On the Perimeter and Area of the Unit Disc

Juan Carlos Álvarez Paiva and Anthony Thompson

1. INTRODUCTION. In the preamble to his fourth problem (presented at the International Mathematical Congress in Paris in 1900) Hilbert suggested a thorough examination of geometries that "stand next to Euclidean geometry" in the sense that they satisfy all the axioms of Euclidean geometry except one. In non-Euclidean geometries the axiom that is usually taken to fail is the famous parallel postulate. This leads to the relatively well-known hyperbolic and elliptic geometries. The significance of these is that, like Euclidean geometry, they are homogeneous (all points have the same status) and isotropic (all directions have the same status).

Another type of geometry that "stands next to Euclidean geometry" is the geometry of normed spaces. Here translating a line segment does not change its length, but the axiom that states that two triangles with equal corresponding sides are congruent no longer holds. These geometries are homogeneous but not isotropic.

In this article we survey some of the most basic results on the geometry of unit discs in two-dimensional normed spaces, while adding a few results and some new proofs of our own. These results answer simple questions about the perimeter of the unit disc, its area, and the relationships between these two quantities.

2. MINKOWSKI PLANES AND DUALITY. A *normed plane* is a pair $(\mathbb{R}^2, \|\cdot\|)$, where the function $\mathbf{q} \mapsto \|\mathbf{q}\|$ is a *norm*:

- $\|\mathbf{q}\| \geq 0$ with equality if and only if $\mathbf{q}$ is the zero vector;
- $\|t\mathbf{q}\| = |t|\|\mathbf{q}\|$;
- $\|\mathbf{q}+\mathbf{x}\| \leq \|\mathbf{q}\| + \|\mathbf{x}\|$.

If we change viewpoint and consider a normed plane as a metric space $(\mathbb{R}^2, d)$ with distance function $d(\mathbf{q},\mathbf{x}) = \|\mathbf{q}-\mathbf{x}\|$, then we call it a *Minkowski plane,* or a *Minkowski geometry.*

Some classical examples of normed and Minkowski planes are provided by p-norms $\|\mathbf{q}\|_p := (|q_1|^p + |q_2|^p)^{1/p}$ $(1 \leq p < \infty)$ and the *maximum* (or *supremum*) norm $\|\mathbf{q}\|_\infty := \max\{|q_1|, |q_2|\}$.

Exercise 1. Consider the 1-norm (or *taxicab norm*) on the plane (i.e., $\|\mathbf{q}\| := |q_1|+|q_2|$) and its corresponding distance function $d(\mathbf{q},\mathbf{x}) = \|\mathbf{q}-\mathbf{x}\|$. Construct two noncongruent, nondegenerate triangles with corresponding sides of length 1, 1, and 2.

In his pioneering work on the geometry of numbers, Minkowski realized that it is often better to consider the unit disc

$$D := \{\mathbf{q} : \|\mathbf{q}\| \leq 1\}$$

as the starting point for the investigation of normed planes and Minkowski geometries. The definition of a norm implies that the unit disc

(1) is a closed, bounded set with $\mathbf{0}$ in its interior;
(2) is symmetric with respect to $\mathbf{0}$ (i.e., if $\mathbf{q}$ belongs to D, then so does $-\mathbf{q}$); and
(3) is convex.

Conversely, if K is a set satisfying the properties (1), (2), and (3), then the function

$$\|\mathbf{q}\|_K := \inf\{t > 0 : \mathbf{q}/t \in K\}$$

is a norm on $\mathbb{R}^2$ for which K is the unit disc.

Condition (3) is crucial for the triangle inequality and this is why we cannot allow $p < 1$ in the definition of $\|\mathbf{q}\|_p$. Figure 1 shows, in increasing order, the nonconvex curve we get for $p = 1/2$ and the unit discs for the p-norms on the plane for $p = 1$, 2, 4, and ∞.

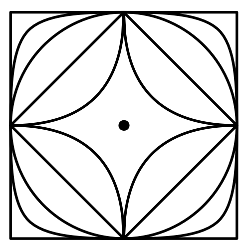

Figure 1.

By considering symmetric convex bodies and the norms that they generate, we obtain a great variety of norms on the plane. The invariance under translations of Minkowski geometries makes them *homogeneous*—every point behaves like every other point. However, these geometries are *not* isotropic except in the case where the norm comes from an inner product (i.e., the unit disc is an ellipse).

Isometries of normed planes. The study of isometries, or distance-preserving maps, between Minkowski planes is greatly simplified by a celebrated theorem of Mazur and Ulam (see [23, p. 76]) that states that if the isometry is surjective, then it is the composition of an invertible linear map and a translation. Such transformations are called *affine transformations,* and the group they form underlies the study of Minkowski geometry.

A ready consequence of the Mazur-Ulam theorem is that two Minkowski planes are isometric if and only if their unit discs are linearly equivalent: there exists an invertible linear transformation from $\mathbb{R}^2$ to itself that takes the unit disc of one Minkowski space to the unit disc of the other. This implies that, apart from translations, a general Minkowski plane does not admit many isometries onto itself.

Exercise 2. Translations, the identity map I, and the symmetry $-I$ are isometries for any Minkowski plane. Show that in general these are the only ones by constructing a convex set K such that the normed space $(\mathbb{R}^2, \|\cdot\|_K)$ has this minimal set of isometries.

Duality in normed planes. A major feature of Minkowski geometry is the notion of duality. The *dual space* of $\mathbb{R}^2$, denoted by $\mathbb{R}^{2*} := (\mathbb{R}^2)^*$, is the space of all linear functionals from $\mathbb{R}^2$ to $\mathbb{R}$. If $\mathbf{p}$ belongs to $\mathbb{R}^{2*}$ and $\mathbf{q}$ to $\mathbb{R}^2$, we denote the pairing of $\mathbf{p}$ and $\mathbf{q}$ by $\mathbf{p} \cdot \mathbf{q}$. A norm $\|\cdot\|$ on $\mathbb{R}^2$ induces a *dual norm* $\|\cdot\|^*$ on $\mathbb{R}^{2*}$ by the formula

$$\|\mathbf{p}\|^* := \sup\{|\mathbf{p} \cdot \mathbf{q}| : \|\mathbf{q}\| \leq 1\} .$$

If K is the unit disc in $(\mathbb{R}^2, \|\cdot\|)$, the unit disc of $(\mathbb{R}^{2*}, \|\cdot\|^*)$ is the *polar* of K and is denoted by K°. It is not hard to show that $(K^\circ)^\circ = K$.

In practice, to draw the polar of a unit disc K we identify $\mathbb{R}^2$ and $\mathbb{R}^{2*}$ by using the standard basis, and plot all the points $\mathbf{p}$ of $\mathbb{R}^{2*}$ for which the line in $\mathbb{R}^2$ with equation $\mathbf{p} \cdot \mathbf{q} = 1$ *supports* K (i.e., intersects the boundary of K, but not its interior).

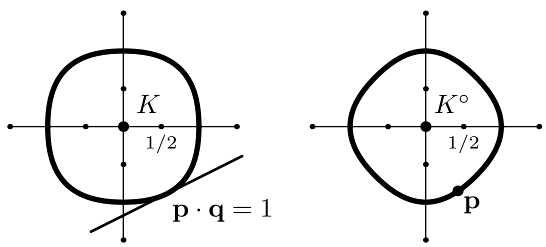

Figure 2. Construction of polars.

Exercise 3. Show that the taxi cab and maximum norms are dual to each other. In general, a celebrated theorem of Minkowski states that the p-norm and the q-norm are dual to each other if and only if $p^{-1} + q^{-1} = 1$. Figure 2 illustrates Minkowski's theorem for $p = 5/2$ and $q = 5/3$.

A more complete account of Minkowski planes can be found in Thompson's book [23] and in the survey [13] by Martini, Swanepoel, and Weiss. Section 7.4 of this last paper served as a motivation for the present, more leisurely, account of the geometry of unit discs in Minkowski planes.

3. BASIC GEOMETRY OF MINKOWSKI PLANES. Having dealt with the fundamental ideas, in this section we look at three topics from Minkowski geometry. These are (i) the perimeter of the unit disc, (ii) the notion of normality (perpendicularity), and (iii) the area of the unit disc.

Perimeter of the disc. On any metric space (X, d) it is possible to define the length of a curve as the supremum over all partitions $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of the curve of the quantity $\sum_{i=1}^{n} d(\mathbf{x}_i, \mathbf{x}_{i-1})$. On a Minkowski plane, we may also compute the length of a differentiable parameterized curve $\gamma(t)$ $(a \leq t \leq b)$ as the integral

$$\int_a^b \|\dot\gamma(t)\| dt.$$

The curve γ is said to be parameterized by *Minkowski arclength* if $\|\dot\gamma(t)\|$ is constantly equal to one. In the sequel, it will be useful to define a *regular curve* in a Minkowski plane as a curve that admits a twice-differentiable parameterization by Minkowski arclength.

If K is a convex body in a Minkowski plane, let ∂K denote its boundary and $\ell(\partial K)$ its Minkowski length. The most obvious curve to consider is the boundary of the unit disc, or *unit circle,* of the Minkowski plane. We stress that the length of the unit circle is measured relative to its defining norm.

Exercise 4. Show that if the unit disc is a parallelogram, its perimeter is eight. If the unit disc is a regular hexagon, its perimeter is six. Construct a family of hexagons H_t $(0 \leq t \leq 1)$ such that when considered as unit discs the perimeter of H_t is $6 + 2t$ (when $t = 1$ the hexagon becomes a parallelogram).

A key remark is that *the perimeter of the unit disc is a linear invariant.* By this we mean that if two symmetric convex bodies D_1 and D_2 are linearly equivalent and we consider each to be the unit disc of a Minkowski plane, then $\ell(D_1) = \ell(D_2)$.

There are two basic theorems about the perimeter of the unit disc in a Minkowski plane. The first theorem, due to Gołąb [7], is relatively well known:

Theorem 1 (Gołąb). *If D is the unit disc of a Minkowski plane, then $6 \leq \ell(\partial D) \leq 8$. Equality holds on the left if and only if D is linearly equivalent to a regular hexagon and on the right if and only if D is a parallelogram.*

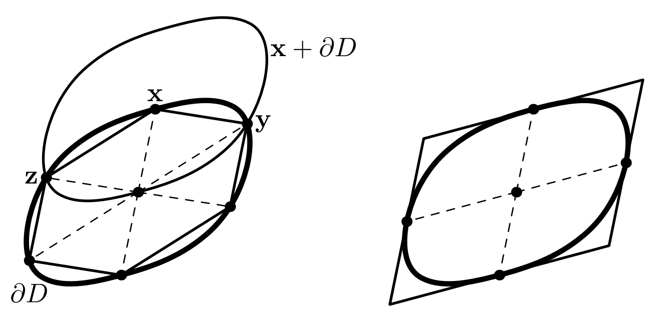

Figure 3. Proof of Gołąb's theorem.

Sketch of the proof. To verify the lower bound one inscribes in D a hexagon by the construction illustrated in Figure 3. Translate a copy of the unit circle ∂D by a generic unit vector $\mathbf{x}$ and mark the two points $\mathbf{y}$ and $\mathbf{z}$ at which ∂D and its translate intersect. The hexagon is given by the convex hull of $\pm\mathbf{x}$, $\pm\mathbf{y}$, and $\pm\mathbf{z}$. Each side of this hexagon is a translate of a unit vector (illustrated by the dashed lines) and, therefore, the length of each side is equal to one.

To establish the upper bound one circumscribes about D a parallelogram P in such a way that the points of tangency bisect its sides. One way of doing this is to take P to be a circumscribing parallelogram of minimal area. In this case the sides of the parallelogram are easily seen to have length two (Figure 3).

The hard part of the proof is to show that the bounds are attained *only* for the affine regular hexagon and the parallelogram. References for this part are Schäffer [20], Petty [16], and Thompson [23]. □

The second basic fact about the perimeter of the unit circle in a Minkowski plane to which we alluded is a result of Schäffer [21] (see section 6 for a proof).

Theorem 2 (**Schäffer**). *If D is the unit disc for a Minkowski plane and if D° is the unit disc in the dual plane, then $\ell(\partial D) = \ell(\partial D^\circ)$.*

Normality in normed planes. One of the ideas that plays a fundamental role in Euclidean geometry is that of orthogonality. Not only does this concept occur in Euclid's axioms themselves, but also in many of the basic theorems (for example, in that staple of high school geometry, Pythagoras's theorem). One of the underlying themes in Minkowski geometry is to look for suitable substitutes for this notion. In what follows we shall be concerned with one of these in particular.

In Euclidean geometry a tangent to a circle is perpendicular to the radius that joins the center to the point of tangency. Equivalently, the shortest segment joining a line L to a point not on L is perpendicular to L. We make this into a definition. The idea goes back at least to Carathèodory, but it is hard to give a precise reference. The most frequently cited reference for the definition is Birkhoff [5].

Definition 1. If $(\mathbb{R}^2, \|\cdot\|)$ is a normed plane with unit disc D, we say that a unit vector $\mathbf{q}$ is *normal to a unit vector* $\mathbf{x}$ if the line joining the origin to $\mathbf{x}$ is parallel to a line supporting D at $\mathbf{q}$ (Figure 4).

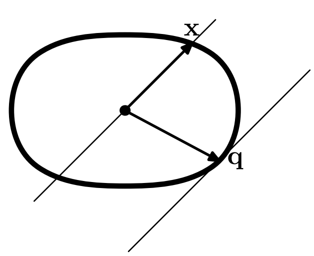

Figure 4. The vector $\mathbf{q}$ is normal to $\mathbf{x}$.

This definition can be extended in an obvious way to the case where $\mathbf{q}$ and $\mathbf{x}$ are arbitrary nonzero vectors. Notice that if ∂D is a regular curve with (Minkowski) arclength parameterization $\gamma(s)$, then $\gamma(s)$ is normal to $\dot{\gamma}(s)$.

If $\mathbf{q}$ is normal to $\mathbf{x}$, it does not follow that $\mathbf{x}$ is normal to $\mathbf{q}$. In fact, for dimensions three and above the only normed spaces for which normality is symmetric are the Euclidean spaces. In dimension two, however, normality is symmetric for a wide class of normed planes studied by Radon in [17] and, thus, known as *Radon planes.*

Definition 2. A *Radon plane* is a normed plane for which normality is symmetric. A curve in $\mathbb{R}^2$ is said to be a *Radon curve* if it is the unit circle of a Radon plane.

Geometrically, if $\mathbf{q}$ and $\mathbf{x}$ are unit vectors in a normed plane, then $\mathbf{q}$ and $\mathbf{x}$ are normal to each other if and only if the line containing $\mathbf{q}$ and the origin is parallel to a line supporting the unit disc at $\mathbf{x}$, and the line containing $\mathbf{x}$ and the origin is parallel to a line supporting the unit disc at $\mathbf{q}$. Thus, it is easy to see that ellipses and regular hexagons are Radon curves, while squares and regular octagons are not (Figure 5).

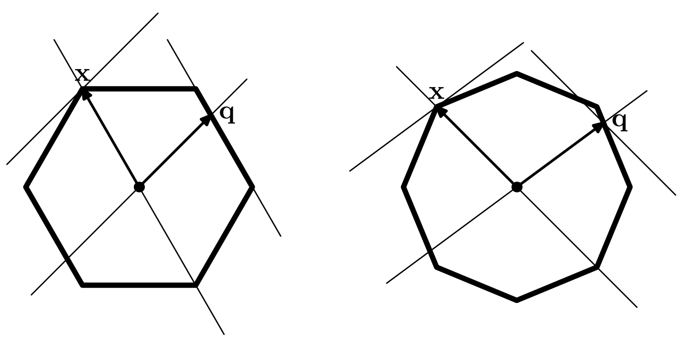

Figure 5. The regular hexagon is a Radon curve, but the regular octagon is not.

In order to complete Figure 5 into a proof that regular hexagons are Radon curves, we just need to remark that

(1) a line that passes through the origin and is parallel to a side of the hexagon intersects it in two of its vertices; and
(2) all the lines supporting the hexagon at a fixed vertex intersect the hexagon in the same two sides.

The following simple theorems present everything the reader has to know about Radon curves in order to understand the theorems and proofs in the rest of the paper. For all the details see [13] and the references therein.

Theorem 3. *The image of a Radon curve by an invertible linear transformation and the polar of a Radon curve are also Radon curves.*

Theorem 4. *Let D be the unit disc of a normed plane such that ∂D is a regular curve and, hence, admits a twice-differentiable parameterization $\gamma(s)$ by Minkowski arclength. The curve ∂D is a Radon curve if and only if the Wronskian $s \mapsto \det(\gamma(s), \dot\gamma(s))$ is constant.*

An interesting generalization of Radon curves that comes up in the theory of area in Minkowski planes has been studied by Martini, Swanepoel, and Weiss (see [13], [14]).

Definition 3. A symmetric, convex curve is *equiframed* if every one of its points is a point of tangency for some circumscribing parallelogram of minimal area.

It is not hard to show that every Radon curve is equiframed, and that every regular equiframed curve is a Radon curve. However, parallelograms and regular octagons are also equiframed. A comprehensive account of equiframed curves that includes their general construction is given by Martini and Swanepoel [14].

For the final part of this discussion of normality we will work with normed planes for which the unit circle ∂D and its polar ∂D° are regular curves.

Definition 4. If $\mathbf{q}$ lies on the unit circle ∂D of a normed plane, then the *Legendre transform* of $\mathbf{q}$ is the unique point $\mathcal{L}(\mathbf{q})$ of ∂D° such that the line with equation $\mathcal{L}(\mathbf{q}) \cdot \mathbf{x} = 1$ is tangent to ∂D at $\mathbf{q}$ (Figure 6).

Note that if $\mathbf{q}$ is a unit vector, then $\mathcal{L}(\mathbf{q}) \cdot \mathbf{y} = 0$ if and only if $\mathbf{q}$ is normal to $\mathbf{y}$. In particular, if $\gamma(t)$ is an any differentiable parameterization of the unit circle, then $\mathcal{L}(\gamma(t)) \cdot \dot\gamma(t) = 0$ for all values of the parameter t.

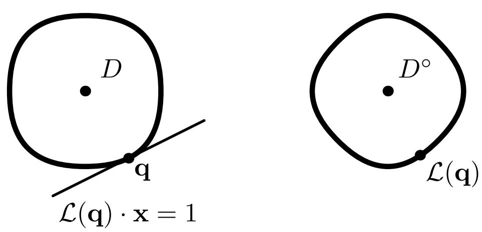

Figure 6. The Legendre transform.

It is easy to see that composing the Legendre transform in a normed plane with the Legendre transform in its dual plane results in the identity transformation. Another property of the Legendre transform is established by the following proposition:

Proposition 1. *If $\mathbf{v}$ and $\mathbf{w}$ are unit vectors in a normed plane $(\mathbb{R}^2, \|\cdot\|)$ that form a positively-oriented basis for $\mathbb{R}^2$, then $\mathcal{L}(\mathbf{v})$ and $\mathcal{L}(\mathbf{w})$ form a positively-oriented basis for $\mathbb{R}^{2*}$.*

Proof. Let us first verify that if the unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$, then $\mathcal{L}(\mathbf{v}_1)$ and $\mathcal{L}(\mathbf{v}_2)$ form a basis for $\mathbb{R}^{2*}$. If this were not the case, then $\mathcal{L}(\mathbf{v}_1)$ would have to equal $\pm\mathcal{L}(\mathbf{v}_2)$. However, since the Legendre transform is injective and odd (i.e., $\mathcal{L}(-\mathbf{q}) = -\mathcal{L}(\mathbf{q})$), this would imply that $\mathbf{v}_1 = \pm\mathbf{v}_2$ and contradict the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis.

Now we make use of a continuity argument. The set of bases for $\mathbb{R}^2$ and $\mathbb{R}^{2*}$ can be considered as open subsets of $\mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbb{R}^{2*} \times \mathbb{R}^{2*}$, respectively. Each of these sets is composed of two connected components, the positively and the negatively-oriented bases. Since $(\mathbf{v}_1, \mathbf{v}_2) \mapsto (\mathcal{L}(\mathbf{v}_1), \mathcal{L}(\mathbf{v}_2))$ is a continuous map that sends bases to bases, we may prove the proposition by showing that the Legendre transform sends *some* positively-oriented basis of $\mathbb{R}^2$ into a positively-oriented basis of $\mathbb{R}^{2*}$.

To construct such a basis, let $\gamma(s)$ be a twice-differentiable parameterization of the unit circle ∂D in $(\mathbb{R}^2, \|\cdot\|)$ by Minkowski arclength that describes ∂D in a counterclockwise fashion. Let t be a value of the parameter s for which the Wronskian $\Delta(s) = \det(\gamma(s), \dot\gamma(s))$ attains its maximum, and set $\mathbf{v}_1 = \gamma(t)$ and $\mathbf{v}_2 = \dot\gamma(t)$. The counterclockwise orientation of the parameterization implies that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a positively-oriented basis for $\mathbb{R}^2$. Moreover, since $0 = \Delta^{'}(t) = \det(\gamma(t), \ddot\gamma(t))$, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are normal to each other. This, together with the definition of the Legendre transform, implies that $\mathcal{L}(\mathbf{v}_i) \cdot \mathbf{v}_j = \delta_{ij}$. In other words, the basis formed by $\mathcal{L}(\mathbf{v}_1)$ and $\mathcal{L}(\mathbf{v}_2)$ is dual to the one formed by $\mathbf{v}_1$ and $\mathbf{v}_2$, hence has the same (positive) orientation.

□

To end this section we exhibit a simple relationship between normality and the Legendre transform that will play a crucial role in our proof of Schäffer's theorem.

Given a unit vector $\mathbf{q}$ in a normed plane, let $\dot{\mathbf{q}}$ denote the unique unit vector such that $\mathbf{q}$ is normal to $\dot{\mathbf{q}}$ and $\{\mathbf{q}, \dot{\mathbf{q}}\}$ is a positively-oriented basis for $\mathbb{R}^2$.

Proposition 2. *If* $\mathbf{q}$ *is a unit vector in a normed plane and* $\mathbf{p} = \mathcal{L}(\dot{\mathbf{q}})$, *then* $\mathcal{L}(\dot{\mathbf{p}}) = -\mathbf{q}$.

Proof. By the definition of the Legendre transform, we have

$$\mathcal{L}(\mathbf{p}) \cdot \dot{\mathbf{p}} = 0, \quad \mathcal{L}(\mathbf{q}) \cdot \dot{\mathbf{q}} = 0\ . \tag{1}$$

Applying $\mathcal{L}$ to both sides of the equality $\mathbf{p} = \mathcal{L}(\dot{\mathbf{q}})$, we obtain $\mathcal{L}(\mathbf{p}) = \dot{\mathbf{q}}$. So the equation on the left in (1) becomes $\dot{\mathbf{p}} \cdot \dot{\mathbf{q}} = 0$. Comparing this with the equation on the right we see that $\mathcal{L}(\mathbf{q}) = \pm\dot{\mathbf{p}}$, hence $\mathbf{q} = \pm\mathcal{L}(\dot{\mathbf{p}})$. Finally, since both $(\mathbf{q}, \dot{\mathbf{q}})$ and $(\mathcal{L}(\mathbf{p}), \mathcal{L}(\dot{\mathbf{p}})) = (\dot{\mathbf{q}}, \mathcal{L}(\dot{\mathbf{p}}))$ are positively-oriented bases, we must have $\mathcal{L}(\dot{\mathbf{p}}) = -\mathbf{q}$. □

Areas in Minkowski planes. In contrast to the measurement of lengths, there is nothing in the definition of a Minkowski plane that fixes a canonical and incontestable way of measuring areas. Nevertheless, a growing body of work in convex, integral, and Finsler geometry demonstrates not only that a theory of areas and volumes in finite-dimensional normed spaces is possible, but also that it provides a common framework for some of the deepest theorems and problems in those fields (see the book [23], the recent survey [2], and the references therein).

The key remark in approaching the problem of measuring areas in a Minkowski plane is that isometries should preserve areas. In particular, the area of a region should be invariant under translation. If we also require—as it is natural to do—that the area of a compact set be finite and that the area of an open set be positive, then a deep theorem of Haar (see [23, p. 37]) implies that area is a Lebesgue measure. By this we mean that for some (linear) system of coordinates (x, y) on the plane the Minkowski area of any open set U equals

$$\iint_U dx\, dy.$$

Since two Lebesgue measures associated with different linear coordinate systems may differ by a multiplicative constant, this still leaves us with the problem of finding a suitable normalization. This normalization cannot be arbitrary. For example, if our unit disc D is an ellipse, then our space is isometric to the Euclidean plane, and we must assign to the ellipse the area π. It doesn't matter how large or small we draw D on the page, its area is π. Thus, "suitable normalization" requires that the area of the unit disc D in our Minkowski plane be a linear invariant of D that takes the value π on ellipses. While there are infinitely many ways of doing this, we shall concentrate on four normalizations that have appeared, sometimes implicitly, in different contexts in convex and differential geometry.

The first normalization, investigated extensively by Busemann, assigns the area π to the unit disc D of any Minkowski plane. It follows that if U is an open set of the normed plane with unit disc D and λ is any Lebesgue measure on the plane, then the *Busemann area* of U, denoted by $\mu_b(U)$, equals $\pi\lambda(U)/\lambda(D)$.

In the second normalization, introduced by Holmes and Thompson [9], the area of the unit disc is equal to its *volume product* divided by π. We recall the definition of this important invariant.

If λ is a Lebesgue measure on $\mathbb{R}^2$ and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ form a basis such that the parallelogram they generate has measure one, we define the *dual measure* λ^* on $\mathbb{R}^{2*}$ as the Lebesgue measure for which the parallelogram generated by the basis dual to $\{\mathbf{e}_1, \mathbf{e}_2\}$ has measure one. It is easily verified that the product measure $\lambda \times \lambda^*$ on $\mathbb{R}^2 \times \mathbb{R}^{2*}$ does not depend on the choice of the Lebesgue measure λ, hence defines a canonical volume on $\mathbb{R}^2 \times \mathbb{R}^{2*}$.

Definition 5. The *volume product* of a symmetric convex body D on the plane, denoted by $\mathrm{vp}(D)$, is the canonical volume of the body $D \times D^\circ$ in $\mathbb{R}^2 \times \mathbb{R}^{2*}$.

From the definition it follows that if U is an open set of the normed plane with unit disc D and λ is any Lebesgue measure on the plane, then the *Holmes-Thompson area* of U, denoted by $\mu_{ht}(U)$, equals $\lambda(U)\lambda^*(D^\circ)/\pi$.

The third normalization we consider is obtained by setting the area of the unit disc to be twice the minimum of the quantities $\lambda(D)/\lambda(P)$, where P ranges over all parallelograms contained in D. A parallelogram on which this minimum is attained is called a *maximal inscribed parallelogram* in D. We may also characterize this normalization by saying that it assigns area 2 to any maximal inscribed parallelogram in the unit disc.

If we set the area of the unit disc to be four times the maximum of the quantities $\lambda(D)/\lambda(P)$, where P ranges over all parallelograms containing D, we obtain a fourth possible normalization. A parallelogram on which this maximum is attained is called a *minimal circumscribing parallelogram* of D. This normalization is characterized by the fact that it assigns area 4 to any minimal circumscribing parallelogram of the unit disc.

These third and fourth notions of area were introduced by Gromov as mass and mass*, respectively, in his landmark paper [8]. Convex geometers will recognize mass* as the *Benson definition of area* (see [3], [4], and [24]). In what follows we denote mass by μ_m and mass* by μ_{m*}.

As the definitions of mass and mass* suggest, definitions of areas on Minkowski planes come in dual pairs.

Definition 6. Two definitions of area μ and μ^* on Minkowski planes are said to be dual if $\mu(D)\mu^*(D^\circ)$ equals the volume product of D whenever D is the unit disc of a Minkowski plane.

It is easy to verify that the area definitions of Busemann and Holmes-Thompson, as well as mass and mass*, are dual to each other. It is not yet clear what role duality plays in the study of volumes and areas in normed spaces, but the following simple proposition will be useful in the sequel.

Proposition 3. *If μ, μ^*, ν, and ν^* are two dual pairs of area definitions for Minkowski planes, then $\mu \geq \nu$ if and only if $\mu^* \leq \nu^*$.*

Proof. Observe that $\mu \geq \nu$ if and only if $\mu(D) \geq \nu(D)$ whenever D is the unit disc of a Minkowski plane. Using the definition of duality we have

$$\mu^*(D) = \frac{\mathrm{vp}(D)}{\mu(D^\circ)} \leq \frac{\mathrm{vp}(D)}{\nu(D^\circ)} = \nu^*(D)$$

and, therefore, $\mu^* \leq \nu^*$. □

4. BOUNDING THE AREA OF THE UNIT DISC. Gołąb's theorem gives precise bounds for the perimeter of the unit disc. In this section we begin the investigation of bounds for its area using the Holmes-Thompson, mass, and mass* definitions. Note that, since μ_b is constant for all unit discs, the question of bounds for the Busemann definition is trivial.

For the Holmes-Thompson definition the question of bounds is equivalent to the following two famous inequalities:

Theorem 5 (**Mahler, Blaschke**). *If K is a symmetric convex body in the plane, then*

$$8 \leq \lambda(K)\lambda^*(K^\circ) \leq \pi^2 \ .$$

Equality holds on the left if and only if K is a parallelogram and on the right if and only if K is an ellipse.

The inequality on the left is due to Mahler [12]. Its conjectured generalization to higher dimensions is a famous open problem of Mahler. The inequality on the right is due to Blaschke [6]. Its higher-dimensional generalization is due to Santaló [18].

From the previous theorem, we immediately obtain the following inequalities:

Corollary 1. *If D is the unit disc of a Minkowski plane, then*

$$8/\pi \leq \mu_{ht}(D) \leq \pi \ .$$

Equality holds on the left if and only if D is a parallelogram and on the right if and only if D is an ellipse.

The analogous inequalities for mass and mass* are given in the next two theorems.

Theorem 6. *If D is the unit disc of a Minkowski plane, then*

$$2 \leq \mu_m(D) \leq \pi \ .$$

Equality holds on the left if and only if D is a parallelogram and on the right if and only if D is an ellipse.

The right-hand inequality is due to Sás [19], but another reference is Macbeath [11].

Theorem 7. *If D is the unit disc of a Minkowski plane, then*

$$3 \leq \mu_{m*}(D) \leq 4 \ .$$

Equality holds on the left if and only if D is linearly equivalent to a regular hexagon and on the right if and only if D is a parallelogram.

The inequality on the left was proved by Petty [16]. These results will be established in the next section, but two of the inequalities are quite simple.

Exercise 5. Show that if D is the unit disc of a Minkowski plane, then $2 \leq \mu_m(D)$ and that $\mu_{m*}(D) \leq 4$. Equality holds in either case if and only if D is a parallelogram.

5. AREA VERSUS PERIMETER. The main theorem in this section relates the perimeter of the unit disc D of a Minkowski plane to the areas of D given by μ_m and μ_{m*}.

Theorem 8. *If D is the unit disc of a Minkowski plane, then*

$$2\mu_m(D) \leq \ell(\partial D) \leq 2\mu_{m*}(D).$$

When ∂D is a regular curve, equality holds on either side if and only if it is a Radon curve.

Proof. We make the simplifying assumption that ∂D is a regular curve. Since any convex body can be approximated in the Hausdorff topology by convex bodies with regular boundaries (see [23, p. 64]) and since the quantities investigated are continuous in that topology, the proof in this particular case and an approximation argument yield the proof in the general case.

Let $\gamma : [0, \ell] \mapsto \mathbb{R}^2$ be a positively-oriented parameterization of ∂D by Minkowski arclength (i.e., for every value of the parameter the vector $\dot{\gamma}(s)$ has magnitude one and the pair of vectors $(\gamma(s), \dot{\gamma}(s))$ is a positively-oriented basis of $\mathbb{R}^2$). By Green's theorem

$$\lambda(D) = \int_D dx\, dy = \frac{1}{2}\int_\gamma x\, dy - y\, dx = \frac{1}{2}\int_0^\ell \det(\gamma(s), \dot{\gamma}(s))\, ds.$$

To prove the inequality $2\mu_m(D) \leq \ell(\partial D)$ we argue as follows. Since $\dot{\gamma}(s)$ is a Minkowski unit vector, the parallelogram P_s with vertices $\gamma(s)$, $\dot{\gamma}(s)$, $-\gamma(s)$, and $-\dot{\gamma}(s)$ is inscribed in D, so its area is no larger than that of a maximal inscribed parallelogram P_i. Moreover, $\det(\gamma(s), \dot{\gamma}(s)) = \lambda(P_s)/2$. Thus we have

$$\lambda(D) = \frac{1}{4}\int_0^\ell \lambda(P_s) ds \leq \frac{1}{4}\int_0^\ell \lambda(P_i) ds = \lambda(P_i)\ell(\partial D)/4.$$

Since $\mu_m(D) = 2\lambda(D)/\lambda(P_i)$, we get the desired inequality.

To prove the inequality $\ell(\partial D) \leq 2\mu_{m*}(D)$ notice that if the Wronskian $\Delta(s) = \det(\gamma(s), \dot{\gamma}(s))$ reaches a minimum at t, then $\gamma(t)$ is parallel to $\ddot{\gamma}(t)$. Indeed, $0 = \Delta^{'}(t) = \det(\gamma(t), \ddot{\gamma}(t))$. In other words, the tangent to ∂D at $\dot{\gamma}(t)$ is parallel to $\gamma(t)$. Thus the tangents to ∂D at $\pm\gamma(t)$ and $\pm\dot{\gamma}(t)$ form a circumscribing parallelogram to D. The area of this parallelogram is $4\det(\gamma(t), \dot{\gamma}(t))$ and cannot be less than that of a minimal circumscribing parallelogram P_c. Therefore, we have $4\det(\gamma(s), \dot{\gamma}(s)) \geq \lambda(P_c)$ for all s. This gives,

$$8\lambda(D) \geq \int_0^\ell \lambda(P_c)\, ds = \ell(\partial D)\lambda(P_c),$$

from which it follows that $\ell(\partial D) \leq 2\mu_{m*}(D)$.

From the proofs of both inequalities we see that equality holds in either case if and only if $\det(\gamma(s), \dot{\gamma}(s))$ is constant. Theorem 4 tells us that this is the case if and only if ∂D is a Radon curve. □

Martini, Swanepoel, and Weiss show in [13] that the unit disc D of a Minkowski plane satisfies the equality $2\mu_m(D) \leq \ell(\partial D)$ if and only if ∂D is a Radon curve and that, on the other hand, it satisfies the equality $\ell(\partial D) = 2\mu_{m*}(D)$ if and only if ∂D is an equiframed curve.

Exercise 6. Show that if the unit circle is a regular octagon (an equiframed curve), then both the perimeter and twice the mass* of the unit disc equal $16\tan(\pi/8)$.

We are now in a position to finish the proof of Theorem 7.

Corollary 2. *If D is the unit disc of a Minkowski plane, then $\mu_{m*}(D) \geq 3$. Equality holds if and only if D is a regular hexagon.*

Proof. By the previous theorem, $2\mu_{m*}(D) \geq \ell(\partial D)$, with equality if ∂D is a Radon curve. On the other hand, Gołąb's theorem states that $\ell(\partial D) \geq 6$, with equality if and only if ∂D is a regular hexagon. Using both inequalities, together with the fact that the regular hexagon is a Radon curve, proves the corollary. □

To end this section we state without proof a result of Moustafaev [15] and use it, in conjunction with Theorem 8, to establish order relations among the four definitions of area. The second of these relations completes the proof of Theorem 6.

Theorem 9 (Moustafaev). *If D is the unit disc of a Minkowski plane, then $2\mu_{ht}(D) \leq \ell(\partial D)$. Equality holds if and only if D is an ellipse.*

Moustafaev's proof requires more machinery than the proofs in this paper. It makes use of the solution of the isoperimetric problem in Minkowski planes and the Blaschke-Santaló inequality.

Corollary 3. *If D is the unit disc of a Minkowski plane, then*

$$\mu_{ht}(D) \leq \mu_{m*}(D) \quad \textit{and} \quad \mu_m(D) \leq \mu_b(D) = \pi\ .$$

Equality holds in either case if and only if D is an ellipse.

Proof. By Theorem 8 and Moustafaev's result, we have

$$2\mu_{ht}(D) \leq \ell(\partial D) \leq 2\mu_{m*}(D).$$

Referring to Proposition 3 and exploiting the information that μ_b, μ_{ht}, and μ_m, μ_{m*} are dual pairs of area definitions, we see that $\mu_m(D) \leq \mu_b(D) = \pi$. □

Exercise 7. Show that $\mu_m(D) \leq \mu_{m*}(D)$ and $\mu_{ht}(D) \leq \mu_b(D)$, and investigate the cases of equality.

6. PROPERTIES OF RADON CURVES. In this section we sharpen the bounds on the perimeter and area of the unit disc in the case where the unit circle is a *Radon curve* (see Definition 2) . We begin by putting together several of the results in the previous section.

Theorem 10. *If D is the unit disc of a Radon plane, then $(\ell(\partial D))^2 = 4\mathrm{vp}(D)$.*

Proof. By duality,

$$4\mathrm{vp}(D) = 4\mu_m(D)\mu_{m*}(D^\circ),$$

while Theorem 8 tells us that $2\mu_m(D) = \ell(\partial D)$ and that $2\mu_{m*}(D^\circ) = \ell(\partial D^\circ)$. We now apply Schäffer's theorem, which asserts that $\ell(\partial D^\circ) = \ell(\partial D)$, to obtain the desired result. $\square$

As a corollary, we have the following result of Lenz [10], which was rediscovered by Yaglom [25].

Corollary 4. *If D is the unit disc of a Radon plane, then $6 \leq \ell(\partial D) \leq 2\pi$.*

Proof. The first inequality follows from Gołąb's theorem. Equality is attained if and only if ∂D is linearly equivalent to a regular hexagon (which is a Radon curve). For the second inequality we have

$$(\ell(\partial D))^2 = 4\mathrm{vp}(D) \leq 4\pi^2$$

from Theorem 10 and the Blaschke inequality in Theorem 5. Here equality holds if and only if D is an ellipse. $\square$

Corollary 5. *If D is the unit disc of a Radon plane, then*

$$3 \leq \mu_m(D) \leq \pi, \quad 3 \leq \mu_{m*}(D) \leq \pi, \quad 9/\pi \leq \mu_{ht}(D) \leq \pi\ .$$

In each case equality holds on the left if and only if D is linearly equivalent to a regular hexagon and on the right if and only if D is an ellipse.

Proof. The first two inequalities are evident from Corollary 4 and the case of equality in Theorem 8. For the third inequality note that according to Corollary 4

$$36 \leq (\ell(\partial D))^2 = 4\mathrm{vp}(D) \leq 4\pi^2.$$

In view of Corollary 4 if we now divide this equation by 4π and use the definition of $\mu_{ht}(D)$, we obtain the result. $\square$

7. PROOF OF SCHÄFFER'S THEOREM. We end the paper by giving a "book proof" of Schäffer's theorem inspired by the symplectic proof of the girth conjecture given in [1].

Theorem 11. *If D is the unit disc for a Minkowski plane and if D° is the unit disc in the dual plane, then $\ell(\partial D) = \ell(\partial D^\circ)$.*

Proof. Consider the closed curve Γ in $\mathbb{R}^2 \times \mathbb{R}^{2*}$ defined by

$$\Gamma = \{(\mathbf{q}, \mathbf{p}) : \mathbf{q} \in \partial D\}\ ,$$

where as in Proposition 2 we set $\mathbf{p} = \mathcal{L}(\dot{\mathbf{q}})$. Then

$$0 = \int_\Gamma d(\mathbf{p} \cdot \mathbf{q}) = \int_\Gamma \mathbf{p} \cdot d\mathbf{q} + \mathbf{q} \cdot d\mathbf{p} = \int_\Gamma \mathbf{p} \cdot d\mathbf{q} + \int_\Gamma \mathbf{q} \cdot d\mathbf{p}.$$

If we first parameterize ∂D by Minkowski arclength s, then Γ acquires a parameterization $\alpha(s) = (\mathbf{q}(s), \mathcal{L}(\dot{\mathbf{q}}(s)))$. Accordingly,

$$\int_\Gamma \mathbf{p} \cdot d\mathbf{q} = \int_0^{\ell_1} \mathcal{L}(\dot{\mathbf{q}}(s)) \cdot \dot{\mathbf{q}}(s)\, ds = \int_0^{\ell_1} ds = \ell_1 \ ,$$

where $\ell_1 = \ell(\partial D)$.

On the other hand, we can parameterize ∂D° by its Minkowski arclength t, in which case by Proposition 2, Γ can be described by a different parameterization, namely, $\beta(t) = (-\mathcal{L}(\dot{\mathbf{p}}(t)), \mathbf{p}(t))$. Then

$$\int_\Gamma \mathbf{q} \cdot d\mathbf{p} = -\int_0^{\ell_2} \mathcal{L}(\dot{\mathbf{p}}(t)) \cdot \dot{\mathbf{p}}(t)\, dt = -\int_0^{\ell_2} ds = -\ell_2,$$

where $\ell_2 = \ell(\partial D^\circ)$. The upshot: $0 = \ell(\partial D) - \ell(\partial D^\circ)$. $\square$

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JUAN CARLOS ALVAREZ PAIVA received his Ph.D. from Rutgers University in 1995 and is now associate professor of mathematics at Polytechnic University, Brooklyn. His research blends symplectic, integral, and convex geometry.
Polytechnic University, Brooklyn, New York, 11201, U.S.A.
jalvarez@duke.poly.edu

ANTHONY (TONY) THOMPSON received his B.Sc. degree from University College London in 1959 and his Ph.D. from the University of Newcastle-upon-Tyne in 1963. He came to Canada and Dalhousie University in 1966 and (apart from sabbaticals in Edinburgh, Nanjing, Toronto, and Freiburg) has remained there. He is now retired from teaching. His main research interests are in geometry, and he has written a book on Minkowski geometry. For many years he was well known for the fact that he rowed to work.
Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5.
tony@mathstat.dal.ca