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Teaching Mathematics for Understanding

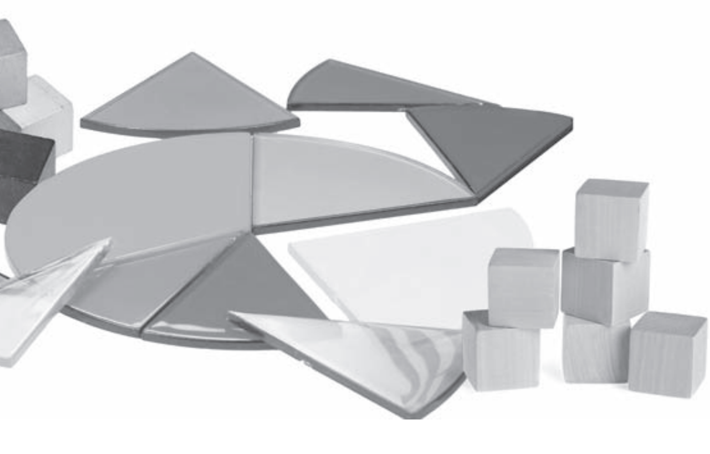
An understanding can never be "covered" if it is to be understood.

Wiggins and McTighe (2005, p. 229)

Teachers generally agree that teaching for understanding is a good thing. But this statement begs the question: What is *understanding*? Understanding is being able to think and act flexibly with a topic or concept. It goes beyond knowing; it is more than a collection of information, facts, or data. It is more than being able to follow steps in a procedure. One hallmark of mathematical understanding is a student's ability to justify why a given mathematical claim or answer is true or why a mathematical rule makes sense (Council of Chief State School Officers, 2010). Although children might *know* their basic multiplication facts and be able to give you quick answers to questions about these facts, they might not *understand* multiplication. They might not be able to justify how they know an answer is correct or provide an example of when it would make sense to use this basic fact. These tasks go beyond simply knowing mathematical facts and procedures. Understanding must be a primary goal for all of the mathematics you teach.

Understanding and Doing Mathematics

Procedural proficiency—a main focus of mathematics instruction in the past—remains important today, but conceptual understanding is an equally important goal (National Council of Teachers of Mathematics, 2000; National Research Council, 2001; CCSSO, 2010). Numerous reports and standards emphasize the need to address skills and understanding in an integrated manner; among these are the *Common Core State Standards* (CCSSO, 2010), a state-led effort coordinated by the National Governors Association Center for Best Practices (NGA Center) and CCSSO that has been adopted by nearly every state and the District of Columbia. This effort has resulted in attention to *how* mathematics is taught, not just *what* is taught.



The National Council of Teachers of Mathematics (NCTM, 2000) identifies the process standards of problem solving, reasoning and proof, representation, communication, and connections as ways to think about how children should engage in learning the content as they develop both procedural fluency and conceptual understanding. Children engaged in the process of *problem solving* build mathematical knowledge and understanding by grappling with and solving genuine problems, as opposed to completing routine exercises. They use *reasoning and proof* to make sense of mathematical tasks and concepts and to develop, justify, and evaluate mathematical arguments and solutions. Children create and use *representations* (e.g., diagrams, graphs, symbols, and manipulatives) to reason through problems. They also engage in *communication* as they explain their ideas and reasoning verbally, in writing, and through representations. Children develop and use *connections* between mathematical ideas as they learn new mathematical concepts and procedures. They also build *connections* between mathematics and other disciplines by applying mathematics to real-world situations. By engaging in these processes, children *learn* mathematics by *doing* mathematics. Consequently, the process standards should not be taught separately from but in conjunction with mathematics as ways of learning mathematics.

Adding It Up (National Research Council, 2001), an influential research review on how children learn mathematics, identifies the following five strands of mathematical proficiency as indicators that someone understands (and can do) mathematics.

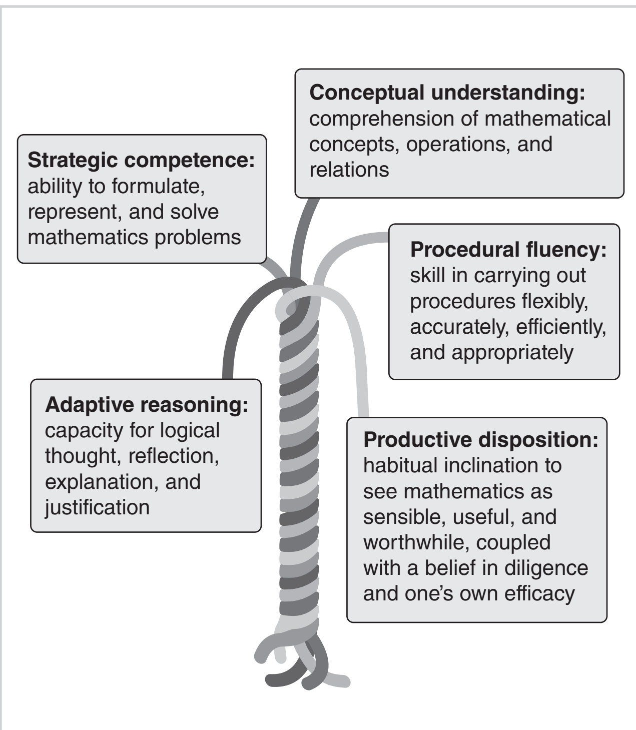
- ***Conceptual understanding***: Comprehension of mathematical concepts, operations, and relations
- ***Procedural fluency***: Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic competence***: Ability to formulate, represent, and solve mathematical problems
- ***Adaptive reasoning***: Capacity for logical thought, reflection, explanation, and justification
- ***Productive disposition***: Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Reprinted with permission from p. 116 of *Adding It Up: Helping Children Learn Mathematics*, 2001, by the National Academy of Sciences, Courtesy of the National Academies Press, Washington, D.C.)

This report maintains that the strands of mathematical proficiency are interwoven and interdependent—that is, the development of one strand aids the development of others (Figure 1.1).

Building on the NCTM process standards and the five strands of mathematical proficiency, the *Common Core State Standards* (CCSSO, 2010) outline the following eight Standards for Mathematical Practice (see Appendix A) as ways in which children can develop and demonstrate a deep understanding of and capacity to do mathematics. Keep in mind that you, as a teacher, have a responsibility to help children develop these practices. Here we provide a brief discussion about each mathematical practice.

Figure 1.1

Interrelated and intertwined strands of mathematical proficiency.



Source: Reprinted with permission from Kilpatrick, J., Swafford, J., & Findell, B. (Eds.), *Adding It Up: Helping Children Learn Mathematics.* Copyright 2001 by the National Academy of Sciences. Courtesy of the National Academies Press, Washington, D.C.

1. *Make sense of problems and persevere in solving them.* To make sense of problems, children need to learn how to analyze the given information, parameters, and relationships in a problem so that they can understand the situation and identify possible ways to solve it. Encourage younger students to use concrete materials or bar diagrams to investigate and solve the problem. Once children learn strategies for making sense of problems, encourage them to remain committed to solving them. As they learn to monitor and assess their progress and change course as needed, they will solve the problems they set out to solve!

2. *Reason abstractly and quantitatively.* This practice involves children reasoning with quantities and their relationships in problem situations. You can support children's development of this practice by helping them create representations that correspond to the meanings of the quantities and the units involved. When appropriate, children should also learn to represent and manipulate the situation symbolically. Encourage children to find connections between the abstract symbols and the representation that illustrates the quantities and their relationships. For example, when children use drawings to show that they made 5 bears from 3 red bears and 2 yellow bears, encourage them to connect their representation to the number sentence $5 = 3 + 2$. Ultimately, children should be able to move flexibly between the symbols and other representations.

3. *Construct viable arguments and critique the reasoning of others.* This practice emphasizes the importance of children using mathematical reasoning to justify their ideas and solutions, including being able to recognize and use counterexamples. Encourage children to examine each others' arguments to determine whether they make sense and to identify ways to clarify or improve the arguments. This practice emphasizes that mathematics is based on reasoning and should be examined in a community—not carried out in isolation. Tips for supporting children as they learn to justify their ideas can be found in Chapter 2.

4. *Model with mathematics.* This practice encourages children to use the mathematics they know to solve problems in everyday life. For younger students this could mean writing an addition or a subtraction equation to represent a given situation or using their number sense to determine whether there are enough plates for all the children in their class. Be sure to encourage children to determine whether their mathematical results make sense in the context of the given situation.

5. *Use appropriate tools strategically.* Children should become familiar with a variety of problem-solving tools that can be used to solve a problem and they should learn to choose which ones are most appropriate for a given situation. For example, second graders should experience using the following tools for computation: pencil and paper, manipulatives, calculator, hundreds chart, and a number line. Then in a situation when an estimate is needed for the sum of 23 and 52, some second graders might consider paper and pencil, manipulatives, and a calculator as tools that would slow down the process and would select a hundreds chart to quickly move from 50 down two rows (20 spaces) to get to 70.

6. *Attend to precision.* In communicating ideas to others, it is imperative that children learn to be explicit about their reasoning. For example, they need to be clear about the meanings of the operations and symbols they use, to indicate the units involved in a problem, and to clearly label the diagrams they provide in their explanations. As children share their ideas, make this expectation clear and ask clarifying questions that help make the details of their reasoning more apparent. Teachers can further encourage

Teaching Tip

Research suggests that children, in particular girls, may tend to continue to use the same tools because they feel comfortable with the tools and are afraid to take risks (Ambrose, 2002). Look for children who tend to use the same tool or strategy every time they work on tasks. Encourage all children to take risks and try new tools and strategies.

children's attention to precision by introducing, highlighting, and encouraging the use of accurate mathematical terminology in explanations and diagrams.

7. *Look for and make use of structure.* Children who look for and recognize a pattern or structure can experience a shift in their perspective or understanding. Therefore, set the expectation that children will look for patterns and structure and help them reflect on their significance. For example, look for opportunities to help children notice that the order in which they add two numbers does not change the sum—they can add $4 + 7$ or $7 + 4$ to get 11. Once they recognize this pattern with other examples, they will have a new understanding and the use of a powerful property of our number system, the commutative property of addition.

8. *Look for and express regularity in repeated reasoning.* Encourage children to step back and reflect on any regularity that occurs in an effort to help them develop a general idea or method or identify shortcuts. For example, as children begin adding numbers together, they will encounter situations in which zero is added to a number. Over time, help children reflect on the results of adding zero to any number. Eventually they should be able to express that when they add or subtract zero to any number, the number is unaffected.

Like the process standards, the Standards for Mathematical Practice should not be taught separately from the mathematics but should instead be incorporated as ways for children to learn and do mathematics. Children who learn to use these eight mathematical practices as they engage with mathematical concepts and skills have a greater chance of developing conceptual understanding. Note that learning these mathematical practices and, consequently, developing understanding takes time. So the common notion of simply and quickly "covering the material" is problematic. The opening quotation states it well: "An understanding can never be 'covered' if it is to be understood" (Wiggins & McTighe, 2005, p. 229). Understanding is an end goal—that is, it is developed over time by incorporating the process standards and mathematical practices and striving toward mathematical proficiency.

How Do Children Learn?

Let's look at a couple of research-based theories that can illustrate how children learn in general: constructivism and sociocultural theory. Although one theory focuses on the individual learner whereas the other emphasizes the social and cultural aspects of the classroom, these theories are not competing; they are actually compatible (Norton & D'Ambrosio, 2008).

Constructivism

At the heart of constructivism is the notion that learners are not blank slates but rather creators (constructors) of their own learning. All people, all of the time, construct or give meaning to things they think about or perceive. Whether you are listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge (existing schemas) to make sense of new information.

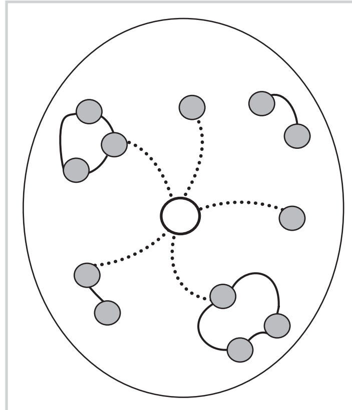
Constructing something in the physical world requires tools, materials, and effort. The tools you use to build understanding are your existing ideas and knowledge. Your materials might be things you see, hear, or touch, or they might be your own thoughts and ideas. The effort required to construct knowledge and understanding is reflective thought.

Through reflective thought people connect existing ideas to new information and in this way modify their existing schemas or background knowledge to incorporate new ideas. Making these connections can happen in either of two ways—*assimilation* or *accommodation*.

Assimilation occurs when a new concept "fits" with prior knowledge and the new information expands an existing mental network. Accommodation takes place when the new concept does not "fit" with the existing network, thus creating a cognitive conflict or state of confusion that causes what theorists call *disequilibrium*. As an example, consider what happens when children start learning about numbers and counting. They make sense of a number by counting a quantity of objects by ones. With larger numbers, such as two-digit numbers, they continue to use this approach to give meaning to the number (assimilation). Eventually, counting large amounts of objects becomes cumbersome and, at the same time, they are likely learning about grouping in tens. Over time they begin to view two-digit numbers differently—as groups of tens and ones—and they no longer have to count to give a number meaning (accommodation). It is through the struggle to resolve the disequilibrium that the brain modifies or replaces the existing schema so that the new concept fits and makes sense, resulting in a revision of thought and a deepening of the learner's understanding.

Figure 1.2

How someone constructs a new idea.



For an illustration of what it means to construct an idea, consider Figure 1.2. The gray and white dots represent ideas, and the lines joining the ideas represent the logical connections or relationships that develop between ideas. The white dot is an emerging idea, one that is being constructed. Whatever existing ideas (gray dots) are used in the construction are connected to the new idea (white dot) because those are the ideas that give meaning to the new idea. The more existing ideas that are used to give meaning to the new one, the more connections will be made.

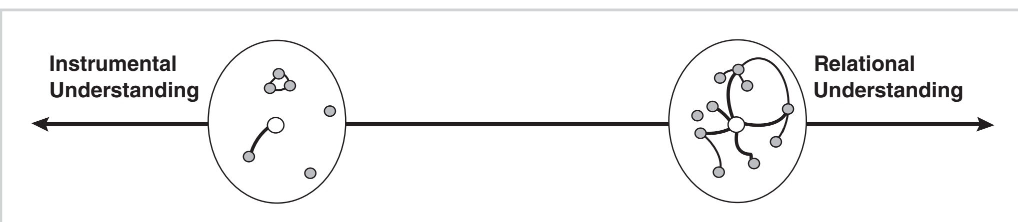
Each child's unique collection of ideas is connected in different ways. Some ideas are well understood and well formed (i.e., connected), others less so as they emerge and build connections. Children's experiences help them develop connections and ideas about whatever they are learning.

Understanding exists along a continuum (Figure 1.3) from an instrumental understanding—knowing something by rote or without meaning (Skemp, 1978)—to a relational understanding—knowing what to do and why. Instrumental understanding, at the left end of the continuum, shows that ideas (e.g., concepts and procedures) are learned, but in isolation (or nearly so) to other ideas. Here you find ideas that have been memorized. Due to their isolation, poorly understood ideas are easily forgotten and are unlikely to be useful for constructing new ideas. At the right end of the continuum is relational understanding. Relational understanding means that each new concept or procedure (white dot) is not only learned, but is also connected to many existing ideas (gray dots), so there is a rich set of connections.

A primary goal of teaching for understanding is to help children develop a relational understanding of mathematical ideas. Because relational understanding develops over time and becomes more complex as a person makes more connections between ideas, teaching for this kind of understanding takes time and must be a goal of daily instruction.

Figure 1.3

Continuum of understanding.



Sociocultural Theory

Like constructivism, sociocultural theory not only positions the learner as actively engaged in seeking meaning during the learning process, but it also suggests that the learner can be assisted by working with others who are "more knowledgeable." Sociocultural theory proposes that learners have their own zone of proximal development, which is a range of knowledge that may be out of reach for individuals to learn on their own but is accessible if learners have the support of peers or more knowledgeable others (Vygotsky, 1978). For example, when young children are learning to measure length, they do not necessarily recognize the significance of placing measurement units end to end. As children measure objects, they may leave gaps between units or overlap units. A more knowledgeable person (a peer or teacher) can draw their attention to this critical idea in measurement.

The best learning for any given child will occur when the conversation of the classroom is within his or her zone of proximal development. Targeting that zone helps teachers provide children with the right amount of challenge, while avoiding boredom on the one hand and anxiety on the other when the challenge is beyond the child's current capability. Consequently, classroom discussions based on children's own ideas and solutions to problems are absolutely "foundational to children's learning" (Wood & Turner-Vorbeck, 2001, p. 186).

Teaching for Understanding

Teaching toward Relational Understanding

To explore the notion of understanding further, let's look into a learner-centered second-grade classroom. In learner-centered classrooms, teachers begin *where the children are*—with the *children's* ideas. Children are allowed to solve problems or to approach tasks in ways that make sense to them. They develop their understanding of mathematics because they are at the center of explaining, providing evidence or justification, finding or creating examples, generalizing, analyzing, making predictions, applying concepts, representing ideas in different ways, and articulating connections or relationships between the given topic and other ideas.

For example, in this second-grade classroom, the children have done numerous activities with the hundreds chart and an open number line. They have counted collections of objects and made many measurements of things in the room. In their counting and measuring, they often count groups of objects instead of counting by ones. Counting by tens has become a popular method for most but not all children. The class has taken big numbers apart in different ways to emphasize relationships between numbers and place value. In many of these activities, the children have used combinations of tens to make numbers. The children in the class have not been taught the typical procedures for addition or subtraction.

The teacher sets the following instructional objectives for the students:

1. Use number relationships (e.g., place-value ideas, such as 36 is 3 groups of 10 and 6 ones; 36 is 4 away from 40; etc.) to add two-digit numbers.
2. Apply flexible methods of addition.

As is often the case, this class begins with a story problem and the children set to work.

When Carla was at the zoo, she saw the monkeys eating bananas. She asked the zookeeper how many bananas the monkeys usually ate in one day. The zookeeper said that yesterday they ate 36 bananas but today they ate only 25 bananas. How many bananas did the monkeys eat in those two days?

Stop and Reflect

Before reading further, see how many different ways you can think of to solve this problem (36 + 25). Then check to see if your ways are alike or different from those that follow. ■

Some children use counters and count by ones. Some use the hundreds chart or base-ten models and others use mental strategies or an open number line. All are expected to use words and numbers and, if they wish, drawings to show what they did and how they thought about the problem. After about 20 minutes, the teacher begins a discussion by having children share their ideas. As the children report, the teacher records their ideas on the board so everyone can see them. Sometimes the teacher asks questions to help clarify ideas but makes no evaluative comments. The teacher asks the children who are listening if they understand or have any questions to ask the presenters. The following solution strategies are common in classrooms where children are regularly asked to generate their own approaches.

Avery: I know that 25 and 25 is 50—like two quarters. And 35 is ten more so that is 60. And then one more is 61.

Teacher: What do you mean when you say "35 is ten more"?

Avery: Well, I used 25 of the 36 and 25 and ten more is 35.

Sasha: I did 30 and 20 is 50 and then 6 + 5 more. Five and five is ten and so 6 + 5 is 11. And then 50 and 11 is 61.

Juan: I counted on using the hundreds chart. I started at 36 and then I had to go 20 from there and so that was 46 and then 56. And then I went five more: 57, 58, 59, 60, 61.

Marie: I used an open number to help me. I started at 36 and went up 4 to 40. Then I went up a jump of 20 and then one more to get to 61. (Figure 1.4)

Teacher: Where is the "25" in your strategy?

Marie: It's above the jumps. 4 + 20 + 1 is the same as 25.

Stop and Reflect

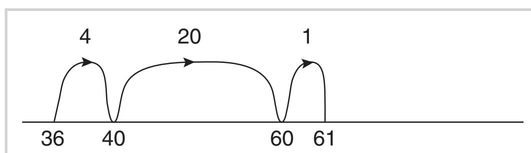
What ideas did you learn from those shared in this example? Try using some of these new ideas to find the sum of 64 and 27. ■

This vignette illustrates that when children are encouraged to solve a problem in their own way (using their own particular set of gray dots or ideas), they are able to make sense of their solution strategies and explain their reasoning. This is evidence of their development of mathematical proficiency.

During the discussion periods in classes such as this one, ideas continue to grow. The children may hear and immediately understand a clever strategy that they could have used but that did not occur to them. Others may begin to create new ideas to use that build from thinking about their classmates' strategies over multiple discussions. Some in the class may hear excellent ideas from their peers that do not make sense to them. These children are simply not ready or do not have the prerequisite concepts (gray dots) to understand these new ideas. On subsequent days there will be similar opportunities for all children to grow at their own pace based on their own understandings.

Figure 1.4

A child uses an open number line to solve 36 + 25 by starting at 36 and then adding 4, 20, and 1. The child wrote the numbers on the number line as the numbers move from 36 to 61.



Teaching toward Instrumental Understanding

In contrast to the lesson just described, in which children are developing concepts (understanding of place value) and procedures (ability to flexibly add) and seeing the relationship between these ideas, let's consider how a lesson with the same basic objective (addition using place-value concepts) might look if the focus is on instrumental understanding.

In this classroom, the teacher introduces only one way to solve multidigit addition problems—by modeling how to add numbers using base-ten materials. The teacher distributes base-ten blocks so that pairs of children have enough materials to solve any problem. The teacher reads to the class the same monkeys and bananas problem that was used earlier. The class quickly agrees that they need to add the two numbers in the problem. Using a projector to demonstrate, the teacher directs the children to make the two numbers on their place-value mats. Care is taken that the 25 is shown with the base-ten blocks beneath the base-ten blocks for 36. The children are directed to begin combining the pieces in the ones place. A series of questions guides them through each step in the standard algorithm.

1. How many ones are there all together?
2. What do we need to do with the 11 ones? (regroup, make a ten)
3. Where do we put the ten?
4. How many tens are there?
5. What is the answer?

Next, the children are given five similar problems to solve using the base-ten blocks. They work in pairs and record answers on their papers. The teacher circulates and helps anyone having difficulty by guiding them through the same steps indicated by the preceding questions.

In this lesson the teacher and children are using manipulatives to illustrate regrouping in addition problems. After engaging in several similar lessons, most children are likely to remember and possibly understand how to add with regrouping using the standard algorithm. Using manipulatives to illustrate why regrouping is needed does build a relational understanding, connecting place value to addition; however, because all children are instructed on one way to solve the problem, the lesson provides fewer opportunities to build connections between mathematical concepts. For example, students are not provided opportunities to use mental counting strategies, the hundreds chart, or the number line to add the numbers. Seeing that all of these methods work helps children build connections between mathematical ideas and across representations—fundamental characteristics of relational understanding. It is important to note that this lesson on the standard algorithm, in combination with other lessons that reinforce other approaches, *can* build a relational understanding, as it adds to children's repertoire of strategies. But if this lesson represents the sole approach to adding, then children are more likely to develop an instrumental understanding of mathematics.

The Importance of Children's Ideas

Let's take a minute to compare these two classrooms. By examining them more closely, you can see several important differences. These differences affect what is learned and who learns. Let's consider the first difference: Who determines the procedure to use?

In the first classroom, the children look at the numbers in the problem, think about the relationships between the numbers, and then *choose* a computational strategy that fits these ideas. They have developed several different strategies to solve addition problems by exploring numbers and various representations, such as the open number line and the hundreds chart. Consequently, they are relating addition to various representations and employing number relationships in their addition strategies (taking numbers apart and putting them together differently). The children in the first classroom are being taught mathematics for understanding—*relational* understanding—and are developing the kinds of mathematical proficiency described earlier.

In the second classroom, the teacher provides one strategy for how to add—the standard algorithm. Although the standard algorithm is a valid strategy, the entire focus of the lesson is on the steps and procedures that the teacher has outlined. The teacher solicits no ideas from individual children about how to combine the numbers and instead is only able to find out who has and who has not been able to follow directions.

When children have more choice in determining which strategies to use, as in the first classroom, they can learn more content and make more connections. In addition, if teachers do not seek out and value children's ideas, children may come to believe that mathematics is a body of rules and procedures that are learned by waiting for the teacher to tell them what to do. This view of mathematics—and what is involved in learning it—is inconsistent with mathematics as a discipline and with the learning theories described previously. Therefore, it is a worthwhile goal to transform your classroom into a mathematical community of learners who interact with each other and with the teacher as they share ideas and results, compare and evaluate strategies, challenge results, determine the validity of answers, and negotiate ideas. The rich interaction in such a classroom increases opportunities for productive engagement and reflective thinking about relevant mathematical ideas, resulting in children developing a relational understanding of mathematics.

A second difference between the two classrooms is the learning goals. Both teachers might write "understand two-digit addition" as the objective for the day. However, what is captured in "understand" is very different in each setting. In the first classroom, the teacher's goals are for children to connect addition to what they already know and to see that two numbers can be combined in many different ways. In the second classroom, understanding is connected to being able to carry out the standard algorithm. The learning goals, and more specifically, how the teacher interprets the meaning behind the learning goals, affect what children learned.

These lessons also differ in terms of how accessible they are—and this, in turn, affects who learns the mathematics. The first lesson is differentiated in that it meets children where they are in their current understanding. When a task is presented as "solve this in your own way," it has multiple entry points, meaning it can be approached in a variety of ways. Consequently, children with different prior knowledge or learning strategies can figure out a way to solve the problem. This makes the task accessible to more learners. Then, as children observe strategies that are more efficient than their own, they develop new and better ways to solve the problem.

In the second classroom, everyone has to do the problem in the same way. Children do not have the opportunity to apply their own ideas or to see that there are numerous ways to solve the problem. This may deprive children who need to continue working on the development of basic ideas of tens and ones as well as children who could easily find one or more ways to do the problem if only they were asked to do so. The children in the second classroom are also likely to use the same method to add all numbers instead of looking for more efficient ways to add based on the relationships between numbers. For example, they are likely to add 29 + 29 using the standard algorithm instead of thinking 30 + 30 and then take away 2. Recall in the discussion of learning theory the importance of building on prior knowledge and learning from others. Student-generated strategies, multiple approaches, and discussion about the problem in the first classroom represent the kinds of strategies that enhance learning for a range of learners.

Children in both classrooms will eventually succeed at finding sums, but what they learn about addition—and about doing mathematics—is quite different. Understanding and doing mathematics involves generating strategies for solving problems, applying those approaches, seeing if they lead to solutions, and checking to see whether answers make sense. These activities were all present in the first classroom but not in the second. Consequently, children in the first classroom, in addition to successfully finding sums, will develop richer mathematical understanding, become more flexible thinkers and better problem solvers, remain more engaged in learning, and develop more positive attitudes toward learning mathematics.

Mathematics Classrooms That Promote Understanding

Three of the most common types of teaching are direct instruction, facilitative methods (also called a *constructivist approach*), and coaching (Wiggins & McTighe, 2005). With direct instruction, the teacher usually demonstrates or models, lectures, and asks questions that are convergent or closed-ended in nature. With facilitative methods, the teacher might use investigations and inquiry, cooperative learning, discussion, and questions that are more open-ended. In coaching, the teacher provides children with guided practice and feedback that highlight ways to improve their performances.

You might be wondering which type of teaching is most appropriate if the goal is to teach mathematics for understanding. Unfortunately, there is no definitive answer because there are times when it is appropriate to engage in each of these types of teaching, depending on your instructional goals, the learners, and the situation. Some people believe that all direct instruction is ineffective because it ignores the learner's ideas and removes the productive struggle or opportunity to learn. This is not necessarily true. A teacher who is striving to teach for understanding can share information via direct instruction as long as that information does not remove the need for children to reflect on and productively struggle with the situation at hand. In other words, regardless of instructional design, the teacher should not be doing the thinking, reasoning, and connection building—it must be the children who are engaged in these activities.

Regarding facilitative or constructivist methods, remember that constructivism is a theory of learning, not a theory of teaching. Constructivism helps explain how children learn—by developing and modifying ideas (schemas) and by making connections between these ideas. Children can learn as a result of different kinds of instruction. The instructional approach chosen should depend on the ideas and relationships children have already constructed. Sometimes children readily make connections by listening to a lecture (direct instruction). Sometimes they need time to investigate a situation so they can become aware of the different ideas at play and how those ideas relate to one another (facilitative). Sometimes they need to practice a skill and receive feedback on their performance to become more accurate (coaching). No matter which type of teaching is used, constructivism and sociocultural theories remind us as teachers to continually wonder whether our children have truly developed the given concept or skill, connecting it to what they already know. By shedding light on what and how our children understand, assessment can help us determine which teaching approach may be the most appropriate at a given time.

The essence of developing relational understanding is to keep the children's ideas at the forefront of classroom activities by emphasizing the process standards, mathematical proficiencies, and the Standards for Mathematical Practice. This requires that the teacher create a classroom culture in which children can learn from one another. Consider the following features of a mathematics classroom that promote understanding (Chapin, O'Conner, & Anderson, 2009; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997; Hoffman, Breyfogle, & Dressler, 2009). In particular, notice who is doing the thinking, the talking, and the mathematics—the children.

- ***Children's ideas are key.*** Mathematical ideas expressed by children are important and have the potential to contribute to everyone's learning. Learning mathematics is about coming to understand the ideas of the mathematical community.
- ***Opportunities for children to talk about mathematics are common.*** Learning is enhanced when children are engaged with others who are working on the same ideas. Encouraging student-to-student dialogue can help children think of themselves as capable of making sense of mathematics. Children are also more likely to question each other's ideas than the teacher's ideas.
- ***Multiple approaches are encouraged.*** Children must recognize that there is often a variety of methods that will lead to a solution. Respect for the ideas shared by others is critical if real discussion is to take place.
- ***Mistakes are good opportunities for learning.*** Children must come to realize that errors provide opportunities for growth as they are uncovered and explained. Trust must be established with an understanding that it is okay to make mistakes. Without this trust, many ideas will never be shared.
- ***Math makes sense.*** Children must come to understand that mathematics makes sense. Teachers should resist always evaluating children's answers. In fact, when teachers routinely respond with "Yes, that's correct," or "No, that's wrong," children will stop trying to make sense of ideas in the classroom and discussion and learning will be curtailed.

Teaching Tip

Listen carefully to children as they talk about what they are thinking and doing as they engage in a mathematical task. If they respond in an unexpected way, try to avoid imposing *your* ideas onto their ideas. Ask clarifying questions to try to make sense of the sense your children are making!

To create a climate that encourages mathematics understanding, teachers must first provide explicit instruction on the ground rules for classroom discussions. Second, teachers may need to model the type of questioning and interaction that they expect from their children. Direct instruction would be appropriate in such a situation. The crucial point in teaching for understanding is to highlight and use children's ideas to promote mathematical proficiency.

Most people go into teaching because they want to help children learn. It is hard to think of allowing—much less planning for—the children in your classroom to struggle. Not showing them a solution when they are experiencing difficulty seems almost counterintuitive. If our goal is relational understanding, however, the struggle is part of the learning, and teaching becomes less about the teacher and more about what the children are doing and thinking.

Keep in mind that you too are a learner. Some ideas in this book may make more sense to you than others. Others may even create dissonance for you. Embrace this feeling of disequilibrium and unease as an opportunity to learn—to revise your perspectives on mathematics and on the teaching and learning of mathematics as you deepen your understanding so that you can help your children deepen theirs.

Stop and Reflect

Look back at the chapter and identify any ideas that make you uncomfortable or that challenge your current thinking about mathematics or about teaching and learning mathematics. Try to determine why these ideas challenge you or make you uncomfortable. Write these ideas down and revisit them later as you read and reflect further. ■