From Ruin Theory to Option Pricing

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Abstract

We examine the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. The time of ruin is analyzed in terms of its Laplace transform, which can naturally be interpreted as discounting. We show how to calculate an expected discounted penalty, which is due at ruin, and may depend on the deficit at ruin and the surplus immediately before ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation. By replacing the penalty at ruin with a payoff at exercise, these results can be applied to price a perpetual American put option on a stock, where the logarithm of the stock price is a shifted compound Poisson process. Because of the stationary nature of the perpetual option, its optimal option-exercise boundary does not vary with respect to the time variable. We have derived an explicit formula for determining the optimal boundary.

Keywords

Collective risk theory, compound Poisson process, surplus process, ruin probability, severity of ruin, deficit at ruin, time of ruin, adjustment coefficient, Lundberg's fundamental equation, Laplace transforms, renewal equation, martingales, optional sampling theorem, stopping time, perpetual American options, put options, continuous junction.

De la théorie de la ruine à l'évaluation du prix d'une option

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Résumé

Nous examinons la distribution conjointe du temps de la ruine, du surplus juste avant la ruine et du déficit au moment de la ruine. Le temps de la ruine est analysé par sa transformée de Laplace, qui peut être interprétée comme une valeur escomptée. Nous montrons comment calculer l'espérance mathématique de la valeur escomptée d'un certain paiement, qui est dû au moment de la ruine, et qui peut être une fonction du déficit et du surplus juste avant la ruine. Cette espérance mathématique, considérée comme fonction du surplus initial, est solution d'une certaine équation de renouvellement. En interprétant le paiement comme le payoff à la date d'exercice, nous pouvons appliquer les résultats pour déterminer le prix d'une option de vente perpétuelle; dans le modèle, le prix de l'action sous-jacente est un processus de Poisson composé avec décalage. A cause de la stationnarité du modèle et du payoff, la frontière d'exercice optimale ne varie pas dans le temps. Nous indiquons une formule explicite pour déterminer cette frontière optimale.

Mots clefs

Théorie du risque collectif, processus de Poisson composé, processus de surplus, probabilité de la ruine, sévérité de la ruine, déficit au moment de la ruine, temps de la ruine, coefficient d'ajustement, équation de Lundberg, transformée de Laplace, équation de renouvellement, martingales, théorème d'arrêt, temps d'arrêt, options américaines perpétuelles, option de vente, jonction continue

1. Introduction

Collective risk theory has started in 1903 with the doctoral thesis of Filip Lundberg, a Swedish actuary, and it has been developed throughout this century. It is now an area rich in useful ideas and sophisticated techniques. Many of its tools can be applied to solve problems in other fields. A recent example is the method of *Esscher transforms,* which was used by Gerber and Shiu (1994a, 1994b, 1996b) to price financial derivatives.

Two particular questions of interest in collective risk theory are (a) the *deficit at ruin,* and (b) the *time of ruin,* both of which have been treated separately in the literature. In this paper the two questions are combined. From a mathematical point of view, a crucial role is played by the amount of surplus immediately before ruin occurs. Hence we examine the joint distribution of three random variables: the surplus immediately before ruin, the deficit at ruin, and the time of ruin. The time of ruin is analyzed in terms of its Laplace transform, which can naturally be interpreted as discounting. We obtain results for the joint distribution by studying the expectation of a discounted *penalty,* which is due at ruin and depends on the deficit at ruin and the surplus immediately prior to ruin. The expected discounted penalty, considered as a function of the initial surplus, satisfies a certain renewal equation.

This paper generalizes and adds to a better understanding of classical ruin theory, which we can retrieve by setting the force of interest (Laplace transform variable) equal to zero. For example, in the classical model, the *adjustment coefficient* is the solution of an implicit equation, which has 0 as the other solution. If the interest rate is positive, the situation is suddenly symmetric: the corresponding equation, called *Lundberg's fundamental equation,* has a positive solution and a negative solution. Both solutions are important and are used to construct exponential martingales.

This paper was motivated by the problem of pricing *perpetual American options.* The classical model uses the geometric Brownian motion to model the stock price process. Such a process has continuous sample paths, which facilitate the analysis of an American option: the option is exercised as soon as the stock price arrives on the optimal exercise boundary, and the price of the option is the expected discounted payoff (Gerber and Shiu 1994b, 1996a). On the other hand, we would like to price an American option in a perhaps more realistic model where the stock price may have jumps, for example drops. The resulting mathematical problem is more intricate, because now, at the time of the exercise, the stock price is not *on* but *beyond* the optimal exercise boundary. If the logarithm of the stock price is modeled by a shifted compound Poisson process, this leads to the type of problems that are discussed in the earlier part of the paper, with "penalty at ruin" replaced by "payoff at exercise."

In our model, perpetual options have the simplicity that the optimal exercise boundary is constant in time; hence it suffices to consider option-exercise strategies with a constant boundary. For the perpetual put option we give an explicit expression for the optimal boundary, and we show that the price can be obtained as the solution of a certain renewal equation. This generalizes results of Michaud (1996, 1997), who uses different methods and treats only the special case of constant jump size.

Formulas (2.33) and (3.3) below are the mathematical keys. They are equivalent formulas. Formula (2.33) is derived analytically in Section 2. Section 3 shows that (3.3) is an immediate consequence of (2.33) and points out that (2.33) can be obtained from (3.3) by probabilistic reasoning. In Section 4, we derive (3.3) by a probabilistic argument.

2. When and How Does Ruin Occur?

We follow the notation in Chapter 12 of *Actuarial Mathematics* (Bowers et al. 1986). Thus $u \geq 0$ is the insurer's initial surplus. The premiums are received continuously at a constant rate c per unit time. The aggregate claims constitute a compound Poisson process, $\{S(t)\}$, with Poisson parameter λ and individual claim amount distribution function $P(x)$, $P(0) = 0$. That is,

$$S(t) = \sum_{j=1}^{N(t)} X_j, \tag{2.1}$$

where $\{N(t)\}$ is a Poisson process with mean per unit time λ and $\{X_j\}$ are independent random variables with common distribution $P(x)$. Then

$$U(t) = u + ct - S(t) \tag{2.2}$$

is the surplus at time t, $t \geq 0$. For simplicity we assume that $P(x)$ is differentiable, with

$$P'(x) = p(x)$$

being the individual claim amount probability density function.

Let T denote the *time of ruin*,

$$T = \inf\{t \mid U(t) < 0\} \tag{2.3}$$

($T = \infty$ if ruin does not occur). We consider the *probability of ultimate ruin* as a function of the initial surplus $U(0) = u \geq 0$,

$$\psi(u) = \Pr[T < \infty \mid U(0) = u]. \tag{2.4}$$

Let p_1 denote the mean of the individual claim amount distribution,

$$p_1 = \int_0^\infty x\, p(x)\, dx = E(X_j).$$

We assume

$$c > \lambda p_1 \tag{2.5}$$

to ensure that $\{U(t)\}$ has a positive drift; hence

$$\lim_{t \to \infty} U(t) = \infty \tag{2.6}$$

with certainty, and

$$\psi(u) < 1. \tag{2.7}$$

Condition (2.5) is a natural assumption in the context of ruin theory; however, it is not crucial for the mathematical development in this paper.

We also consider the random variables U(T-), the *surplus immediately before ruin,* and U(T), the *surplus at ruin.* See Figure 1. For given $U(0) = u \geq 0$, let $f(x, y, t \mid u)$ denote the joint probability density function of U(T-), |U(T)| and T. Then

$$\int_0^\infty\int_0^\infty\int_0^\infty f(x, y, t \mid u)\, dxdydt = \Pr[T < \infty \mid U(0) = u] = \psi(u). \tag{2.8}$$

Because of (2.7), $f(x, y, t \mid u)$ is a *defective* probability density function. We remark that, for $x > u + ct$,

$$f(x, y, t \mid u) = 0,$$

and that

$$f(u{+}ct, y, t \mid u)\, dxdydt = e^{-\lambda t}\lambda p(u{+}ct{+}y)dydt.$$

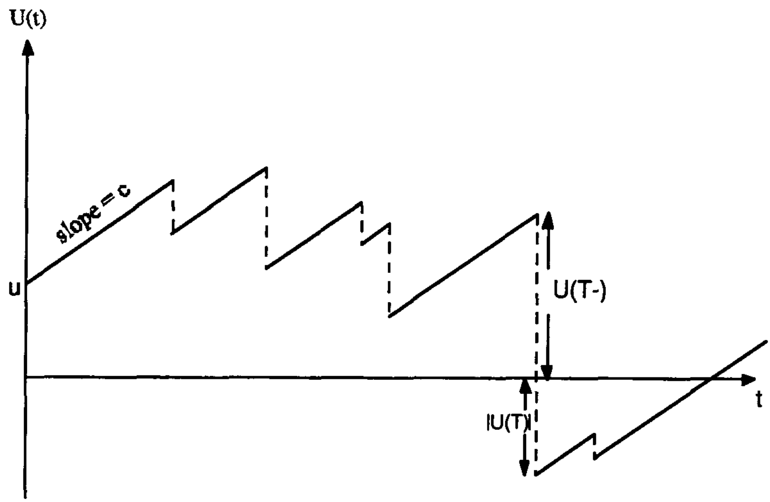

Figure 1. The Surplus Immediately before and at Ruin

It is easier to analyze the following function, the study of which is a central theme in this paper. For $\delta \geq 0$, define

$$f(x, y \mid u) = \int_0^\infty e^{-\delta t} f(x, y, t \mid u)\, dt. \tag{2.9}$$

Here δ can be interpreted as a force of interest, or, in the context of Laplace transforms, as a dummy variable. For notational simplicity, the symbol $f(x, y \mid u)$ does not exhibit the dependence on δ. If $\delta = 0$, (2.9) is the defective joint probability density function of U(T-) and |U(T)|, given U(0) = u. Also, if $\delta > 0$, then

$$e^{-\delta T} = e^{-\delta T} I(T < \infty),$$

where I denotes the indicator function, i.e., I(A) = 1 if A is true and I(A) = 0 if A is false.

Let w(x, y) be a nonnegative function of x > 0 and y > 0. We consider, for $u \geq 0$, the function $\phi(u)$ defined as

$$\phi(u) = E[w(U(T-), |U(T)|) e^{-\delta T} I(T < \infty) \mid U(0) = u] \qquad (2.10)$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty w(x, y) e^{-\delta t} f(x, y, t \mid u) \, dt \, dx \, dy \qquad (2.11)$$

$$= \int_0^\infty \int_0^\infty w(x, y) f(x, y \mid u) \, dx \, dy. \qquad (2.12)$$

Note that the symbol $\phi(u)$ does not exhibit the dependence on the parameter δ and the function w(x, y). For $x_0 > 0$ and $y_0 > 0$, if w(x, y) is a *generalized function* with mass 1 for $(x, y) = (x_0, y_0)$ and 0 for other values of (x, y), then

$$\phi(u) = f(x_0, y_0 \mid u).$$

Hence the analysis of the function f(x, y | u) is included in the analysis of the function $\phi(u)$.

If we interpret δ as a force of interest and w as some kind of penalty when ruin occurs, then $\phi(u)$ is the expectation of the discounted penalty. If w is interpreted as the benefit amount of an insurance (or reinsurance) payable at the time of ruin, then $\phi(u)$ is the single premium of the insurance. We should clarify that, while it can be very helpful to consider δ as a force of interest, we are dealing with the classical collective risk model here; the surplus does not earn any interest.

An interesting example of a penalty function is

$$w(x, y) = (1 - e^{-\rho y})/\delta,$$

where ρ is the positive solution of Lundberg's fundamental equation (to be discussed later in this section). Then $\phi(u)$ is the expected present value of a deferred continuous annuity at a rate of 1 per unit time, starting at the time of ruin and ending as soon as the surplus rises to zero. This example is discussed in Gerber and Shiu (1997). In the context of option pricing, penalty at ruin is replaced by payoff at exercise. The payoff function considered in Section 5 is

$$w(x, y) = \max(K - e^{a-y}, 0)$$
$$= (K - e^{a-y})_+,$$

where K is the exercise price of a put option, and e^a is the value of an option-exercise boundary.

Our immediate goal is to derive a functional equation for $\phi(u)$ by applying the law of iterated expectations to the right-hand side of (2.10). For h > 0, consider the time interval (0, h), and condition on the time t and the amount x of the first claim in this time interval. Note that the probability that there is no claim up to time h is $e^{-\lambda h}$, the probability that the first claim occurs between time t and time t + dt is $e^{-\lambda t}\lambda dt$, and

$$x > u + ct$$

means that ruin has occurred with the first claim. Hence

$$\phi(u) = e^{-(\delta+\lambda)h}\phi(u+ch) + \int_0^h \Big[\int_0^{u+ct}\phi(u+ct-x)p(x)dx\Big]e^{-(\delta+\lambda)t}\lambda dt$$
$$+ \int_0^h \Big[\int_{u+ct}^{\infty} w(u+ct, x-u-ct)p(x)dx\Big]e^{-(\delta+\lambda)t}\lambda dt. \tag{2.13}$$

Differentiating (2.13) with respect to h and setting h = 0, we obtain

$$0 = -(\delta+\lambda)\phi(u) + c\phi'(u) + \lambda\int_0^u \phi(u-x)\,p(x)\,dx$$
$$+ \lambda\int_u^{\infty} w(u, x-u)\,p(x)\,dx$$
$$= -(\delta+\lambda)\phi(u) + c\phi'(u) + \lambda\int_0^u \phi(u-x)\,p(x)\,dx + \lambda\omega(u), \tag{2.14}$$

where

$$\omega(u) = \int_u^{\infty} w(u, x-u)\,p(x)\,dx \tag{2.15}$$

$$= \int_0^{\infty} w(u, y)\,p(u+y)\,dy. \tag{2.16}$$

For further analysis, we use the technique of *integrating factors.* Let

$$\phi_\rho(u) = e^{-\rho u}\phi(u), \tag{2.17}$$

where ρ is a nonnegative number to be specified later. Multiplying (2.14) with $e^{-\rho u}$, applying the product rule for differentiation, and rearranging yields

$$c\phi_\rho'(u) = (\delta+\lambda-c\rho)\phi_\rho(u) - \lambda\int_0^u \phi_\rho(u-x)e^{-\rho x}p(x)dx - \lambda e^{-\rho u}\omega(u). \tag{2.18}$$

Define

$$\ell(\xi) = \delta + \lambda - c\xi; \tag{2.19}$$

hence the coefficient of $\phi_\rho(u)$ in (2.18) is $\ell(\rho)$. In this paper we let $\hat{f}$ denote the Laplace transform of a function f,

$$\hat{f}(\xi) = \int_0^{\infty} e^{-\xi x} f(x)\,dx. \tag{2.20}$$

The Laplace transform of p, $\hat{p}(\xi)$, is defined for all nonnegative numbers ξ, and is a decreasing convex function because

$$\hat{p}'(\xi) = -\int_0^{\infty} e^{-\xi x}\,x\,p(x)\,dx < 0$$

and

$$\hat{p}''(\xi) = \int_0^{\infty} e^{-\xi x}\,x^2\,p(x)\,dx > 0.$$

Consider the equation

$$\ell(\xi) = \lambda\hat{p}(\xi). \tag{2.21}$$

Since the linear function $\ell(\xi)$ has a negative slope and

$$\ell(0) = \delta + \lambda \geq \lambda = \lambda\hat{p}(0),$$

equation (2.21) has a unique nonnegative root, say ξ_1. Furthermore, if the individual claim amount density function p is sufficiently regular, equation (2.21) has one more root, say ξ_2, which is negative. This negative root will be denoted as $-R$. As we shall see in Section 4, both roots are related to the construction of exponential martingales. When $\delta = 0$, R is the *adjustment coefficient* in classical risk theory. Equation (2.21) is equivalent to Beekman (1974, p. 41, top equation), Panjer and Willmot (1992, Eq. 11.7.8), and Seal (1969, Eq. 4.24). Lundberg (1932, p. 144) points out that the equation is "fundamental to the whole of collective risk theory," and Seal (1969, p. 111) calls it "Lundberg's (1928) 'fundamental' equation." [Seal (1969, p. 112) asserts incorrectly that the second root is also positive.]

The trick for solving (2.18) is to choose

$$\rho = \xi_1, \tag{2.22}$$

so that (2.18) becomes

$$\begin{aligned} c\phi_\rho'(u) &= \lambda\hat{p}(\rho)\phi_\rho(u) - \lambda\int_0^u \phi_\rho(u-x)e^{-\rho x}p(x)dx - \lambda e^{-\rho u}\omega(u) \\ &= \lambda\Big[\hat{p}(\rho)\phi_\rho(u) - \int_0^u \phi_\rho(x)e^{-\rho(u-x)}p(u-x)dx - e^{-\rho u}\omega(u)\Big]. \end{aligned} \tag{2.23}$$

For $z > 0$, we integrate (2.23) from $u = 0$ to $u = z$. After a division by λ, the resulting equation is

$$\begin{aligned} &\frac{c}{\lambda}[\phi_\rho(z) - \phi_\rho(0)] \\ &= \hat{p}(\rho)\int_0^z\phi_\rho(u)du - \int_0^z\Big[\int_0^u \phi_\rho(x)e^{-\rho(u-x)}p(u-x)dx\Big]du - \int_0^z e^{-\rho u}\omega(u)du \\ &= \hat{p}(\rho)\int_0^z\phi_\rho(u)du - \int_0^z\Big[\int_x^z e^{-\rho(u-x)}p(u-x)du\Big]\phi_\rho(x)dx - \int_0^z e^{-\rho u}\omega(u)du \\ &= \int_0^z\phi_\rho(x)\Big[\int_{z-x}^\infty e^{-\rho y}p(y)dy\Big]dx - \int_0^z e^{-\rho u}\omega(u)du. \end{aligned} \tag{2.24}$$

For $z \to \infty$, the first terms on both sides of (2.24) vanish, which shows that

$$\phi_\rho(0) = \frac{\lambda}{c}\int_0^\infty e^{-\rho u}\omega(u)du = \frac{\lambda}{c}\,\hat{\omega}(\rho). \tag{2.25}$$

Substituting (2.25) in (2.24) and simplifying yields

$$\phi_\rho(z) = \frac{\lambda}{c}\Big\{\int_0^z\phi_\rho(x)\Big[\int_{z-x}^\infty e^{-\rho y}p(y)dy\Big]dx + \int_z^\infty e^{-\rho u}\omega(u)du\Big\}, \quad z \ge 0. \tag{2.26}$$

Multiplying (2.26) with $e^{\rho z}$ and applying (2.17), we have

$$\phi(z) = \frac{\lambda}{c}\Big\{\int_0^z\phi(x)\Big[\int_{z-x}^\infty e^{\rho(z-x-y)}p(y)dy\Big]dx + \int_z^\infty e^{\rho(z-u)}\omega(u)du\Big\}. \tag{2.27}$$

For two integrable functions f_1 and f_2 defined on $[0, \infty)$, the *convolution* of f_1 and f_2 is the function

$$(f_1*f_2)(x) = \int_0^x f_1(y)\, f_2(x-y)\, dy, \qquad x \ge 0. \tag{2.28}$$

Note that

$$f_1*f_2 = f_2*f_1.$$

With the definitions

$$g(x) = \frac{\lambda}{c}\int_x^\infty e^{-\rho(y-x)}\, p(y)\, dy \tag{2.29}$$

$$= \frac{\lambda}{c}\int_0^\infty e^{-\rho z}\, p(x+z)\, dz, \qquad x \ge 0, \tag{2.30}$$

and

$$h(x) = \frac{\lambda}{c}\int_x^\infty e^{-\rho(u-x)}\, \omega(u)\, du \tag{2.31}$$

$$= \frac{\lambda}{c}\int_x^\infty\int_0^\infty e^{-\rho(u-x)}\, w(u, y)\, p(u+y)\, dy\, du, \quad x \ge 0, \tag{2.32}$$

equation (2.27) can be written more concisely as

$$\phi = \phi * g + h. \tag{2.33}$$

In the literature of integral equations, (2.33) is classified as a *Volterra equation of the second kind.* The function g is a nonnegative function on $[0, \infty)$ and hence may be interpreted as a (not necessarily proper) probability density function; in probability theory, (2.33) is known as a *renewal equation* for the function ϕ.

The solution of (2.33) can be expressed as an infinite series of functions, sometimes called a *Neumann series,*

$$\phi = h + g*h + g*g*h + g*g*g*h + g*g*g*g*h + \cdots . \tag{2.34}$$

One may obtain (2.34) from (2.33) by the method of successive substitution.

Remarks (i) It follows from the conditional probability formula,

$$\Pr(A \cap B) = \Pr(A)\Pr(B \mid A),$$

that the joint probability density function of U(T-), |U(T)| and T at the point (x, y, t) is the joint probability density function of U(T-) and T at the point (x, t) multiplied by the conditional probability density function of |U(T)| at y, given that U(T-) = x and T = t. The latter does not depend on t and is

$$\frac{p(x+y)}{\int_0^\infty p(x+y)\, dy} = \frac{p(x+y)}{1 - P(x)}, \qquad y \ge 0.$$

Hence

$$f(x, y, t \mid u) = \left[\int_0^\infty f(x, z, t \mid u)\, dz\right]\frac{p(x+y)}{1 - P(x)}. \tag{2.35}$$

With the definition

$$f(x \mid u) = \int_0^\infty f(x, y \mid u)\, dy$$

$$= \int_0^\infty\int_0^\infty e^{-\delta t}\, f(x, y, t \mid u)\, dt\, dy, \tag{2.36}$$

multiplying (2.35) with $e^{-\delta t}$ and then integrating with respect to t yields

$$f(x, y \mid u) = f(x \mid u)\,\frac{p(x+y)}{1 - P(x)}. \tag{2.37}$$

With $\delta = 0$, (2.37) was pointed out by Dufresne and Gerber (1988, Eq. 3); another proof can be found in Dickson and Egídio dos Reis (1994).

(ii) It follows from an integration by parts that

$$\hat{p}(\xi) = 1 - \xi\int_0^\infty e^{-\xi x}[1 - P(x)]\,dx,$$

with which we can rewrite Lundberg's fundamental equation (2.21) as

$$\begin{aligned}\delta &= c\xi - \lambda[1 - \hat{p}(\xi)] \\ &= \xi\{c - \lambda\int_0^\infty e^{-\xi x}[1 - P(x)]\,dx\}.\end{aligned} \tag{2.38}$$

Hence

$$\begin{aligned}\lim_{\delta\to 0}\frac{\delta}{\rho} &= c - \lambda\int_0^\infty [1 - P(x)]\,dx \\ &= c - \lambda p_1,\end{aligned} \tag{2.39}$$

which is the drift of $\{U(t)\}$.

3. Zero initial surplus

In this section we study functions such as $f(x \mid 0)$ [defined by (2.36)] and $f(x, y \mid 0)$ [defined by (2.9)]. With initial surplus $U(0) = u = 0$, some very explicit results can be obtained. Since ϕ satisfies the renewal equation (2.33), it follows that

$$\phi(0) = h(0). \tag{3.1}$$

Applying (2.12) and (2.32) to (3.1) yields

$$\int_0^\infty\int_0^\infty w(x, y)\, f(x, y \mid 0)\,dx\,dy = \frac{\lambda}{c}\int_0^\infty\int_0^\infty e^{-\rho x}\, w(x, y)\, p(x + y)\,dx\,dy. \tag{3.2}$$

Because this identity holds for an arbitrary function w, it follows that

$$f(x, y \mid 0) = \frac{\lambda}{c} e^{-\rho x}\, p(x + y), \qquad x > 0, y > 0. \tag{3.3}$$

We just saw that (3.3) follows from (2.33). The two formulas are really equivalent, since (2.33) can be obtained immediately from (3.3) by probabilistic reasoning. It suffices to look at the first time when the surplus falls below the initial level u, distinguish according to whether or not ruin takes place at that time, and apply the law of iterated expectations; see Section 5 of Gerber and Shiu (1997).

Formula (3.3) plays a central role. An alternative proof and additional insight will be given in Section 4. Some immediate consequences can be obtained by integrating over x, y, and both:

$$\begin{aligned}\int_0^\infty f(x, y \mid 0)\,dx &= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}\, p(x + y)\,dx \\ &= g(y),\end{aligned} \tag{3.4}$$

as defined by (2.30);

$$\begin{aligned}f(x \mid 0) &= \int_0^\infty f(x, y \mid 0)\,dy \\ &= \frac{\lambda}{c}e^{-\rho x}\int_0^\infty p(x + y)\,dy \\ &= \frac{\lambda}{c}e^{-\rho x}[1 - P(x)];\end{aligned} \tag{3.5}$$

$$E[e^{-\delta T} I(T<\infty) \mid U(0)=0] = \int_0^\infty \int_0^\infty f(x, y \mid 0)\, dy\, dx$$
$$= \frac{\lambda}{c}\int_0^\infty e^{-\rho x}[1 - P(x)]\, dx. \tag{3.6}$$

As a check, note that (3.3) and (3.5) satisfy (2.37) with $u = 0$.

With $\delta = 0$, and hence $\rho = 0$, (3.3) reduces to a result of Dufresne and Gerber (1988, Eq. 10). In particular,

$$f(x, y \mid 0) = f(y, x \mid 0). \tag{3.7}$$

Dickson (1992) has pointed out that this symmetry can be explained in terms of *duality*. Further discussion can be found in Dickson and Egídio dos Reis (1994), and Gerber and Shiu (1997). For $\delta > 0$, formula (3.7) does not hold any longer.

For $\delta = 0$, (3.6) reduces to the famous formula

$$\psi(0) = \frac{\lambda}{c}\int_0^\infty [1 - P(x)]\, dx$$
$$= \lambda p_1/c. \tag{3.8}$$

For $\delta > 0$, we can use (3.6) and the fact that ρ is a solution of (2.38) to see that

$$E[e^{-\delta T} \mid U(0)=0] = E[e^{-\delta T} I(T<\infty) \mid U(0)=0]$$
$$= 1 - \frac{\delta}{c\rho}. \tag{3.9}$$

Formula (3.8) can be obtained as a limiting case of (3.9) because of (2.39).

Example Let us look at the case of an exponential individual claim amount distribution,

$$p(x) = \beta e^{-\beta x}, \qquad x \geq 0, \tag{3.10}$$

with $\beta > 0$ and $c > \lambda p_1 = \frac{\lambda}{\beta}$. The number ρ is ξ_1, the nonnegative solution of (2.21), which now is

$$\delta + \lambda - c\xi = \frac{\lambda\beta}{\beta + \xi},$$

or

$$c\xi^2 + (c\beta - \delta - \lambda)\xi - \beta\delta = 0. \tag{3.11}$$

Hence

$$\rho = \xi_1$$
$$= \frac{\lambda + \delta - c\beta + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}{2c}. \tag{3.12}$$

(Note that, if $\delta = 0$, then $\rho = \xi_1 = 0$.) Then

$$f(x, y \mid 0) = \frac{\lambda\beta}{c} e^{-(\rho+\beta)x - \beta y} = \frac{\lambda}{c} e^{-(\rho+\beta)x} p(y);$$
$$g(y) = \frac{\lambda\beta}{c(\beta+\rho)} e^{-\beta y} = \frac{\lambda}{c(\beta+\rho)} p(y); \tag{3.13}$$
$$f(x \mid 0) = \frac{\lambda}{c} e^{-(\rho+\beta)x};$$

$$E[e^{-\delta T} I(T < \infty) \mid U(0) = 0] = \int_0^\infty\int_0^\infty f(x, y \mid 0)\, dy\, dx$$

$$= \frac{\lambda}{c(\beta + \rho)} \tag{3.14}$$

$$= \frac{2\lambda}{c\beta + \delta + \lambda + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}. \tag{3.15}$$

An alternative to (3.14) and (3.15) is formula (3.9), which is simple and general at the same time. In Section 4 we shall show that

$$E[e^{-\delta T} I(T < \infty) \mid U(0) = u] = E[e^{-\delta T} I(T < \infty) \mid U(0) = 0]\, e^{\xi_2 u}, \tag{3.16}$$

where ξ_2 is the negative root of (3.11); see (4.14) and (4.19). Finally, we note that (2.37) can be simplified to

$$f(x, y \mid u) = f(x \mid u)\, p(y), \qquad u \geq 0, x > 0, y > 0. \tag{3.17}$$ ||||

4. Martingales

Further insight can be obtained with *martingale theory*. Let ξ be a number. Because $\{U(t)\}_{t\geq 0}$ is a stochastic process with stationary and independent increments, a process of the form

$$\{e^{-\delta t + \xi U(t)}\}_{t\geq 0} \tag{4.1}$$

is a martingale if and only if, for each $t > 0$, its expectation at time t is equal to its initial value, i.e., if and only if

$$E[e^{-\delta t + \xi U(t)} \mid U(0) = u] = e^{\xi u}. \tag{4.2}$$

Since

$$E[e^{-\delta t + \xi U(t)} \mid U(0) = u] = \exp(-\delta t + \xi u + \xi ct + \lambda t[\hat{p}(\xi) - 1]),$$

the martingale condition is that

$$-\delta + c\xi + \lambda[\hat{p}(\xi) - 1] = 0,$$

which is again Lundberg's fundamental equation (2.21). Thus, for (4.1) to be a martingale, the coefficient of U(t) in (4.1) is either $\xi_1 = \rho \geq 0$ or $\xi_2 = -R < 0$.

With such a ξ, (4.2) holds for each fixed t, $t \geq 0$. However, if we replace t by a *stopping time* which is a random variable, then there is no guarantee that (4.2) will hold. Fortunately, it holds in two important cases, as we shall see in this and the next paragraph. If the stopping time is T, the time of ruin, then the *optional sampling theorem* is applicable to the martingale with $\xi = -R$. For $0 \leq t < T$,

$$\delta t + RU(t) \geq 0,$$

and hence

$$0 < e^{-\delta t - RU(t)} \leq 1.$$

With $\{e^{-\delta t - RU(t)}\}_{0\leq t<T}$ being bounded, it follows from the optional sampling theorem that

$$E[e^{-\delta T - RU(T)} \mid U(0) = u] = e^{-Ru}. \tag{4.3}$$

Furthermore, because of (2.6), we have

$$E[e^{-\delta T - RU(T)} I(T = \infty) \mid U(0) = u] = 0$$

even if $\delta = 0$. Consequently, we can rewrite (4.3) as

$$e^{-Ru} = E[e^{-\delta T - RU(T)} I(T < \infty) \mid U(0) = u], \qquad \delta \geq 0, u \geq 0. \tag{4.4}$$

We now show that the quantity $e^{-\rho(x-u)}$, which appears in the last two sections (usually with $u = 0$), has a probabilistic interpretation. For $x > U(0) = u$, let

$$T_x = \min \{t \mid U(t) = x\} \tag{4.5}$$

be the first time when the surplus reaches the level x. We can use equality to define the stopping time T_x because the process $\{U(t)\}$ is jump-free upward. Then, for $0 \leq t \leq T_x$,

$$e^{-\delta t + \rho U(t)} \leq e^{\rho x}. \tag{4.6}$$

Hence we can apply the optional sampling theorem to the martingale $\{e^{-\delta t + \rho U(t)}\}$ to obtain

$$\begin{aligned} e^{\rho u} &= E[e^{-\delta T_x + \rho U(T_x)} \mid U(0) = u] \\ &= E[e^{-\delta T_x} \mid U(0) = u]\, e^{\rho x}, \end{aligned}$$

or

$$e^{-\rho(x-u)} = E[e^{-\delta T_x} \mid U(0) = u]. \tag{4.7}$$

With δ interpreted as a force of interest, the quantity $e^{-\rho(x-u)}$ is the expected discounted value of a payment of 1 due at the time when $U(t) = x$ for the first time. We note that (4.7) remains valid even if u is negative. The required condition is $x > u$; the condition $u \geq 0$ is not needed anywhere in the derivation. Formula (4.7) was probably first given by Kendall (1957, Eq. 14), although he did not provide a complete proof. It can also be found in Prabhu (1961; 1980, p. 79, Theorem 5(i); 1980, p. 105, #4), Cox and Miller (1965, p. 245, Eq. 184), Takács (1967, p. 88, Theorem 8), and Gerber (1990, Eq. 11).

Formula (4.7) can be used to give an alternative proof of the important formula (3.3). For $x > u = U(0)$, let $\pi_1(x, t \mid u)$, $t > 0$, denote the probability density function of the random variable T_x. Hence (4.7) is

$$\int_0^\infty e^{-\delta t} \pi_1(x, t \mid u)\, dt = e^{-\rho(x-u)}. \tag{4.8}$$

The differential $\pi_1(x, t \mid u)dt$ is the probability that the surplus process upcrosses level x between t and t+dt and that then this happens for the first time. We remark that, for $t < (x - u)/c$,

$$\pi_1(x, t \mid u) = 0,$$

and that

$$\pi_1(x, (x-u)/c \mid u)\, dt = e^{-\lambda(x-u)/c},$$

reflecting the point mass of the distribution of T_x.

For $U(0) = u \geq 0$, $x > 0$, let $\pi_2(x, t \mid u)$, $t > 0$, be the function defined by the condition that $\pi_2(x, t \mid u)dt$ is the probability that ruin does not occur by time t and that there is an upcrossing of the surplus process at level x between t and t+dt. It can be proved by duality that

$$\pi_1(x, t \mid 0) = \pi_2(x, t \mid 0), \qquad x > 0, t > 0. \tag{4.9}$$

Now, $f(x, y, t \mid u)dtdxdy$ can be interpreted as the probability of the event that ruin does not take place by time t, that the surplus process upcrosses through level x between time t and time t+dt, but does not attain level x+dx, i.e., that there is a claim within $\frac{dx}{c}$ time units after T_x, and that the size of this claim is between x+y and x+y+dy. Thus

$$f(x, y, t \mid u)\, dt\, dx\, dy = [\pi_2(x, t \mid u)\, dt]\, [\lambda \tfrac{dx}{c}]\, [p(x + y)\, dy], \tag{4.10}$$

from which it follows that

$$f(x, y, t \mid u) = (\lambda/c)\, p(x + y)\, \pi_2(x, t \mid u). \tag{4.11}$$

This formula is particularly useful if $u = 0$: then it follows from (4.9) that

$$f(x, y, t \mid 0) = (\lambda/c)\, p(x + y)\, \pi_1(x, t \mid 0). \tag{4.12}$$

If we multiply (4.12) by $e^{-\delta t}$, integrate from $t = 0$ to $t = \infty$, and apply (4.8) with $u = 0$, we obtain (3.3),

$$f(x, y \mid 0) = (\lambda/c)\, p(x + y)\, e^{-\rho x},$$

once again.

Example Again, consider the case where the individual claim amount distribution is exponential, $p(x) = \beta e^{-\beta x}$. Then R is given by

$$R = -\xi_2 = \frac{c\beta - \delta - \lambda + \sqrt{(c\beta - \delta - \lambda)^2 + 4c\beta\delta}}{2c}. \tag{4.13}$$

Applying (3.17) to (4.4) yields

$$\begin{aligned} e^{-Ru} &= \left[\int_0^\infty e^{Ry} p(y) dy\right] E[e^{-\delta T}\, I(T < \infty) \mid U(0) = u] \\ &= \frac{\beta}{\beta - R} E[e^{-\delta T}\, I(T < \infty) \mid U(0) = u]. \end{aligned}$$

Hence

$$E[e^{-\delta T} I(T < \infty) \mid U(0) = u] = \frac{\beta - R}{\beta} e^{-Ru}. \tag{4.14}$$

To reconcile (4.14) for $u = 0$ with (3.9) we need to show that, for $\delta > 0$,

$$\frac{R}{\beta} = \frac{\delta}{c\rho}. \tag{4.15}$$

Equation (4.15) holds because the product of the two roots of the quadratic equation (3.11) is $-\beta\delta/c$. As a further check, we want to see that (4.14) with $u = 0$ is consistent with (3.14); here we need the identity

$$\frac{\beta - R}{\beta} = \frac{\lambda}{c(\beta + \rho)}. \tag{4.16}$$

Because ρ and –R are the roots of (3.11), we have

$$c(\beta + \rho)(\beta - R) = c(-\beta)^2 + (c\beta - \delta - \lambda)(-\beta) - \beta\delta$$
$$= \lambda\beta,$$

which is (4.16). It follows from (3.13) and (4.16) that

$$g(y) = (\beta - R)e^{-\beta y} = \frac{p(y)}{\hat{p}(-R)}. \tag{4.17}$$

In the particular case where w(x, y) = w(y), a function not depending on x, we can apply (3.17) and (4.14) to obtain an explicit expression for $\phi(u)$:

$$\phi(u) = E[e^{-\delta T} w(|U(T)|)\, I(T < \infty) \mid U(0) = u]$$
$$= \left[\int_0^\infty w(y)p(y)dy\right] E[e^{-\delta T} I(T < \infty) \mid U(0) = u]$$
$$= \left[\int_0^\infty w(y)e^{-\beta y}dy\right](\beta - R)e^{-Ru}$$
$$= \hat{w}(\beta)(\beta - R)e^{-Ru}. \tag{4.18}$$

In particular,

$$\phi(u) = \phi(0)e^{-Ru}. \tag{4.19}$$ ||||

5. Application to Option Pricing

For $t \geq 0$, let A(t) be the price of a traded asset, typically a stock, at time t. We assume that {ln[A(t)]} is a stochastic process of the form {U(t)} given by (2.2), i.e.,

$$A(t) = e^{u + ct - S(t)}, \qquad t \geq 0. \tag{5.1}$$

(However, we do not assume that condition (2.5) holds.) In this model, the stock price process has downward discontinuities; the times and amounts of the drops are random. We assume that the riskless force of interest is a positive constant and denote it as δ. Of course, we suppose that $c > \delta$, because otherwise nobody would invest in the stock.

We assume that the *price* of a derivative security is the expectation of its discounted payoff. The expectation is taken with respect to a so-called *risk-neutral probability measure* or *equivalent martingale measure,* which is not unique in general. There are various methods to obtain a risk-neutral measure from the physical probability measure. One is the method of Esscher transforms, where the Poisson parameter and the jump amount distribution are simultaneously adjusted; another method is to adjust only the Poisson parameter. With the assumption that the asset pays no dividends, the discounted asset price is a martingale under the risk-neutral probability measure, which is the condition that the current price of the asset is the discounted expectation of the price at any future time t,

$$A(0) = e^{-\delta t}E[A(t)],$$

or

$$e^u = \exp(-\delta t + u + ct + \lambda t[\hat{p}(1) - 1]). \tag{5.2}$$

It follows from (5.2) that

$$-\delta + c + \lambda[\hat{p}(1) - 1] = 0, \qquad (5.3)$$

showing that the positive solution of Lundberg's fundamental equation (2.21) is

$$\xi_1 = \rho = 1.$$

See also the first paragraph of Section 4.

Now consider a *perpetual American put option* with exercise price K. (The word "perpetual" means that the option has no expiry date, and hence it cannot be a European option. Although the word "American" is not necessary, it is added here to emphasize that the option holder can exercise the option at any time.) The payoff function is

$$\Pi(s) = \max(K - s, 0)$$
$$= (K - s)_+, \qquad s \geq 0. \qquad (5.4)$$

If the option holder chooses to exercise the option at time t, $t \geq 0$, he receives

$$\Pi(A(t)) = (K - A(t))_+.$$

Because of the stationary nature of the perpetual option, its optimal exercise boundary does not vary with respect to the time variable. Hence it is sufficient to consider option-exercise strategies of the form

$$T = \inf\{t \mid A(t) < e^a\}, \qquad (5.5)$$

with $a \leq \text{Min}(u, \ln K)$. Let $V(u; a)$ denote the expected discounted payoff of the option-exercise strategy,

$$V(u; a) = E[e^{-\delta T} I(T < \infty)\, \Pi(A(T)) \mid A(0) = e^u], \quad u \geq a. \qquad (5.6)$$

With the definition

$$w(x, y) = \Pi(e^{a-y}), \qquad (5.7)$$

we have, for $u \geq a$,

$$V(u; a) = \phi(u - a). \qquad (5.8)$$

Hence it follows from (2.33) that the function

$$\varphi(z) = V(a + z; a), \qquad z \geq 0,$$

satisfies the defective renewal equation

$$\varphi = \varphi * g + h,$$

where, by (2.32),

$$h(z) = \frac{\lambda}{c} \int_z^\infty \Big[\int_0^\infty e^{-(x-z)}\, \Pi(e^{a-y})\, p(x + y)\, dy\Big]\, dx. \qquad (5.9)$$

Since

$$\Pi(e^{a-y}) = (K - e^{a-y})_+$$
$$= K - e^{a-y}, \qquad y \geq 0,$$

formula (5.9) is

$$h(z) = \frac{\lambda e^z}{c} \Big\{ K \int_z^\infty e^{-x}[1 - P(x)]dx - e^a \int_z^\infty \Big[\int_0^\infty e^{-x-y}\, p(x + y)\, dy\Big]\, dx \Big\}. \qquad (5.10)$$

With the change of variables $\zeta = x + y$ and $x = x$, the double integral in (5.10) is

$$\int_z^\infty \Big[\int_z^\zeta e^{-\zeta}\, p(\zeta)\, dx\Big] d\zeta = \int_z^\infty (\zeta - z)\, e^{-\zeta}\, p(\zeta)\, d\zeta. \qquad (5.11)$$

Because

$$V(a; a) = \varphi(0) = h(0),$$

we can obtain an explicit expression for V(a; a). Applying (5.10), (5.11), and (2.38) [with $\xi = \rho = 1$], we have

$$\begin{aligned} V(a; a) &= \frac{\lambda}{c}\{K\int_0^\infty e^{-x}[1 - P(x)]dx - e^a\int_0^\infty \zeta\, e^{-\zeta} p(\zeta)\, d\zeta\} \\ &= \frac{\lambda}{c}[K(c - \delta)/\lambda + e^a \hat{p}'(1)]. \end{aligned} \tag{5.12}$$

This result enables us to determine $\tilde{a}$, the optimal value of a (thus $e^{\tilde{a}}$ is the optimal option-exercise stock price). If

$$V(a; a) < \Pi(e^a),$$

we gather that $\tilde{a} > a$, and if

$$V(a; a) > \Pi(e^a),$$

we conclude that $\tilde{a} < a$. Hence $\tilde{a}$ is determined by the *continuous junction condition*

$$\begin{aligned} V(\tilde{a}; \tilde{a}) &= \Pi(e^{\tilde{a}}) \\ &= K - e^{\tilde{a}}. \end{aligned} \tag{5.13}$$

We substitute (5.12) with $a = \tilde{a}$ in the left-hand side of (5.13) and obtain a linear equation for $e^{\tilde{a}}$. The solution is

$$e^{\tilde{a}} = K\frac{\delta}{c + \lambda\hat{p}'(1)}. \tag{5.14}$$

Hence, for $u \ge \tilde{a}$, the price of the perpetual put option is $V(u; \tilde{a}) = \phi(u - \tilde{a})$. For $u < \tilde{a}$, the option is exercised immediately and its price is simply $K - e^u$.

Remark It is intuitively clear that the optimal option-exercise stock price has to be less than the exercise price, i.e., $e^{\tilde{a}} < K$. An algebraic proof that the fraction on the right-hand side of (5.14) is between 0 and 1 goes as follows. Using (5.3) and the inequality that

$$1 - e^{-z} > z\, e^{-z}, \qquad z > 0,$$

we see that

$$\begin{aligned} 0 < \delta &= c - \lambda[1 - \hat{p}(1)] \\ &= c - \lambda[1 - \int_0^\infty e^{-z} p(z)\, dz] \\ &= c - \lambda\int_0^\infty (1 - e^{-z})\, p(z)\, dz \\ &< c - \lambda\int_0^\infty z\, e^{-z} p(z)\, dz, \end{aligned} \tag{5.15}$$

which is the denominator on the right-hand side of (5.14). Hence the fraction in (5.14) is indeed between 0 and 1.

Example Suppose that the jumps are exponentially distributed,

$$P(x) = 1 - e^{-\beta x}, \qquad x \ge 0.$$

By (5.14), the optimal option-exercise stock price is

$$e^{\tilde{a}} = K\frac{\delta}{c - \lambda\beta(1+\beta)^{-2}}. \tag{5.16}$$

On the other hand, it follows from (5.8) [with w defined by (5.7) and (5.4)] and (4.18) that, for u ≥ a,

$$\begin{aligned} V(u; a) &= \phi(u-a) \\ &= \Big[\int_0^\infty (K - e^{a-y})_+ e^{-\beta y} dy\Big](\beta - R)e^{-R(u-a)} \\ &= \Big[\frac{K}{\beta} - \frac{e^a}{\beta+1}\Big](\beta - R)e^{-R(u-a)}, \end{aligned} \tag{5.17}$$

which is maximal at

$$e^{\tilde{a}} = K\frac{\beta+1}{\beta}\frac{R}{R+1}. \tag{5.18}$$

Let us reconcile (5.18) with (5.16). It follows from (4.15), with ρ = 1, that

$$R = \delta\beta/c. \tag{5.19}$$

Substituting (5.19) in (5.18) and then applying (5.3), we have

$$\begin{aligned} e^{\tilde{a}} &= K\frac{\beta+1}{\beta}\frac{\delta\beta}{\delta\beta+c} \\ &= K(\beta+1)\frac{\delta}{[c-\lambda(1-\hat{p}(1))]\beta + c}. \end{aligned} \tag{5.20}$$

Since

$$\hat{p}(1) = \beta/(\beta+1),$$

(5.20) is the same as (5.16).

Finally, let κ denote the ratio of the exercise price to the stock price,

$$\kappa = K/A(0) = Ke^{-u}. \tag{5.21}$$

Then it follows from (5.17) and (5.18) that, for u > ã, the ratio of the option price to the stock price is

$$\begin{aligned} \frac{V(u;\tilde{a})}{A(0)} &= \frac{\kappa}{\beta}\Big[1 - \frac{R}{R+1}\Big](\beta - R)e^{-R(u-\tilde{a})} \\ &= \frac{\kappa(\beta - R)}{\beta(R+1)}\left[\frac{\kappa(\beta+1)R}{\beta(R+1)}\right]^R. \end{aligned}$$

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