

Radiometry, BRDF and Photometric Stereo

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1 Radiometry

1.1 Solid Angle

Solid angle is a very important concept. Solid angle is defined by the projected area of a surface patch onto a unit sphere of a point, meaning that **a solid angle is subtended by a point and a surface patch**, which is shown in Figure 1. \mathbf{n} is the normal of the surface

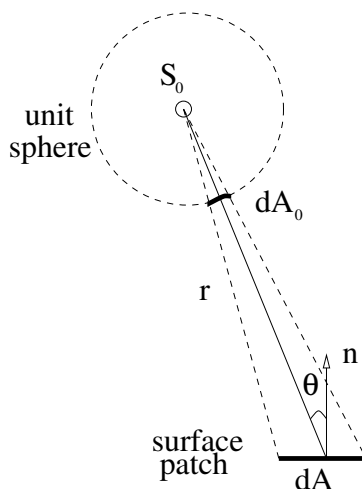


Figure 1: Solid angle subtended by S_0 and surface patch dA

patch dA , and S_0 is the point centered at a unit sphere. The distance between S_0 and dA is r , and the angle between surface normal \mathbf{n} and the point direction is θ . dA_0 is the projection of the surface patch on the unit sphere. So,

$$d\omega = dA_0 = \frac{dA \cos\theta}{r^2} \quad (1)$$

Where $dA \cos\theta$ is the foreshortened area of dA on the direction of $dA-S_0$. The unit of solid angle is steradians, or *sr* for short.

Question: what is the solid angle of the whole sphere? (Answer: 4π . Why?)

1.2 Radiance and Irradiance

Both Radiance and Irradiance are measures for lighting intensities. Frequently, we use radiance to measure the lighting sources, but use irradiance to measure the lighting intensity received by a patch. By definition, the radiance of an **area lighting source** is:

$$\text{Radiance} = \frac{\text{power}}{\text{solid_angle_source} \times \text{foreshortened_area_of_source}}$$

i.e.,

$$L_r(\mathbf{x}, \theta_r, \psi_r) = \frac{d\phi}{d\omega_r \cos\theta_r dA_r} \quad (2)$$

where dA_r is the lighting source surface, \mathbf{x} is the location of the source, and θ_r, ψ_r is used to represent the lighting direction. Denote illuminated surface by dA_i and its foreshortening angle by θ_i , then the solid angle

$$d\omega_r = \frac{dA_i \cos\theta_i}{r^2}. \quad (3)$$

Then

$$L_r(\mathbf{x}, \theta_r, \psi_r) = \frac{r^2 d\phi}{\cos\theta_i \cos\theta_r dA_i dA_r}. \quad (4)$$

Note the foreshortened area is for the area lighting source, and the solid angle is that one subtended by the source and another illuminated surface dA_i . So, radiance basically means that how much lighting emitted from the source per area of the source patch per solid angle of the illuminated patch.

Obviously, the power emitted from the source to the illuminated patch is:

$$d\phi = L_r(\mathbf{x}, \theta_r, \psi_r) (\cos\theta_r dA_r) d\omega_r \quad (5)$$

If the source is a **point source**, by definition, we use radiance intensity:

$$\text{Radiance_Intensity} = \frac{\text{power}}{\text{solid_angle_source}}$$

i.e.,

$$I = \frac{d\phi}{d\omega_r} = \frac{r^2 d\phi}{\cos\theta_i dA_i} \quad (6)$$

Let's have an example: an area source dA_1 illuminating a surface patch dA_2 as shown in Figure 2 Obviously, the foreshortened area for source dA_1 is $dA_1 \cos\theta_1$, and the solid angle

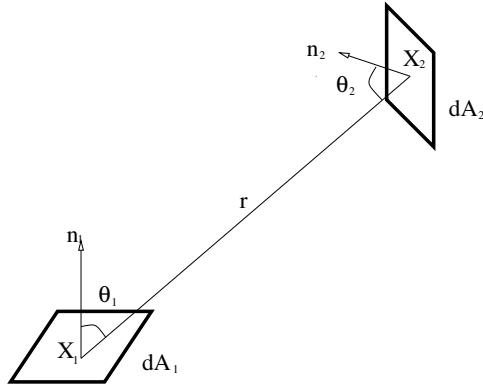


Figure 2: An example of calculating radiance

subtended by A_1 and patch dA_2 is $d\omega_{1(2)} = \cos\theta_2 dA_2 / r^2$. So, the power emitted from the source patch dA_1 is:

$$d\phi_{1 \rightarrow 2} = L(x_1, x_1 \rightarrow x_2) (\cos\theta_1 dA_1) d\omega_{1(2)} = L(x_1, x_1 \rightarrow x_2) \frac{\cos\theta_1 \cos\theta_2 dA_1 dA_2}{r^2}$$

On the other hand, we can also calculate the power received by the illuminated surface patch dA_2 , noticing that the foreshortened area for the surface patch is $dA_2 \cos \theta_2$, and the solid angle subtended by A_2 and dA_1 is $d\omega_{2(1)} = \cos \theta_1 dA_1 / r^2$:

$$d\phi_{2 \leftarrow 1} = L(x_2, x_2 \leftarrow x_1) (\cos \theta_2 dA_2) d\omega_{2(1)} = L(x_2, x_2 \leftarrow x_1) \frac{\cos \theta_1 \cos \theta_2 dA_1 dA_2}{r^2}$$

Obviously, since $d\phi_{1 \rightarrow 2} = d\phi_{2 \leftarrow 1}$, so $L(x_1, x_1 \rightarrow x_2) = L(x_2, x_2 \leftarrow x_1)$.

On the other hand, we use irradiance to represent the lighting received by a surface patch. By definition,

$$\text{Irradiance} = \frac{\text{power}}{\text{not_foreshortened_area_of_patch}}$$

i.e.,

$$L_i = \frac{d\phi}{dA_i} \quad (7)$$

So, for point sources,

$$L_i = \frac{I \cos \theta_i}{r^2}, \quad (8)$$

while for area sources,

$$L_i = \frac{L_r(x, \theta_r, \psi_r) dA_r \cos \theta_r d\omega_r}{dA_i} = \frac{L_r(x, \theta_r, \psi_r) dA_r \cos \theta_r \cos \theta_i}{r^2} = L_r(x, \theta_r, \psi_r) \cos \theta_i d\omega_i. \quad (9)$$

Again, let's examine an example: calculating the irradiance of a surface patch dA_0 illuminated by a plate source O' , where the surface patch is parallel to the plate, as shown in Figure 3. Apparently, the foreshortened area of a small patch on the plate is $dA \cos \theta =$

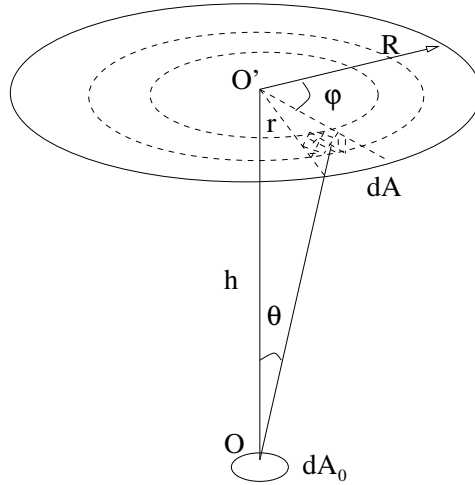


Figure 3: An example of calculating irradiance

$r \cos \theta dr d\varphi$, and the solid angle subtended by A_0 and such a small lighting patch is $d\omega = dA_0 \cos \theta / PO^2 = dA_0 \cos \theta / (h^2 + r^2)$. Noticing that $\cos \theta = h / \sqrt{h^2 + r^2}$, we have the power emitted from such a small lighting patch dA to the surface patch dA_0 is:

$$d\phi = L_r dA \cos \theta d\omega = L_r \frac{h^2 dA_0}{(h^2 + r^2)^2} r dr d\varphi$$

So, the total power emitted from the plate to the surface patch dA_0 is an integral of all such a small lighting patches, i.e.,

$$\phi_e = \int_0^R \int_0^{2\pi} d\phi = L_r \frac{\pi R^2}{h^2 + R^2} dA_0$$

In result, the irradiance of the surface patch is:

$$L_i = \frac{\phi_e}{dA_0} = L_r \frac{\pi R^2}{h^2 + R^2}$$

2 The Relationship between Image Intensity and Object Illuminance

We can see objects because they reflect lights. Intuitively, if the camera receive more lights reflected from the object, the image should look brighter. So, what is the exact relationship between the image intensity and the object illuminance? First, we can examine the power

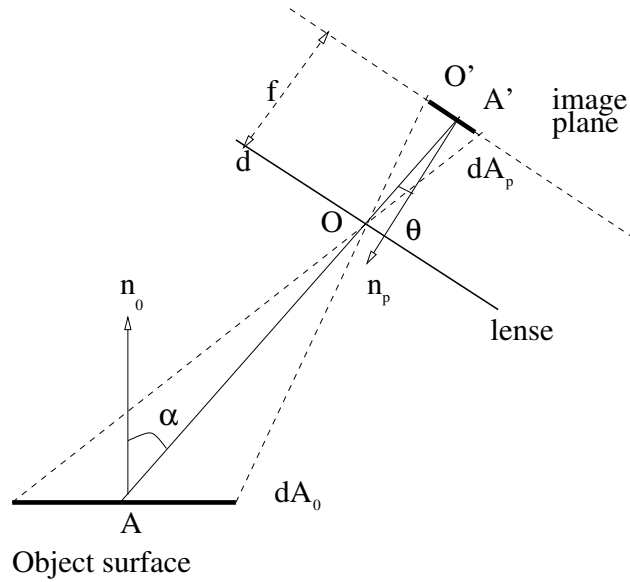


Figure 4: The relationship between image intensity and object illuminance

emitted received by the lenses (with diameter d). We assume the power received by the image patch is the same as that received by the lenses with no loss. Here, the power emitted to lenses is:

$$d\phi = L_r dA_0 \cos\alpha d\omega_0$$

where $d\omega_0$ is the solid angle subtended by the source A and the lenses:

$$d\omega_0 = \frac{\pi d^2}{4r^2} \cos\theta$$

where $r = |\mathbf{OA}|$. Since the solid angle subtended by optic center O and surface patch dA_0 is the same as the one subtended by O and image patch dA_p , we have:

$$\frac{dA_0 \cos \alpha}{r^2} = \frac{dA_p \cos \theta}{\mathbf{OA}'^2}$$

where $\mathbf{OA}' = f / \cos \theta$. So,

$$d\phi = L_r r^2 d\omega_0 \frac{dA_p \cos \theta}{\mathbf{OA}'^2} = \frac{\pi}{4} L_r \left(\frac{d}{f}\right)^2 \cos^4 \theta dA_p$$

It follows that the irradiance of the image patch is:

$$L_i = \frac{d\phi}{dA_p} = \frac{\pi}{4} L_r \left(\frac{d}{f}\right)^2 \cos^4 \theta$$

Since generally the field of view of camera is quite narrow, i.e., θ is small, we can let $\cos \theta = 1$. Approximately, we have:

$$L_i = \frac{\pi}{4} L_r \left(\frac{d}{f}\right)^2 \quad (10)$$

This equation means that the irradiance, thus image intensity, is proportional to the radiance of the object, which follows exactly our intuition.

3 Surfaces and BRDF

3.1 BRDF

Different surface patches have different properties of reflecting lights. We define a *bidirectional reflectance distribution function* to representing the relationship in terms of energy between incident light and reflected light. BRDF is defined by the ratio of outgoing radiance and incident irradiance, i.e.,

$$\rho_{bd}(\theta_r, \varphi_r, \theta_i, \varphi_i) = \frac{L_r(x, \theta_r, \varphi_r)}{L_i(x, \theta_i, \varphi_i)} = \frac{L_r(x, \theta_r, \varphi_r)}{L_s(x, \theta_s, \varphi_s) \cos \theta_i d\omega_i} \quad (11)$$

The unit of BRDF is sr^{-1} .

3.2 Lambertian Surface, Diffusion and Albedo

A Lambertian surface has a constant BRDF, which means that a Lambertian surface will look equally bright from any view direction, since its BRDF is independent of outgoing directions. Lambertian surface is also called ideal diffusion surface.

We define *diffuse reflectance* or *Albedo* by:

$$\rho_d = \int_{\Omega} \rho_{bd}(\theta_r, \varphi_r, \theta_i, \varphi_i) \cos \theta_r d\omega_r = \int_{\Omega} \rho \cos \theta_r d\omega_r = \pi \rho$$

In other words, for a Lambertian surface, the diffused radiance L_d is proportional to the incident radiance, i.e., $L_d = r_d L_i$, where r_d is a constant. Since for a point source, the irradiance of the surface is $L_i = I \cos \theta_i / r^2$, so the diffusion radiance is $L_d = r_d I \cos \theta_i / r^2$. As a result, image intensity will be

$$I_d = I r_d \frac{\pi}{4} \left(\frac{d}{f} \right)^2 \frac{1}{r^2} \cos \theta_i = k_d \cos \theta_i,$$

where θ_i is the illumination direction (or the incident light direction, i.e., the angle between lighting source and the surface), and k_d is the diffusion coefficient for a particular surface. Such an equation means that:

- a Lambertian surface is equally bright from any view direction with fixed source;
- the image intensities of the surface only changes with the illumination directions.

3.3 Specular Surface

Specular surfaces are such that behaves like mirrors. For an incident light, specular surfaces only reflect light along specific directions. If the view directions are not the same as the specular reflection direction, the camera will not see the specular reflected lights. The image intensity can be written by:

$$I_s = k_s \delta(\theta_c - \theta_i) \delta(\varphi_c - \varphi_i)$$

4 Shading Model of Point Source

In this section, we only consider point light sources. Suppose \mathbf{n} is the normal of a surface patch, and \mathbf{s} is the lighting direction. So, the radiosity of such a patch is

$$B(x) = \int_{\Omega} L(x, \theta, \varphi) \cos \theta d\omega = \rho_d(x) \frac{\mathbf{n} \cdot \mathbf{s}}{r(x)^2}$$

where $r(x)$ is the distance between the light source and the point, which is a function of the point x . Also, ρ_d, \mathbf{n} and \mathbf{s} all are function of the point position x .

If the point light source is located at infinity, we have

$$B(x) = \rho_d(x) (\mathbf{n} \cdot \mathbf{s})$$

since we can treat \mathbf{n} and \mathbf{s} constants for all points on the patch. Obviously, the albedos of each point are different. Furthermore, if we have multiple point sources, we have:

$$B(x) = \sum_{k \in \text{visible sources}} \rho_d(x) (\mathbf{n}(x) \cdot \mathbf{s}_k)$$

5 Photometric Stereo

5.1 The Problem

The problem of photometric stereo is quite interesting: if we are given a set of images of the same scene taken under different given lighting sources (the camera and the scene are kept intact), can we recover the 3D (shape) of the scene?

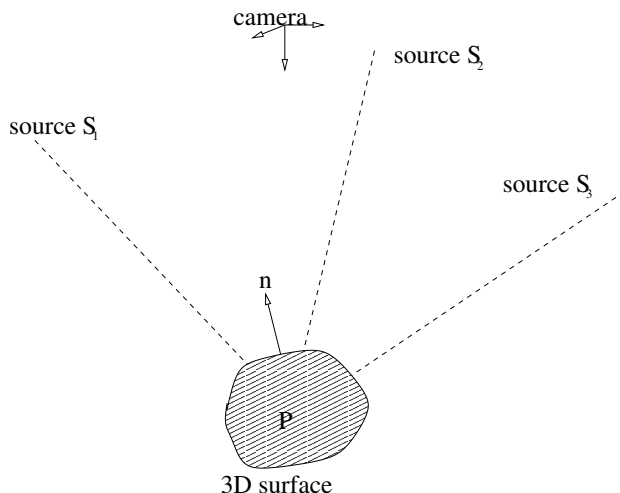


Figure 5: The setting of photometric stereo

For simplicity, we assume Lambertian for all surfaces, and the light sources are point light sources at infinity.

5.2 Recover Surface Normals

From previous sections, we have known that the image intensity can be written as:

$$I = k_d \cos\theta = k_d \mathbf{s}^T \mathbf{n}$$

Suppose we are given three point light sources, i.e., $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$. We have taken three image under such three lighting condition, respectively:

$$I_i = k_d \mathbf{s}_i^T \mathbf{n}, \quad i = 1, 2, 3$$

Since we keep the camera and the scene intact, each image pixel of the three images correspond to the same 3D point, meaning that for any particular image pixel, I_i is only a function of lighting direction \mathbf{s}_i because k_d and \mathbf{n} are unchanged. Also we assume the intensities for different sources are the same, and the distances from different lighting sources are the same also. So, we stack I_i up to get a vector for each pixel:

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = k_d \begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \mathbf{s}_3^T \end{bmatrix} \mathbf{n} = k_d \mathbf{S} \mathbf{n}$$

where \mathbf{S} is a 3×3 matrix. Since the directions for each lighting source are given, i.e., \mathbf{S} is given, we have:

$$k_d \mathbf{n} = \mathbf{S}^{-1} \mathbf{I}$$

Since \mathbf{n} is normalized unit vector, we have:

$$\mathbf{n} = \frac{\mathbf{S}^{-1} \mathbf{I}}{\|\mathbf{S}^{-1} \mathbf{I}\|}$$

Apparently, if we have more than three sources, we can use least square fitting to get the solution, i.e.,

$$\mathbf{n} = \frac{\mathbf{S}^\dagger \mathbf{I}}{\|\mathbf{S}^\dagger \mathbf{I}\|}$$

Question: what if we do not know the lighting directions? (Possible answer: we can calibrate or estimate lighting directions by giving the normals of least three 3D points. The procedure follows the same as above.)

5.3 From Normals to Shape

Till now, we have solved the normals of the points on the surface, but we have not figured out its 3D shape represented by the 3D coordinates of these 3D points. So, let's see how to solve shape from normals.

A 3D surface is represented as $Z = f(X, Y)$. In the camera coordinate system, image coordinates can be written as

$$u = \frac{fX}{Z}, v = \frac{fY}{Z}, f \text{ is focal length}$$

When we use weak perspective (or scaled orthographic) camera model, we have

$$u = kX, v = kY, k = \frac{f}{Z_0}$$

So, $Z = f(kX, kY)$, and

$$\begin{aligned} \mathbf{n} &= \frac{1}{\sqrt{1 + \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2}} \left[-\frac{\partial Z}{\partial X}, -\frac{\partial Z}{\partial Y}, 1 \right]^T \\ &= \frac{1}{\sqrt{k^2 + \left(\frac{\partial Z}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial v}\right)^2}} \left[-\frac{\partial Z}{\partial u}, -\frac{\partial Z}{\partial v}, k \right]^T \end{aligned}$$

Since $\mathbf{n} = [n_x, n_y, n_z]^T$ and we have figured it out before, we can write,

$$\begin{aligned} \frac{\partial Z}{\partial u} &= -k \frac{n_x}{n_z} \\ \frac{\partial Z}{\partial v} &= -k \frac{n_y}{n_z} \end{aligned}$$

Since we do not know Z_0 , we can assume $\bar{Z} = Z/k$, and we have

$$\frac{\partial \bar{Z}}{\partial u} = -\frac{n_x}{n_z} = n_2(u, v) \quad (12)$$

$$\frac{\partial \bar{Z}}{\partial v} = -\frac{n_y}{n_z} = n_1(u, v) \quad (13)$$

And Z can be solved from such a set of partial differential equations. However, since images are just discrete points, we can avoid solving differential equations by looking at all the image pixels:

$$\begin{aligned} \bar{Z}(u+1, v) - \bar{Z}(u, v) &= n_1(u, v) \\ \bar{Z}(u, v+1) - \bar{Z}(u, v) &= n_2(u, v) \end{aligned}$$

If we have M pixels, we will have $2M$ equations to solve M unknowns. We can again use least square fitting to figure them out. Please notice, our solution is up to a scale factor, since what we solve is not true Z , but \bar{Z} , which is a scaled version of Z .

6 Illumination Cone (optional)

If an image consists of N pixels, we can treat this image as a point in space \mathcal{R}^N . If we fix the camera and the object, but change the lighting conditions arbitrarily, for example, we can move the light sources, change the lighting intensities, and even add or remove lighting sources, the question we ask is: *what is the set of images of an object under ALL lighting conditions?* In other words, what should be the distribution of all the images in \mathcal{R}^N ?

Do those images sweep whole \mathcal{R}^N ? Let's guess . . .

What is the dimensionality of such a set of images? You can have guess again . . .

If we are give several images of the same object taken under different lighting conditions, can we synthesis some other images of this object under other lighting conditions? You surely answer yes, but how?

To get the answers, let's read Peter Belhumeur and David Kriegman "*What is the Set of Images of an Object Under All Possible Lighting Conditions*", which is the best paper of CVPR'96. If interested, please read another paper by A. Georghiades, D. Kriegman and P. Belhumeur, "*Illumination Cones for Recognition Under Variable Lighting: Faces*", in CVPR'98.