THE CARDINALITY OF HAMEL BASES
OF BANACH SPACES

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Abstract

We show that in an infinite dimensional Banach space, every Hamel base has the cardinality of the Banach space, which is at least the cardinality of the continuum.

1 Facts concerning Hamel bases

In this article we investigate by set theoretical methods the cardinality of Hamel bases (also called "algebraic bases") of Banach spaces. In this text, a Banach space E is a complete normed vector space over a field $K \subset \mathbb{R}$ (or $K \subset \mathbb{C}$), and to exclude the trivial case, we always assume $E \neq \{0\}$. Notice, that a Banach space over $\mathbb{R}$ can be considered as a Banach space over any subfield $K \subseteq \mathbb{R}$, simply by restricting the scalars to K. Conversely, a Banach space E over $K \subseteq \mathbb{R}$ can be considered as a Banach space over $\mathbb{R}$. This is done by extending the multiplication with scalars to $\mathbb{R}$: If $v \in E$ and $r \in \mathbb{R}$, let $rv := \lim_{r_i \to r} r_i v \in E$, where $r_i \in K$. This is possible, since any subfield K of $\mathbb{R}$ contains $\mathbb{Q}$ and is hence dense in $\mathbb{R}$. It is readily checked that this definition makes E being a Banach space over $\mathbb{R}$.

With the aid of the axiom of choice, one can prove that every vector space has a Hamel base (cf. [Ha] and [Hd 1, p. 295]). Furthermore, the axiom of choice is necessary for the existence of Hamel bases of vector spaces (cf. [La]). The proof of the existence of a Hamel base is not constructive, but since the axiom of choice is consistent with the other axioms of set theory (cf. [Gö]), it is consistent to assume the existence of a Hamel base in every vector space. If the continuum hypothesis (see Section 2) holds—which is by [Gö] consistent to assume—we have, as an easy consequence of Baire's Category Theorem, that every Hamel base of an infinite dimensional Banach space has at least the cardinality of the continuum. But it is also consistent to assume that the continuum hypothesis fails (cf. [Co]), and in this case, Baire's Category Theorem only implies that a Hamel base of an infinite dimensional Banach space must be uncountable (see also [Kr], [Ts] and [Li]). It is therefore a natural question to ask what cardinality a Hamel base can have in this case (and the answer is given in Section 3). The continuum hypothesis can also be characterized by Hamel bases; namely,

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if one considers $\mathbb{R}$ as a Banach space over $\mathbb{Q}$, then the continuum hypothesis is equivalent to the statement, that $\mathbb{R}$ can be covered by countably many Hamel bases (see [Si] and [EK]). These examples show that properties of vector spaces are closely related to the axioms of set theory and conversely, that set theory has implications in functional analysis on a very fundamental level.

2 Some set theory

In this section, we summarize some set theoretical notations and definitions. All the notations and definitions are standard and are in accordance with [Je] or [Ku].

A set x is **transitive** if every element of x is a subset of x. A relation R **well-orders** a set x, or $\langle R, x\rangle$ is a **well-ordering**, if $\langle R, x\rangle$ is a total ordering and every non-empty subset of x has an R-least element. The axiom of choice is equivalent to the statement, that every set can be well-ordered. A set x is an **ordinal number**, if x is transitive and well-ordered by $\in$. The axiom of choice is also equivalent to the statement, that for every set x there exists an ordinal number α and a bijection $f : \alpha \to x$. The class of all ordinal numbers is transitive and well-ordered by $\in$. The set of all natural numbers is equal to the set of all finite ordinal numbers and is denoted by ω. (A natural number n is the set of all natural numbers which are smaller than n, e.g. $0 = \emptyset$.)

For a set x, the **cardinality** of x, denoted by $|x|$ is the least ordinal number α for which there exists a bijection $f : \alpha \to x$. A set x is called finite, if $|x| \in \omega$, otherwise it is called infinite. Further it is called countable, if $|x| \leq |\omega| =: \aleph_0$. For a set x, $\mathcal{P}(x)$ denotes the power set of x. There exists a bijection between $\mathbb{R}$ and $\mathcal{P}(\omega)$, hence $|\mathbb{R}| = |\mathcal{P}(\omega)|$, and we denote this cardinality by $\mathfrak{c}$. The continuum hypothesis states $\mathfrak{c} = |\omega_1| =: \aleph_1$, where ω_1 denotes the least ordinal number which is not countable. Finally let $[x]^\omega := \{y \in \mathcal{P}(x) : |y| = \aleph_0\}$ and $[x]^{<\omega} := \{y \in \mathcal{P}(x) : |y| < \aleph_0\}$. If x is infinite, then $|[x]^{<\omega}| = |x|$. We use the same symbol for a set $y \in \mathcal{P}(x)$ and for its characteristic function, i.e., we write $y(z) = 1$ if $z \in y$ and $y(z) = 0$ otherwise.

In the next section, we will use an uncountable *independent family* over ω:

Let $\mathcal{I}\subseteq[\omega]^\omega$; then $\mathcal{I}$ is called an **independent family** (i.f.) if and only if whenever $m, n \in \omega$ and $x_0, \ldots, x_m, y_0, \ldots, y_n$ are distinct members of $\mathcal{I}$, then

$$\big|x_0 \cap \cdots \cap x_m \cap (\omega \setminus y_0) \cap \cdots \cap (\omega \setminus y_n)\big| = \aleph_0 .$$

Notice that this is equivalent to

$$\Big|\bigcap_{i\leq m} x_i \setminus \bigcup_{j\leq n} y_j\Big| = \aleph_0 .$$

There is always an i.f. of cardinality $\mathfrak{c}$ (cf. [FK], [Hd 2] or [Ku, Exercise (A6)]) which can be constructed even without using the axiom of choice.

3 The cardinality of Hamel bases of Banach spaces

By the axiom of choice, every vector space E over a field K possesses a Hamel base. A Hamel base is hence a set H of vectors such that

(i) H spans E, i.e. $E = \langle H \rangle$ (which denotes the set of all finite K-linear combinations of vectors of H) and

(ii) H is finitely linearly independent over K, i.e., finitely many vectors in H are linearly independent over K.

This is equivalent to saying that H is a minimal set with property (i) or that H is a maximal set with property (ii).

By Baire's Category Theorem it is easy to see that a Hamel base in an infinite dimensional real or complex Banach space E cannot be countable. In this section, we will show that if E is an infinite dimensional Banach space over a field K, where $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$, then every Hamel base of E has the cardinality $|E|$, which is at least the cardinality of the continuum. We start by recalling the well-known

FACT 3.1 *If E is an infinite dimensional vector space over a field K and H_1 and H_2 are two Hamel bases of E, then $|H_1| = |H_2|$.*

The next lemma summarizes a few simple facts which will be useful later.

LEMMA 3.2 *(a) If E is at the same time a vector space over a field K_1 and over a field $K_2 \subseteq K_1$ and if H_i is a Hamel base of E with respect to K_i ($i \in \{1, 2\}$), then $|H_1| \leq |H_2|$.*

(b) If E is a Banach space over a field K, then $|E| \geq \mathfrak{c}$.

(c) If E is a vector space over an infinite field K and if H is a Hamel base of E, then $|E| = \max\{|K|, |H|\}$.

(d) If K_1 and K_2 are two fields with $\mathbb{Q} \subseteq K_1 \subseteq K_2 \subseteq \mathbb{C}$ such that K_1 is dense in K_2, and if E is a Banach space over K_1, then there exists a K_1-linear homeomorphism from E to a Banach space $E' = E$ over K_2.

PROOF: (a) Obviously, H_1 can be extended to a Hamel base $H \supseteq H_1$ with respect to K_2, and therefore, using Fact 3.1, $|H_2| = |H| \geq |H_1|$.

(b) This follows from the completeness of Banach spaces.

(c) Since H is a Hamel base of E, we get $|E| = |[K]^{<\omega} \times [H]^{<\omega}|$, and because K is infinite, no matter if H is infinite or finite, we get $|[K]^{<\omega} \times [H]^{<\omega}| = \max\{|K|, |H|\}$.

(d) We define on the set $E' := E$, equipped with the same additive group and the same norm as on E, the following multiplication:

$$(K_2, E) \to E, \quad (\lambda, x) \mapsto \lambda x := \lim_{K_1 \ni \lambda_i \to \lambda} \lambda_i x.$$

It is easy to check, that (K_2, E) is a normed vector space and that $(K_1, E) \to (K_2, E)$, $x \mapsto x$, is a K_1-linear homeomorphism. $\dashv$

Remark: The Banach spaces in Lemma 3.2(d) need not have the same dimension. A drastic example is $\mathbb{R}$, which is an infinite-dimensional Banach space over $\mathbb{Q}$, but only a one-dimensional Banach space over itself.

The following lemma is due to Mazur (see [LT, Lemma 1.a.6]):

Lemma 3.3 *Let E be an infinite dimensional Banach space over K. Let $F \subseteq E$ be a finite dimensional subspace and let $\varepsilon > 0$. Then there is an $x \in E$ with $\|x\| = 1$ so that $\|y\| \leq (1+\varepsilon)\,\|y + \lambda x\|$ for every $y \in F$ and every scalar $\lambda \in K$*

Proof: Confer the proof of Lemma 1.a.6 of [LT]. $\dashv$

Now we are ready to prove

Lemma 3.4 *If $K \subseteq \mathbb{C}$ is a field and E is a Banach space over K such that $\dim(E) = \infty$, then every Hamel base of E has at least cardinality $\mathfrak{c}$.*

Proof: We consider two cases. In the first case $|K| < \mathfrak{c}$, and in the second case $|K| = \mathfrak{c}$. For both cases, let $H = \{X_\iota : \iota < \kappa < \mathfrak{c}\} \subseteq E$ be a family (of cardinality $\kappa < \mathfrak{c}$) of vectors of E. We will show that H is not a Hamel base of E.

1. case: Assume $|K| < \mathfrak{c}$ and that H is a Hamel base of E. By Lemma 3.2 we have $\mathfrak{c} \leq |E| = \max\{|K|, |H|\} < \mathfrak{c}$, which is a contradiction.

2. case: Assume $|K| = \mathfrak{c}$ and that H is a Hamel base of E. Lemma 3.3 is usually used in order to construct a subspace of E which possesses a Schauder base. Since we did not assume that K is complete, the construction does not lead to a complete subspace, nevertheless, the resulting sequence is sufficient for our purposes: We start with a unit vector $x_0 \in E$. Then we construct iteratively the sequence $\{x_i\}_{i\in\omega}$ with $\|x_i\| = 1$ such that

$$\|y\| \leq (1+\varepsilon_n)\,\|y + \lambda x_{n+1}\|$$

for all $y \in \langle x_0, \ldots, x_n \rangle$ and all $\lambda \in K$. Here, we choose the sequence of positive numbers $\{\varepsilon_n\}_{n\in\omega}$ such that $\prod_{n=0}^{\infty}(1+\varepsilon_n) \leq 1+\varepsilon$ for some $\varepsilon > 0$. Now, we claim that $\sum_{n=0}^{\infty} \lambda_n x_n = 0$

implies that $\lambda_n = 0$ for all $n \in \omega$. If not, we find a first index i with $\lambda_i \neq 0$. Then we have

$$\begin{aligned} \|\lambda_i x_i\| &\leq (1+\varepsilon_i) \left\|\lambda_i x_i + \lambda_{i+1} x_{i+1}\right\| \\ &\leq (1+\varepsilon_i)(1+\varepsilon_{i+1}) \left\|\lambda_i x_i + \lambda_{i+1} x_{i+1} + \lambda_{i+2} x_{i+2}\right\| \\ &\leq \prod_{k=i}^{n} (1+\varepsilon_k) \left\|\sum_{k=i}^{n+1} \lambda_k x_k\right\| \end{aligned}$$

Since the first factor is uniformly bounded in n and the second factor converges to 0 as $n \to \infty$, we obtain $\|\lambda_i x_i\| = 0$, which contradicts $\lambda_i \neq 0$.

Let $\mathcal{I}$ be an i.f. of cardinality $\mathfrak{c}$ (remember that such a family always exists) and let us consider the injective map

$$\zeta : \mathcal{I} \to E, \quad z \mapsto \sum_{i=0}^{\infty} z(i) 2^{-i} x_i.$$

We recall the notation $z(i) = 1$ if $i \in z$ and $z(i) = 0$ otherwise. Notice that, by construction, the vectors in $\{\zeta(z) : z \in \mathcal{I}\}$ are finitely linearly independent over K: In fact, if we take distinct $z_0, \ldots, z_m \in \mathcal{I}$, then for any number $k \in \omega$ and for any $i \leq m$ we find a $k' > k$ such that $z_i(k') \neq z_j(k')$ for all $j \neq i$, hence, the vectors $\zeta(z_0), \ldots, \zeta(z_m)$ are linearly independent over K.

For each $Y \in E$, there are finitely many uniquely determined $X_{\iota_0}, \ldots, X_{\iota_{n(Y)}} \in H$, $\iota_i < \iota_{i+1}$ and $s_0, \ldots, s_{n(Y)} \in K \setminus \{0\}$, so that $Y = \sum_{k=0}^{n(Y)} s_k X_{\iota_k}$. Thus, the function

$$\begin{aligned} \varphi : E &\to [K]^{<\omega} \times [H]^{<\omega} \\ Y &\mapsto \big\langle \langle s_0, \ldots, s_{n(Y)} \rangle, \langle X_{\iota_0}, \ldots, X_{\iota_{n(Y)}} \rangle \big\rangle \end{aligned}$$

is a bijection and the composed function $\varphi \circ \zeta : \mathcal{I} \to [K]^{<\omega} \times [H]^{<\omega}$ is injective. On the other hand, since $|\mathcal{I}| = \mathfrak{c}$ and $|[H]^{<\omega}| = |H| < \mathfrak{c}$, we find by the pigeonhole principle (see [Je, p. 321]) an infinite set $\mathcal{C} \subseteq \mathcal{I}$ such that $\mathrm{pr}_2 \circ \varphi \circ \zeta : \mathcal{C} \to [H]^{<\omega}$ is constant (pr_2 denotes the projection $\mathrm{pr}_2(\langle a, b \rangle) := b$). So, let $H_0 = \langle \mathrm{pr}_2 \circ \varphi \circ \zeta(\mathcal{C}) \rangle$ denote the corresponding finite dimensional subspace. Since ζ is injective, $\zeta(\mathcal{C}) \subseteq H_0$ is an infinite set of linearly independent vectors, which is a contradiction. $\dashv$

Now we can give the main result of this section.

THEOREM 3.5 *If $K \subseteq \mathbb{C}$ is a field and E is a Banach space over K such that* $\dim(E) = \infty$, *then every Hamel base of E has cardinality $|E|$.*

PROOF: Let E be a Banach space over K such that $\dim(E) = \infty$ and let H be a Hamel base of E. By Lemma 3.2 we have $|E| = \max\{|K|, |H|\}$. By Lemma 3.4 we have $|H| \geq \mathfrak{c}$, and because $|K| \leq \mathfrak{c}$, we get $|E| = |H|$. $\dashv$

Remark: It is worth mentioning that the previous result for F-spaces follows directly from Martin's Axiom (the definition and some consequences can be found in [Ku, Ch. II]): Let E be an F-space, i.e., a topological vector space whose topology is induced by a complete invariant metric d. If $H_0 \subseteq H \subseteq E$ with $|H_0| = \aleph_0$, $|H| < \mathfrak{c}$, we may consider the countable set $A := \langle H_0 \rangle_{\mathbb{Q}}$, the set of all finite rational linear combinations of vectors of H_0. Let P be the set $\{B_{1/n}(a_i) : a_i \in A, n \in \omega\}$, where $B_{1/n}(a_i) := \{x \in E : d(x, a_i) < \frac{1}{n}\}$. Let $\mathbb{P} = \langle P, \subseteq \rangle$, then $\mathbb{P}$ is a partially ordered set in which every anti-chain is countable. A set $D \subseteq P$ is called dense, if for every $p \in P$ there exists a $q \in D$ such that $q \subseteq p$. For every finite dimensional K-linear subspace $V \subseteq E$, the set $D_V := \{p \in P : p \cap V = \emptyset\}$ is dense. Since $|H| < \mathfrak{c}$ we have strictly less than $\mathfrak{c}$ many dense sets of this form and Martin's Axion gives a descending chain in P such that for every dense set D_V we find an element in this chain, which is contained in D_V. Since E is a complete space, this chain converges to a point which does not belong to any of the finite dimensional subspaces spanned by H. Hence, H is not a Hamel base.

As a corollary, we obtain a slightly stronger version of a theorem in [Ja, Chapter 9]:

Corollary 3.6 *The set E^f of all linear functions $E \to \mathbb{R}$ on an infinite dimensional Banach space E has cardinality $2^{|E|}$.*

Proof: It is easy to see that $|E^f| = |\mathfrak{c}^H| = 2^{|H|}$, where H is a Hamel base of E, and therefore $|E^f| = 2^{|E|}$. $\dashv$

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