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A FAMILY OF FACE PRODUCTS OF MATRICES AND ITS PROPERTIES

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As is well known, there existed periods in the history of mathematics when technological progress introduced noticeable correctives in the natural course of the development of abstract mathematical theories. In recent years, such a stage has come in the theory of matrices. This time, it has affected rather fundamental principles of linear algebra, supplementing it by new operations of composition of matrix data.

In addition to the classical set of these operations, due to the objective demands for technological applications of systems analysis a lot of unconventional procedures of matrix multiplication should be introduced in practice. In particular, we mean the generalized Kronecker products with respect to columns and rows [1, 2], left and right almost Kronecker products [2], and also the family of face products of matrices [3–5].

The face multiplication, which was primarily proposed for processing signals of digital antenna arrays, may be also used for analysis of other complicated systems. Therefore, the objective of this article is to generalize its properties and consider new varieties of such a procedure for block matrices.

According to [3–5], we will call the $p \times gs$ matrix $A \square B$ specified by the equality

$$A \square B = [a_{ij} B_i] \tag{1}$$

the face product of the $p \times g$ matrix $A = [a_{ij}]$ and $p \times s$ matrix B that is represented as the block-matrix of the rows B_i $(B = [B_i],\ i = 1, \ldots, p)$.

Example 1.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix};$$

$$A \square B = \left[\begin{array}{ccc|ccc} a_{11} \cdot b_{11} & a_{11} \cdot b_{12} & a_{11} \cdot b_{13} & a_{12} \cdot b_{11} & a_{12} \cdot b_{12} & a_{12} \cdot b_{13} \\ a_{21} \cdot b_{21} & a_{21} \cdot b_{22} & a_{21} \cdot b_{23} & a_{22} \cdot b_{21} & a_{22} \cdot b_{22} & a_{22} \cdot b_{23} \\ a_{31} \cdot b_{31} & a_{31} \cdot b_{32} & a_{31} \cdot b_{33} & a_{32} \cdot b_{31} & a_{32} \cdot b_{32} & a_{32} \cdot b_{33} \end{array}\right].$$

If the matrix A is considered in this example as the totality of the column vectors corresponding to the initial coordinates of several points moving in a Cartesian coordinate system, then the semantic aspect of the product $A \square B$ can be described as a geometric transformation B that equally changes the like (i.e., of the same name) coordinates of all the points of the object A.

In essence, before the introduction of the face product in matrix algebra, the following two extreme variants of matrix-data composition existed: the Hadamard and Kronecker products, which reflect the elementary and extremely general levels of decomposition of matrices as systems of numbers. In the former case, the elementwise multiplication is executed, and in the latter and its generalizations [1, 2], almost the entire matrix is considered as a multiplier. At

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the same time, the face product represents an intermediate level of the decomposition of matrices that consists in their rowwise fragmentation. The term "face" figuratively reflects the fact that the end face of the right matrix is, as it were, split into rows before multiplying by the elements of the left matrix.

The symmetric complement of the face variant of multiplication is its transposed modification. According to [3–5], the transposed face product (TFP) of the $g \times p$ matrix $A = [a_{ij}]$ and the $s \times p$ matrix B that is represented as a block matrix of the columns ($B = [B_j]$, $j = 1, \ldots, p$), is the $gs \times p$ matrix $A \blacksquare B$ specified by the equality

$$A \blacksquare B = [a_{ij} \cdot B_j]. \tag{2}$$

Example 2.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix};$$

$$A \blacksquare B = \begin{bmatrix} a_{11} \cdot b_{11} & a_{12} \cdot b_{12} & a_{13} \cdot b_{13} \\ a_{11} \cdot b_{21} & a_{12} \cdot b_{22} & a_{13} \cdot b_{23} \\ \cdots & \cdots & \cdots \\ a_{21} \cdot b_{11} & a_{22} \cdot b_{12} & a_{23} \cdot b_{13} \\ a_{21} \cdot b_{21} & a_{22} \cdot b_{22} & a_{23} \cdot b_{23} \end{bmatrix}.$$

It is worthy of note that, for expressions (1) and (2), we can write the law of inversion of the order of matrices [5]. This law is similar to the well-known property of transposition of generalized Kronecker products with respect to rows and columns [1]

$$(A \square B)^T = A^T \blacksquare B^T, \; (A \blacksquare B)^T = A^T \square B^T. \tag{3}$$

The first things to notice are the following associative and distributive properties of face products of matrices [3–5]:

$$(A \square B) \square C = A \square (B \square C), \quad (A \blacksquare B) \blacksquare C = A \blacksquare (B \blacksquare C), \tag{4}$$

$$(A+B) \square C = A \square C + B \square C, \; (A+B) \blacksquare C = A \blacksquare C + B \blacksquare C, \tag{5}$$

$$A \square (B+C) = A \square B + A \square C, \; A \blacksquare (B+C) = A \blacksquare B + A \blacksquare C, \tag{6}$$

$$\begin{gathered} (A+B) \square (C+D) = A \square C + B \square C + A \square D + B \square D, \\ (A+B) \blacksquare (C+D) = A \blacksquare C + B \blacksquare C + A \blacksquare D + B \blacksquare D. \end{gathered} \tag{7}$$

Naturally, it is assumed that the forms of the matrices are concordant in expressions (4)–(7). The relations for TFP can be easily obtained by transposing the corresponding expressions for the face product. Therefore, in what follows, we will restrict ourselves, whenever possible, to consideration of the properties of face multiplication, except for the cases where the representation contains both varieties of this multiplication.

Taking account of what has been said above, let us recall that the face products are not commutative [5], i.e.,

$$A \square B \neq B \square A, \tag{8}$$

though the commutativity of vectors and the products of vectors and matrices is admissible [3], i.e.,

$$a \square B = B \square a. \tag{9}$$

In the particular case where a and b are vectors, the following easily checked properties take place [3]:

$$a^T \square b^T = a^T \otimes b^T \tag{10}$$

(a is a p-vector and b is a g-vector);

$$a \square b = a \bigcirc b \tag{11}$$

(a and b are p-vectors, "$\bigcirc$" is the symbol the of Hadamard product).

The procedure of face product makes it possible to reduce the number of computing operations in prevalent problems of multiplication of the diagonal matrix $A = \text{diag}\,[a_1\, a_2\, \ldots\, a_p]$ and $p{\times}s$ matrix B: $\text{diag}\,[a_1\, a_2\, \ldots\, a_p]\cdot B = [a_1\, a_2\, \ldots\, a_p]^T \square B$. In this case, according to [3], in contrast to the ordinary product, the multiplication operations are reduced in number by a factor of p for the face product, and the execution of $ps(p-1)$ additions is not necessary at all.

The consideration of interrelation of the face products with the multiplication methods that are well known in matrix theory is of most interest in the case where one expression contains several of their varieties. In this case, according to [5],

$$A \otimes (B \square C) = (A \otimes B) \square C, \tag{12}$$

$$A \bigcirc (B \square C) \neq (A \bigcirc B) \square C. \tag{13}$$

The adaptation of the face product with the Hadamard one is possible within the framework of the identity [3]

$$(A \bigcirc B) \square (C \bigcirc D) = (A \square C) \bigcirc (B \square D), \tag{14}$$

which holds true if A and B are $m{\times}n$ matrices, and C and D have the dimension $m{\times}p$. The Hadamard product on the right side of (14) is defined, since its constituents $A \square C$ and $B \square D$ are $m{\times}np$ block matrices.

In view of the property of commutativity of the elementwise multiplication, relation (14) may be rewritten in the form

$$(A \bigcirc B) \square (C \bigcirc D) = (A \square D) \bigcirc (B \square C)$$

$$= (B \square D) \bigcirc (A \square C) = (B \square C) \bigcirc (A \square D).$$

For the m-vector b, $m{\times}r$matrix S, and $m{\times}r$matrix F, it is easy to verify the validity of the substitution formula $(S^T \blacksquare F^T)\cdot b = \text{vec}\,[(b^T \blacksquare F^T)\cdot S\,]$, where "vec" is an operator representing the $m{\times}r$ matrix by means of a mr-column vector [6].

Since a combination of the face and transposed face products is also possible within the framework of matrix representation, we can make use of the theorem announced in [3] and given below.

THEOREM 1. If all matrix products are defined, then the face and transposed face products of matrices satisfy the equality

$$(A \square B)\cdot(C \blacksquare D) = (A\cdot C) \bigcirc (B\cdot D). \tag{15}$$

Proof. Let the dimensions of A, B, C, and D be equal to $j{\times}k$, $j{\times}z$, $k{\times}p$, and $z{\times}p$, respectively. Since the matrix $A \square B$ consists of $k{\times}z$ columns and the matrix $C \blacksquare D$ consists of $k{\times}z$ rows, the matrix product $(A \square B)\cdot(C \blacksquare D)$ is defined. It consists of j rows and p columns. The element that is situated at the intersection of the jth row and the pth column is equal to $\sum\limits_k \sum\limits_z a_{jk}\cdot b_{jz}\cdot c_{kp}\cdot d_{zp}$.

Since the matrices $A{\cdot}C$ and $B{\cdot}D$ contain j rows and p columns, respectively, the Hadamard product $(A{\cdot}C) \bigcirc (B{\cdot}D)$ is the $j \times p$ matrix and its element that is situated at the intersection of the jth row and pth column is equal to

$$\sum_k a_{jk}{\cdot}c_{kp}{\cdot}\sum_z b_{jz}{\cdot}d_{zp} = \sum_k \sum_z a_{jk}{\cdot}b_{jz}{\cdot}c_{kp}{\cdot}d_{zp},$$

and this completes the proof.

The theorems on the absorption of the Kronecker products by the face ones are easily proved by analogy.

THEOREM 2. For the matrices $A, B, C,$ and D whose dimensions are, respectively, $j{\times}k$ (j is the number of rows and k is the number of columns), $m{\times}z$, $k{\times}p$, and $z{\times}p$, the following equality is valid:

$$(A \otimes B){\cdot}(C \blacksquare D) = (A{\cdot}C) \blacksquare (B{\cdot}D). \tag{16}$$

THEOREM 3. For the matrices $A, B, C,$ and D whose dimensions are $p{\times}k$, $p{\times}z$, $k{\times}j$, and $z{\times}m$, respectively, the following equality is valid:

$$(A \square B){\cdot}(C \otimes D) = (A{\cdot}C) \square (B{\cdot}D). \tag{17}$$

The proof of Theorem 3 may be also constructed on the basis of transpositions of expression (16). Indeed,

$$[(A \otimes B){\cdot}(C \blacksquare D)]^T = (C \blacksquare D)^T{\cdot}(A \otimes B)^T = (C^T \square D^T){\cdot}(A^T \otimes B^T).$$

On the other hand,

$$[(A \otimes B){\cdot}(C \blacksquare D)]^T = [(A{\cdot}C) \blacksquare (B{\cdot}D)]^T = (C^T{\cdot}A^T) \square (D^T{\cdot}B^T).$$

Thus, $(C^T \square D^T){\cdot}(A^T \otimes B^T) = (C^T{\cdot}A^T) \square (D^T{\cdot}B^T)$.

Having replaced the above designations, it is easy to obtain (17), which is what had to be proved.

A consequence of Theorems 1–3 is one more property of "absorption":

$$(A \square L)(B \otimes M){\cdot}(C \otimes N){\cdot} \ldots {\cdot}(J \otimes S){\cdot}(K \blacksquare T) = (A{\cdot}B{\cdot}C{\cdot} \ldots {\cdot}J{\cdot}K) \bigcirc (L{\cdot}M{\cdot}N{\cdot} \ldots {\cdot}S{\cdot}T),$$

which completes the enumeration of the properties of the face product that are known to date.

In statistical analysis of a collection of subsystems of the same type that form a complicated system, for example, a multiposition radio-locating network consisting of several receiving stations, block matrices are used. As is well known [5], rules (1)–(16) are also applicable to them; however, in some cases, there is a need for special block modifications of face products, which were first considered in [5].

One of them is the block face product (BFP), which, for the $bp{\times}cs$ matrix $A = [A_{ij}]$ and $bp{\times}cg$ matrix $B = [B_{ij}]$ ($i = 1, \ldots b$; $j = 1, \ldots c$) with a concordant partition into blocks, whose size is equal to $p{\times}s$ and $p{\times}g$, respectively, is determined by the following equality [5]:

$$A \Uparrow B = [A_{ij} \square B_{ij}]. \tag{18}$$

Example 3.

$$P = \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right] = \left[\begin{array}{cc|cc} p_{111} & p_{121} & p_{112} & p_{122} \\ p_{211} & p_{221} & p_{212} & p_{222} \\ p_{311} & p_{321} & p_{312} & p_{322} \end{array}\right],$$

$$T = \left[\begin{array}{c|c} T_1 & T_2 \end{array}\right] = \left[\begin{array}{ccc|ccc} t_{111} & t_{121} & t_{131} & t_{112} & t_{122} & t_{132} \\ t_{211} & t_{221} & t_{231} & t_{212} & t_{222} & t_{232} \\ t_{311} & t_{321} & t_{331} & t_{312} & t_{322} & t_{332} \end{array}\right],$$

$$P \Uparrow T = \left[P_1 \square T_1 \;\middle|\; P_2 \square T_2 \right].$$

If each block of the matrix P is treated as the totality of column vectors describing the parameters of the state of two subsystems of an object in one of independently varying reference systems, then a certain two-component system functioning simultaneously in two different reference systems can be assigned to the entire matrix P in the example considered. The block face product $A \Uparrow T$ makes it possible to formalize the dynamics of evolution of such a system at successive points of time (three in this example) during the transformation of the reference systems that identically change the like parameters of column vectors in each reference system. It is especially important that the laws of parameter transformation are assumed to be dependent on reference systems and variable in time.

In the case where the partition into blocks is discordant, the face product of matrices (1) can be also used instead of BFP. In particular, for the engineering object P considered in the preceding example, a transformation may be specified that identically changes the parameters of the states of the subsystems with respect to all the reference systems. Its result will be written according to rule (1) as follows:

$$P = \left[P_1 \;\middle|\; P_2 \right],\quad T = \begin{bmatrix} t_{111} & t_{121} & t_{131} \\ t_{211} & t_{221} & t_{231} \\ t_{311} & t_{321} & t_{331} \end{bmatrix},\quad P \square T = \left[P_1 \square T \;\middle|\; P_2 \square T \right].$$

The above holds true for the transposed BFP [5]. By definition, the transposed block face product of the $cs \times bp$ matrix $A = [A_{ij}]$ and $cg \times bp$matrix $B = [B_{ij}]$ ($j = 1, \ldots, c$; $i = 1, \ldots, b$) with a concordant partition into blocks, whose size is equal to $s \times p$ and $g \times p$, respectively, is the matrix $A \Rightarrow B$ specified by the equality $A \Rightarrow B = [A_{ij} \blacksquare B_{ij}]$.

Note that the block types of products have many properties that are inherent in the face multiplication and its transposed variant; therefore, they are not discussed here.

In conclusion, let us consider one more modification of the face product that can be useful in technological applications. Its essence lies in using a series of numbers located in a dimension, which is additional relative to the left matrix, as the rows by which the right matrix is to "be split." In this case, the penetrating face product of the $p \times g$ matrix $A = [a_{ij}]$ and n-dimentional matrix B ($n \geq 3$) that is unfolded in the block row with $p \times g$ blocks ($B = [B_r]$) is a matrix of size B of the form

$$A \odot B = \left[A \circ B_1 \;\middle|\; A \circ B_2 \;\middle|\; \cdots \;\middle|\; A \circ B_r \;\middle|\; \cdots \right], \tag{19}$$

where $A \circ B_r$ is the Hadamard product.

Example 4.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},\quad B = \left[\begin{array}{cc|cc|cc} b_{111} & b_{121} & b_{112} & b_{122} & b_{113} & b_{123} \\ b_{211} & b_{221} & b_{212} & b_{222} & b_{213} & b_{223} \\ b_{311} & b_{321} & b_{312} & b_{322} & b_{313} & b_{323} \end{array}\right],$$

$$A \odot B$$

$$= \left[\begin{array}{cc|cc|cc} a_{11} \cdot b_{111} & a_{12} \cdot b_{121} & a_{11} \cdot b_{112} & a_{12} \cdot b_{122} & a_{11} \cdot b_{113} & a_{12} \cdot b_{123} \\ a_{21} \cdot b_{211} & a_{22} \cdot b_{221} & a_{21} \cdot b_{212} & a_{22} \cdot b_{222} & a_{21} \cdot b_{213} & a_{22} \cdot b_{223} \\ a_{31} \cdot b_{311} & a_{32} \cdot b_{321} & a_{31} \cdot b_{312} & a_{32} \cdot b_{322} & a_{31} \cdot b_{313} & a_{32} \cdot b_{323} \end{array}\right].$$

Treating B as a matrix of Cartesian coordinates of the points that form a geometric object containing two points on each of three planes, the transformation $A \odot B$ may be interpreted as a space deformation for which the coordinates of the like points of an object, which belong to different planes, are transformed identically.

In this situation, the face product (1) produces a weighted "duplicating" of the three-dimensional matrix, repeating it according to the number of columns of the two-dimensional one. The distinctive feature of the penetrating face product is the "penetration" of the two-dimensional matrix through the three-dimensional one without changing the dimension of the latter (an unfolded representation in (19) and in Example 4 is given for the sake of obviousness). Thus, the operation introduced in (19) permits one to formalize the process of "penetration" of discrete sets through sets of greater dimensions; mathematical modeling of this process is frequently necessary in systems analysis.

It is remarkable that if a p-vector C is concordant with a two-dimensional matrix B with respect to the number of rows, then the identity $C \square B = C \odot B$ is valid. It is relevant to remark that one of the properties of the penetrating face multiplication is its commutativity: $A \odot B = B \odot A$.

The examples presented above are only a perfunctory illustration of the capabilities of the new types of matrix operations. Using these operations, the author has already succeeded in obtaining the Cramer–Rao bounds to estimate the potential accuracy of multisignal direction finding, spectral selection, and impulse distance measurement [5, 7], in constructing a series of analytical models of conformal digital antenna arrays [4], and also in synthesizing methods based on these models for the hyper Rayleigh resolution of point sources. The progress in these domains has been in many respects hindered because of the imperfection of the traditional matrix algebra. The author believes that there exist other problems in systems analysis whose solution may be supported by the approach proposed.

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