deviation.

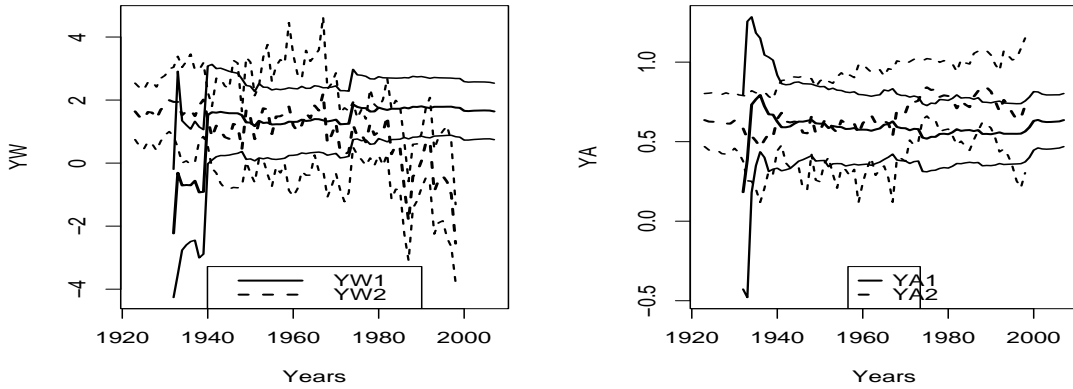


Figure 8: Recursive estimates of *YW* and *YA* with 95% confidence intervals for earlier and later sub-periods, 1923-2007

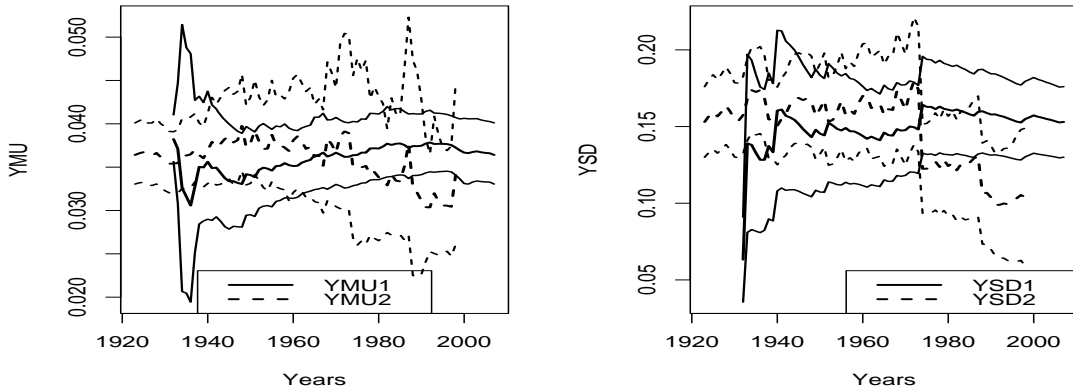


Figure 9: Recursive estimates of *YMU* and *YSD* with 95% confidence intervals for earlier and later sub-periods, 1923-2007

## 5 SHARE DIVIDENDS AND SHARE PRICES

### Original Model

The indices for share dividends and share prices come from the same source as the share dividend yields. The original model for share dividends, where  $D(t)$  is the value of a dividend index on ordinary shares at time  $t$  and  $K(t)$  is the annual change in the logarithm, is:

$$\begin{aligned}
K(t) &= \ln D(t) - \ln D(t-1) \\
&= DW.DM(t) + DX.I(t) + DMU \\
&\quad + DY.YE(t-1) + DB.DE(t-1) + DE(t) \\
DM(t) &= DD.I(t) + (1 - DD).DM(t-1) \\
DE(t) &= DSD.DZ(t) \\
DZ(t) &\sim iidN(0,1)
\end{aligned}$$

Wilkie constrained  $DX$  to equal  $1 - DW$ , so that a change in  $\ln Q$  ultimately produced the same change in  $\ln D$ ; the transfer function had 'unit gain'.

Hence, a model for  $P(t)$ , the value of a price index of ordinary shares at time  $t$  can be obtained as:

$$\begin{aligned}
P(t) &= D(t)/Y(t) \\
\ln P(t) &= \ln D(t) - \ln Y(t)
\end{aligned}$$

The model explains the change in the logarithm of the dividend index as a function of current and past values of inflation, plus a mean real dividend growth (which in Wilkie (1986) was taken as zero), plus an influence from last year's dividend yield innovation, plus an influence from last year's dividend innovation, plus a random innovation. The term  $DB.DE(t-1)$ , involving last year's dividend innovation, makes the basic model for 'real dividends' into a moving average order one (MA(1)) model.

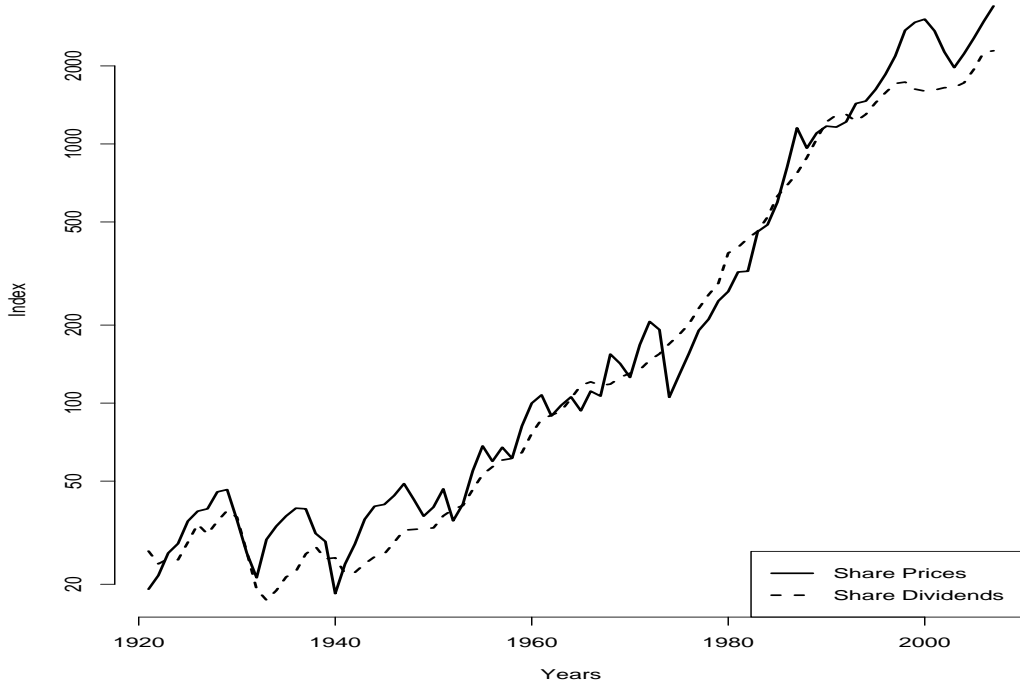


Figure 10: Share price index and 22 x Dividend index, 1921-2007

Figure 10 shows the share price index and 22 times the share dividend index on the same graph, on a vertical logarithmic scale. Both graphs have the appearance of non-stationary series. Further, the way they move close to one another over the whole period

$\ln D(t)$	1923-1994	1923-2007
$DW$	0.5793 (0.2138)	0.5779 (0.2077)
$DD$	0.1344 (0.0798)	0.1441 (0.0783)
$DMU$	0.0157 (0.0124)	0.0142 (0.0114)
$DY$	-0.1761 (0.0438)	-0.1507 (0.0377)
$DB$	0.5734 (0.1286)	0.6070 (0.1074)
$DSD$	0.0671 (0.0056)	0.0654 (0.0050)
$r_z(1)$	-0.023	-0.008
$r_{z^2}(1)$	0.224	0.217
skewness $\sqrt{\beta_1}$	-0.61	-0.55
kurtosis $\beta_2$	3.97	4.02
Jarque-Bera $\chi^2$	8.16	8.77
$p(\chi^2)$	0.01689	0.01245

Table 4: Parameter estimates for the model for  $\ln D$  over 1923-1994 and 1923-2007

suggests that they are co-integrated.

### Updating and Rebasing to 1923-2007

Table 4 shows the estimated parameters and their standard errors. The value of the smoothing parameter  $DD$  is not much more than one standard error away from zero, and it therefore could be thought that it should be zero. Wilkie investigated what happens if he omits the influence of inflation by setting both  $DW$  and  $DD$  to zero. Since the log likelihood is worsened substantially, and, in addition, the crosscorrelation between the residuals  $DE$  and the residuals from inflation  $QE$  is large he decided to keep these parameters. Moreover, he found it economically necessary taking into account the direct transfer from retail prices to dividends.

On the other hand, the estimated value of  $DMU$ , the mean rate of growth of real dividends, is not much more than one standard error away from zero for both of the periods. It can, therefore, be set to zero as in the model in Wilkie (1986). However, since the real rate of growth of dividends is an important element in the total return on shares Wilkie preferred keeping this parameter.

Diagnostic tests of the residuals for the model show no remaining autocorrelation. The first autocorrelations are negative but not significant. The residuals are not conspicuously non-normal, but the skewness coefficients  $\sqrt{\beta_1} = -0.61$  and  $-0.55$  and kurtosis  $\beta_2 = 3.97$  and  $4.02$  are outside two standard errors away from their expected values. The Jarque-Bera statistics are  $8.16$  and  $8.77$ , with  $p(\chi^2) = 0.01689$  and  $0.01245$ , so there is some evidence of fat-tailedness.

### Parameter Stability

We examine the stability of the parameters by calculating the recursive estimates on incrementally larger data sets as we did in the previous sections. Figure 11, Figure 12 and Figure 13 present the recursive estimates and 95% confidence intervals of the model parameters  $DW$ ,  $DD$ ,  $DMU$ ,  $DY$ ,  $DB$  and  $DSD$  for earlier sub-periods (data sets starting in 1923) and later sub-periods (data sets ending in 2007). Since we have six parameters for this model, we have omitted the sub-periods less than 20 years in order to use enough data to get reasonable estimates.

In Figure 11 both graphs for  $DW$  and  $DD$  indicate nonstationarity for about 30 or

40 years with large and unstable confidence intervals. Especially for these periods, the parameters highly depend on the initial values. We used the R programming language to estimate these parameters and for some of the sub-periods the algorithm did not converge. In these cases, the hessian matrices have both negative and positive eigenvalues which indicate saddle points instead of local minimum or maximum. When we cannot find the standard errors of estimates, we either change slightly the initial values or change the optimization method from Nelder-Mead to quasi-Newton method or conjugate gradients methods which might give more reasonable values in some cases. Both of these caused great changes in the estimates which might show instability of the parameters. Moreover, we tried beta prior distribution for  $DW$  with the parameters  $\alpha = 1.1$  and  $\beta = 1.1$  to restrict it in the range 0 and 1. We chose these parameters to use a relatively straight curve for  $DW$  which is close to a uniform distribution. Although the R programming language could not find solutions for the short sub-periods, the estimates are very close to the original ones when we use more than 40 years as the sub-period. We believe that using prior distributions for the parameters deserves more work.

Figure 12 indicates that the mean level of dividends,  $DMU$ , and the dependence parameter on the dividend yield innovation,  $DY$ , have more stable shapes. They seem stationary when the sub-periods include data more than 30 years. The confidence intervals become smaller as the sub-periods extend. The significant value for  $DY$  shows that an unexpected change in the dividend yield in one year forecasts changes in dividends in the forthcoming year in a negative way, i.e. a forecast rise in dividends causes a rise in share prices and a fall in dividend yields.

The  $DB1$  parameter in Figure 13 is also quite stable after the 1960s. The significance of this parameter justifies the idea that companies pay out only part of any additional earnings in dividend in one year, with a further part in the following year. On the other hand, the standard deviation of the dividend model  $DSD$  has a decreasing trend after about 30 years, which is the period which gives reasonable values for the estimated parameters. As the sub-period increases, the confidence intervals get smaller.

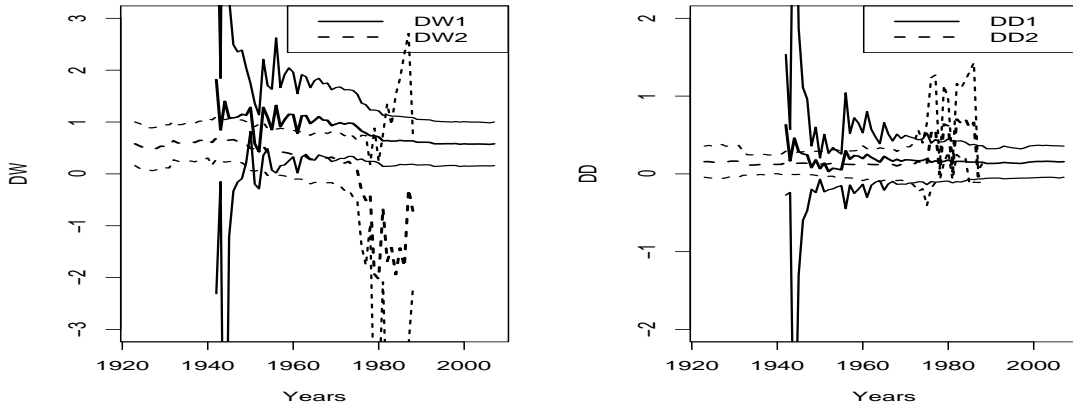


Figure 11: Recursive estimates of  $DW$  and  $DD$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

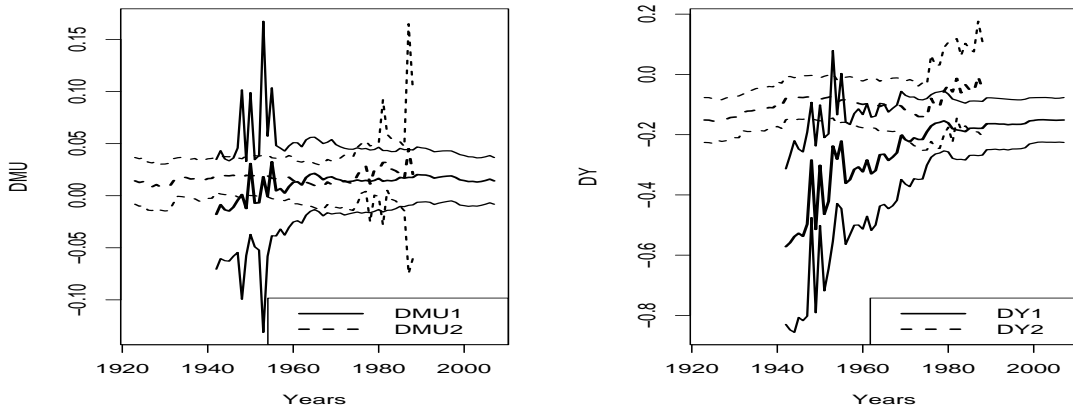


Figure 12: Recursive estimates of  $DMU$  and  $DY$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

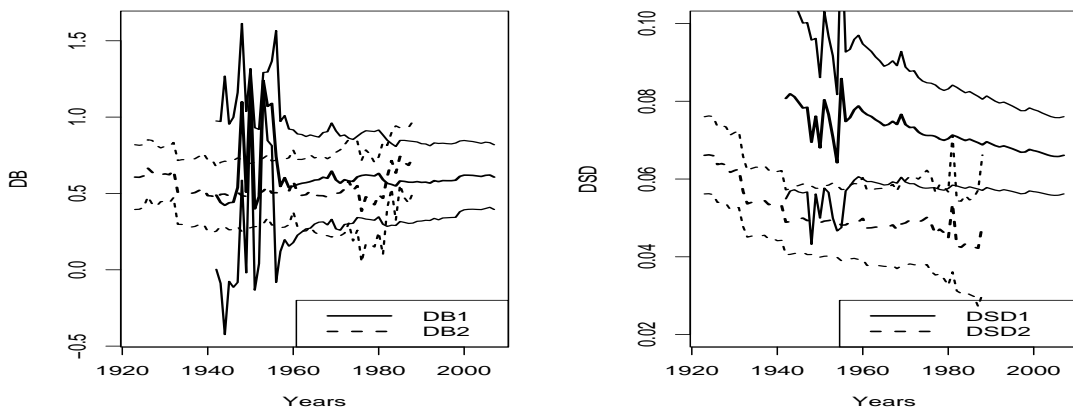


Figure 13: Recursive estimates of  $DB$  and  $DSD$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007



## 6 LONG-TERM INTEREST RATES

### Original Model

Wilkie used consols (originally short for consolidated annuities or consolidated stock) for the long-term bond yields which are a form of British government bond, dating originally from the 18th century. Consols are one of the rare examples of an actual perpetuity, although they may be redeemed by the issuer.

For long term bond yields,  $C(t)$ , the earlier values are the yield on 2<sup>1/2</sup>% Consols, and the later are the yield on the FTSE-Actuaries BGS Indices irredeemables index, which is now purely the yield on 3<sup>1/2</sup>% War Stock (War Loan).

The original model for the yield on 'consols' based on annual data, where  $C(t)$  is the yield on consols at time  $t$ , is:

$$\begin{aligned}C(t) &= CW.CM(t) + \overbrace{CMU.\exp(CN(t))}^{CR(t)} \\CM(t) &= CD.I(t) + (1 - CD).CM(t - 1) \\CN(t) &= CA.CN(t - 1) + CY.YE(t) + CE(t) \\CE(t) &= CSD.CZ(t) \\CZ(t) &\sim iidN(0, 1)\end{aligned}$$

The 'real' part of the consols yield is:

$$CR(t) = C(t) - CW.CM(t)$$

so that:

$$\ln CR(t) = \ln CMU + CN(t)$$

the model for the logarithm of  $CR(t)$  was originally an AR(3) model, but even in 1986 Wilkie suggested that a possible simplification was to use an AR(1) model with only one  $CA$  parameter.

The model is composed of two parts: an expected future inflation and a real yield. The inflation part of the model is a weighted moving average model. The real part is essentially an autoregressive model of order one with a contribution from the dividend yield. This model, with  $CW = 1$ , fully takes into account the 'Fisher effect', in which the nominal yield on bonds reflects both expected inflation over the life of the bond and a 'real' rate of interest.

Figure 14 shows the graphs of consols yields and bank rates. It illustrates that Bank Rate (which is used for short-term interest rates) is clearly connected with the consols yield (which is used for long-term interest rates).

### Updating and Rebasing to 1923-2007

Wilkie fixed  $CW = 1$  and  $CD = 0.045$  as parameters in the consols yield model. He also suggested investigating alternative values of  $CD$  and  $CW$  and as another option to allow the effect of inflation to be omitted before some date and to include it gradually after that date. This could represent better the way investors may actually have thought: before a certain time, inflation was not considered to be permanent and the nominal rate was taken as the real one; an appreciation of the difference only affected the market slowly.

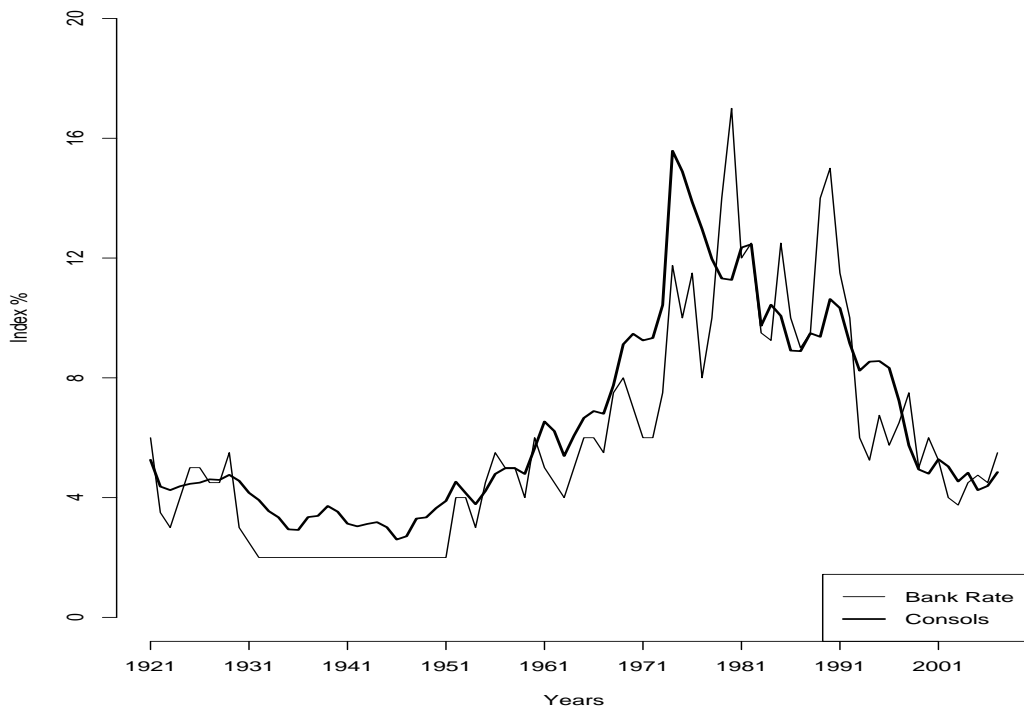


Figure 14: Yield on consols and bank rate, 1921-2007

Wilkie tried various time-varying models of this kind and suggested the above model as the best among them.

When we applied the suggested model to updated data using the fixed values of  $CW$  and  $CD$ , we got negative real interest rates, which is not allowed by the structure of the model. The real interest rate is defined as the difference between the nominal interest rate and expected future inflation. Wilkie modelled expected future inflation as a weighted moving average model,  $CM$ , which takes into account all previous years' inflation, with exponentially decreasing weights. With suggested fixed values, we have negative real interest rates for five years after 1998 (1999, 2000, 2003, 2005, 2006) because  $CM$  exceeds  $C$ . This means that the expected future inflation does not take into account the previous years' inflation as assumed by the model. Therefore we modified the model by introducing a  $min$  value which is called  $CMIN$  and we redefined  $CM(t)$  as:

$$CM(t) = \min(CD.I(t) + (1 - CD).CM(t - 1), C(t) - CMIN)$$

where  $CMIN = 0.005$

It must be emphasized that this adjustment does not affect the  $CM$  term before 1998 and hence does not affect the parameters previously obtained for 1923-1994.

By introducing the  $CMIN$  term, we avoid negative real interest rates, but, as we will discuss below, the model standard deviation increased a lot and the residuals do not satisfy the normality assumption any more; this is an undesirable feature.

Table 5 shows the parameters and the diagnostic test results for the two different periods. As we explained above, in order to apply the model on updated data, we have to modify it to avoid having negative real interest rates. When we examine the parameters for these two periods, we see that the mean level of consols decreased from 3.05% to 2.33% and the dependence on the residuals of the current year's dividend yields increased. Although  $CA$  remained almost the same, indicating strong autocorrelation, the standard deviation of the model increased significantly. Standard errors show that all the parameters are significantly different from zero. For the first period, the autocorrelation coefficients of the standardized residuals indicate that they are uncorrelated and we fail to

$C(t)$	1923-1994	1923-2007
$CW$	1	1
$CD$	0.045	0.045
$CMU\%$	3.05% (0.65%)	2.33% (0.63%)
$CA$	0.8974 (0.0443)	0.8954 (0.0435)
$CY$	0.3371 (0.1454)	0.4690 (0.1910)
$CSD$	0.1853 (0.0154)	0.2568 (0.0197)
$r_z(1)$	0.165	0.076
$r_{z^2}(1)$	0.093	0.073
skewness $\sqrt{\beta_1}$	-0.53	-1.16
kurtosis $\beta_2$	3.58	6.26
Jarque-Bera $\chi^2$	4.88	60.95
$p(\chi^2)$	0.0873	0.0000

Table 5: Parameter estimates for model for  $C$  over 1923-1994 and 1923-2007

reject the normality assumption for a 0.05 significance level. On the other hand, when we fit the model to updated data, though the residuals seem uncorrelated, the Jarque-Bera statistic indicates strong non-normality. Thus, even the modified consols yield model is not so satisfactory for the updated data.

### Parameter Stability

We examine the stability of the parameters of the modified consols yield model by calculating the recursive estimates on incrementally larger data sets as we did in the previous sections. Figure 15 and Figure 16 present the recursive estimates and 95% confidence intervals of the mean level of consols yield  $CMU$ , the autoregressive parameter of the consols model,  $CA$ , the dependence on the previous year's dividend yield innovation,  $CY$ , and the standard deviation,  $CSD$ , respectively, for the earlier (data sets starting in 1923) and later sub-periods (data sets ending in 2007).

The figures indicate that some of the parameters are not stationary over time. The algorithm has not converged for the sub-periods including 30 years or less. The parameters are very sensitive to the initial values, the optimization method used. Especially the  $CMU1$  parameter shows big jumps and large confidence intervals for the first 10 to 30 years and in the 1970s due to the effect of oil crises. It seems stationary for the last 30 years. When we look at  $CMU2$ , it is relatively stationary.

The autoregressive parameter,  $CA$  is quite stationary after the first 20 years. The jump in  $CA1$  in 1970s indicates that the increase in consols yield causes an increase in its dependence on its previous value.

The  $CY$  parameter measures the dependence of consols yield on the current year's dividend yield innovation. Two jumps in  $CY1$  in the 1970s and the late 1990s show that the dependence increased after these years.

When we examine the  $CSD$  graph, we see that it gradually increases over the whole period, but especially during the periods of the Second World War and from 1998 onwards.

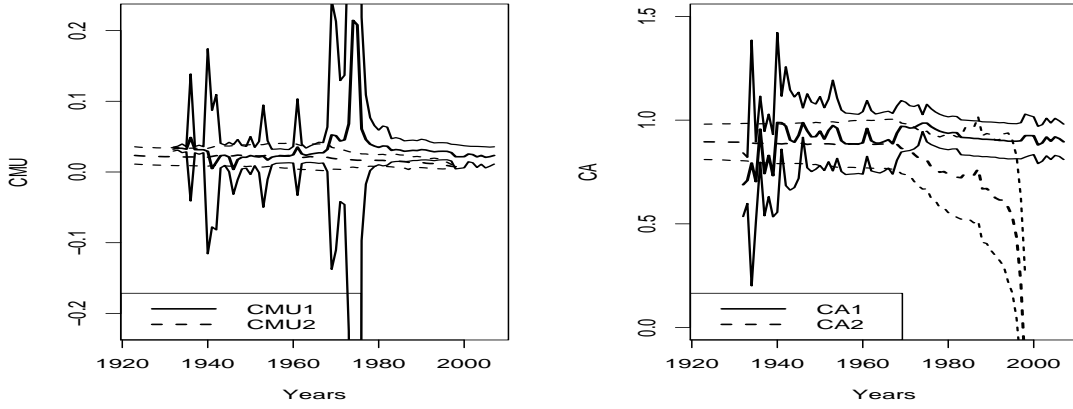


Figure 15: Recursive estimates of  $CMU$  and  $CA$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

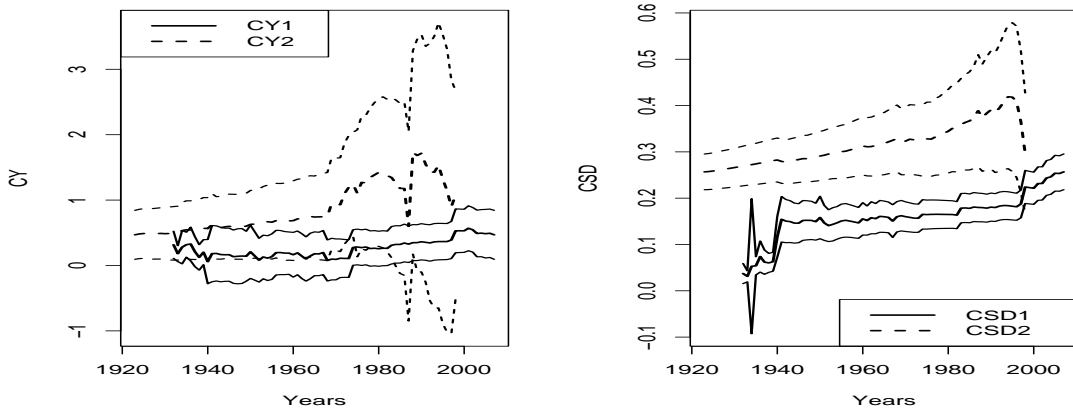


Figure 16: Recursive estimates of  $CY$  and  $CSD$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

## 7 SHORT-TERM INTEREST RATES

### Original Model

Wilkie used bank rate or bank base rate series to model short-term interest rates. The values are plotted in Figure 14, along with the yields on consols.

Short-term interest rates are clearly connected with long-term ones, as shown in Figure 14. Wilkie's approach was to model the difference between the logarithms of the difference of these series where  $B(t)$  is the value of bank rate at time  $t$ :

$$\ln C(t) - \ln B(t) = -\ln(B(t)/C(t))$$

i.e. the logarithm of the ratio of the rates.

Inspection of the data shows that the starting model is an AR(1) model for the log ratio. The short-term rate of interest at time  $t$  is defined as  $B(t)$  and we put:

$$B(t) = C(t) \cdot \exp(-BD(t))$$

where:

$B(t)$	1923-1994	1923-2007
$BMU$	0.2173 (0.0827)	0.1695 (0.0727)
$BA$	0.7420 (0.0804)	0.7275 (0.0757)
$BSD$	0.1808(0.0151)	0.1824 (0.0140)
$r_z(1)$	0.046	0.021
$r_{z^2}(1)$	0.083	0.060
skewness $\sqrt{\beta_1}$	-0.31	-0.27
kurtosis $\beta_2$	3.26	2.96
Jarque-Bera $\chi^2$	1.58	1.07
$p(\chi^2)$	0.4546	0.5852

Table 6: Parameter estimates for model for  $B$  over 1923-1994 and 1923-2007

$$\begin{aligned}
BD(t) &= BMU + BA.(BD(t-1) - BMU) + BE(t) \\
BE(t) &= BSD.BZ(t) \\
BZ(t) &\sim iidN(0, 1)
\end{aligned}$$

Note that  $BD$  has a minus sign in front of it, because short-term yields are, on average, lower than long-term ones.

Wilkie allowed an extra influence from the residuals of the consols yield as a possible elaboration but he decided to take it as zero in the U.K. model, although it was useful for other countries.

### Updating and Rebasing to 1923-2007

Parameter estimates for the short-term interest rate model for 1923-1994 along with estimates for model for 1923-2007 are shown in Table 6.

Table 6 shows that the parameters  $BMU$  and  $BA$  decreased a bit and  $BSD$  increased a bit. All three parameters are significantly different from zero. The autocorrelation coefficients of the residuals do not indicate any dependence in either of the periods. The skewness and kurtosis improved, which resulted in a decrease in the Jarque-Bera statistic. It is clearly seen that we fail to reject the normality assumption. Therefore, when we updated the data we have had a better fit for short-term interest rates.

### Parameter Stability

The stability of the parameters is examined using the same method as in previous sections. Figure 17 and Figure 18 present the recursive estimates and 95% confidence intervals of the mean rate of the ratio,  $BMU$ , the autoregressive parameter,  $BA$ , and the standard deviation,  $BSD$ , respectively, for the earlier (data sets starting in 1923) and later sub-periods (data sets ending in 2007).

Figure 17 shows that the mean rate parameter is stationary over the whole period. When we look at the  $BA$  graphs, we can say that  $BA1$  is quite stable and the confidence interval is shrinking as larger data is considered (through to the right hand end). The right hand end of  $BA2$  indicates a lower dependence on the previous year's ratio (i.e.  $-\ln(B(t)/C(t))$ ).

In Figure 18, after a sharp decrease until the late 1940s,  $BSD1$  has had two jumps but

still seems stationary for the rest of the period and  $BSD2$  is relatively stationary over the whole period.

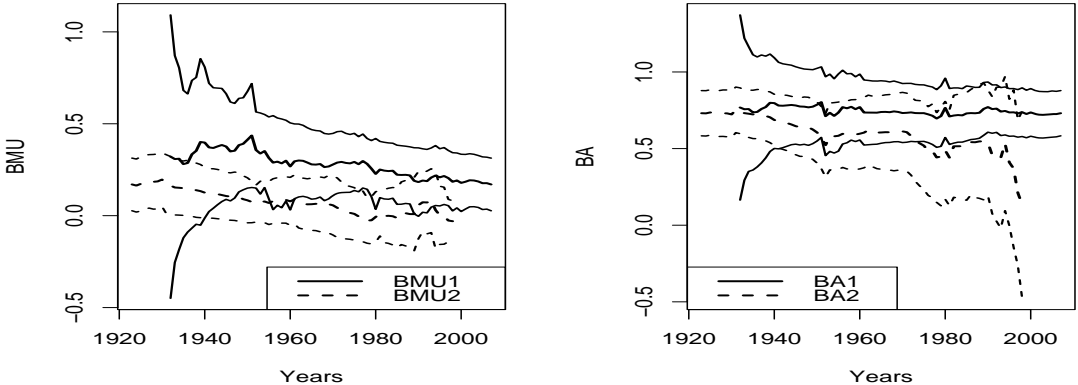


Figure 17: Recursive estimates of  $BMU$  and  $BA$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

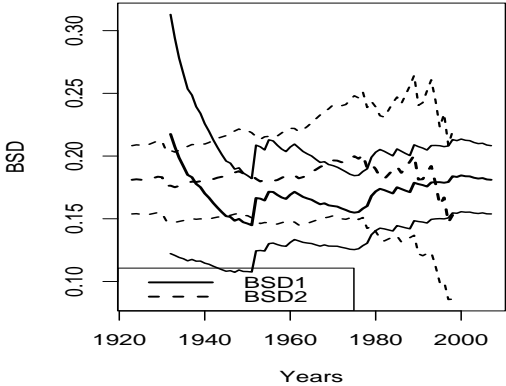


Figure 18: Recursive estimates of  $BSD$  with 95% confidence intervals for earlier and later sub-periods, 1923-2007

## 8 CONCLUSION

In this paper we have examined the Wilkie investment model from a statistical viewpoint and we have re-estimated the parameters by updating the data to 2007. We have considered models for retail prices, including an ARCH model, share dividend yields, share dividends and prices, long-term interest rates and short-term interest rates. We have also recursively estimated the parameters on incrementally larger data sets and have analysed their stability using graphical representations.

The updated parameters of the retail prices model have not changed significantly. Because of low and stable inflation during last 15 years, the mean level of inflation  $QMU$  and the standard deviation  $QSD$  have decreased slightly. The model still does not satisfy the normality assumption and especially the two parameters  $QMU$  and  $QSD$  are not stable over time.

Although the ARCH model satisfies the normality assumption for the 1923-1994 data, its performance gets worse on the updated data and the residuals are not normally distributed any more. The parameters have not changed significantly but especially the  $QSA$  and  $QSB$  parameters highly depend on the initial values and the data and hence these parameters are not stationary.

The share dividend yield model is still satisfactory and the parameters are relatively stable over time.

The performance of the share dividend model is almost the same but its parameters are not stationary. The  $DW$ ,  $DD$  and  $DB$  parameters and their confidence intervals are highly unstable and change greatly as the sub-periods change.

We had to modify the long-term interest rate model to apply it to the updated data by introducing a fixed parameter called  $CMIN$  which is equal to 0.005. In order to avoid negative real interest rates, we used the modified model for 1923-2007 to estimate the parameters. The value of  $CMU$  decreased, and  $CY$  and  $CSD$  increased significantly in the model with updated data. The residuals of the modified model are not normally distributed according to the Jarque-Bera test statistic, and except for  $CY$ , the parameters are not stationary either.

The short-term interest rates model is the best model among them all. It satisfies all the diagnostic tests and fits the data better over the interval 1923-2007. Moreover, its parameters are quite stable.

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