ON THE ASSUMPTION OF RIEMANN[1]

Marcel Riesz

STOCKHOLM

1°. In this Note, I propose to represent the function $\frac{\Gamma(1-\frac{s}{2})}{\zeta(s)}$ by an integral of the form

$$\int_0^\infty F(x)x^{-\frac{s}{2}+1}dx.$$

$F(x)$ denotes the integral function given explicitly. As one knows, the assumption of Riemann consists in the statement that the function $\zeta(s)$ does not have zeros in the half-plane $\Re(s) > \frac{1}{2}$. Our representation will provide a condition necessary and sufficient for the validity of this assumption of Riemann. I cannot decide yet if this condition will facilitate the checking of the assumption.

2°. Let us form the quantities

$$C_k = \sum_{n=1}^{\infty} n^{-2k} \qquad (k = 1, 2, \ldots)$$

and the integral function

$$F(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{C_k\Gamma(k)}.$$

According to the calculation of the residues, there is

$$F(x) = \frac{1}{2}\int_{a-i\infty}^{a+i\infty} \frac{x^z}{\zeta(2z)\Gamma(z)\sin(\pi z)}dz, \qquad \left(\frac{1}{2} \leq a \leq 1\right)^2 \qquad (1)$$

the integral being absolutely and uniformly convergent for

$$-\frac{\pi}{2} + \delta \leq \text{Arg } x \leq \frac{\pi}{2} - \delta \qquad (2)$$

[1] Acta mathematica. **40**. Printed on December 20, 1916.

[2] With regard to the order of magnitude of $1/\zeta(s)$ cf. Landau: Handbuch der Lehre der Verteilung der Primzahlen (Teubner, 1909) T. I p. 177-178.

It results from it immediately, while putting $a = \frac{1}{2}$, that one has uniformly

$$\lim_{x\to\infty} \frac{F(x)}{x^{\frac{1}{2}}} = 0$$

or by adopting a notation due Mr. LANDAU[3]

$$F(x) = o(x^{1/2}) \tag{3}$$

x tending towards the infinity inside an angle of the form (2).

3°. Let us form now the integral

$$\phi(s) = \int_0^\infty x^{-(\frac{s}{2}+1)} F(x) dx \tag{4}$$

the path of integration being the positive real axis. Under the terms of (3), this integral converges absolutely and uniformly in any band of the form

$$z + \delta \leq \Re(s) \leq 2 - \delta \qquad (\delta > 0),$$

and it thus represents an analytical function there. In addition, one can write

$$n^{-s}\phi(s) = \int_0^\infty x^{-(\frac{s}{2}+1)} F\left(\frac{x}{n^2}\right) dx$$

$$\zeta(s)\phi(s) = \sum_{n=1}^{\infty} \int_0^\infty x^{-(\frac{s}{2}+1)} F\left(\frac{x}{n^2}\right) dx$$

It is legitimate to invert the order of the summation and integration. That is obvious for the integral taken between 0 and 1, e.g. terms of the series being of about size n^{-2}. For the integral taken between 1 and ∞, the formula (1) gives us

$$\left|F\left(\frac{x}{n^2}\right)\right| < M\left(\frac{x}{n^2}\right)^a,$$

M indicating a positive number depending only on a. While choosing $\frac{1}{2} < a < \frac{\Re(s)}{2}$, one sees that the inversion in question is legitimate. However, the series

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n^2}\right)$$

being uniformly convergent in any limited domain and having consequently the value

$$xe^{-x}$$

[3] As usually, the notation $\Phi(x) = o(x^a)$ means $\frac{\Phi(x)}{x^a}$ tends towards zero and $\Phi(x) = \mathcal{O}(x^a)$ means that $\frac{\Phi(x)}{x^a}$ remain finite, when x tends to the infinity.

one has finally

$$\zeta(s)\phi(s) = \int_0^\infty x^{-(\frac{s}{2}+1)} x e^{-x} dx = \Gamma\left(1 - \frac{s}{2}\right)$$

i.e.

$$\frac{\Gamma(1-\frac{s}{2})}{\zeta(s)} = \int_0^\infty x^{-(\frac{s}{2}+1)} F(x) dx. \tag{5}$$

4°. If one had for x tending towards the infinity along the positive real axis,

$$F(x) = \mathcal{O}\left(x^{\frac{1}{4}+\delta}\right) \tag{6}$$

for any positive number δ, the integral (5) would converge absolutely and uniformly in any band of the form

$$\frac{1}{2} + \delta \leq \Re(s) \leq 2 - \delta.$$

Thus, with the help of the assumption (6), the function $\zeta(s)$ of Riemann does not have zeros in the half-plane $\Re(s) > \frac{1}{2}$.

In addition, by using a theorem of Mr. CARATHODORY, M. LANDAU had shown that with the help of the assumption of Riemann, one has

$$\frac{1}{\zeta(s)} = \mathcal{O}(|s|^r) \quad \text{for} \quad \Re(s) \geq \frac{1}{2} + \delta,$$ [4]

r denoting a positive number depending only on δ. Therefore, by admitting that the assumption of Riemann is true, one could put in (1) $a = \frac{1}{4} + \epsilon$, which would involve (6).

One concludes from it that (6) is *the necessary and sufficient condition for the validity of the assumption of Riemann.*

Let us notice that the formula (1) highlights that the relation (6) is fulfilled in any angle (2), as soon as it is on the positive real axis.

Conversely, one sees without pain, by combining (3) with a known theorem of Misters PHRAGMN and LINDELF, that (6) being fulfilled on some right-hand semi-plane (side) located in the angle (2), it will be valid on the positive real axis.

[4] LANDAU, *loc. cit* v. II p. 870. Mr. LITTLEWOOD showed that with the help of the same hypothesis, one has $\frac{1}{\zeta(s)} = \mathcal{O}(|s|^\epsilon))$, ϵ indicating an arbitrarily small positive number. (Some consequences of the assumption that the function $\zeta(s)$ of Riemann does not have zeros in half-plane $\Re(s) > 1/2$. Comptes Rendus, January 29, 1912.)

One could replace $F(x)$ by other functions of similar structure. It is seen that the method that we followed also makes it possible to express the reciprocal value of an unspecified function represented by a series of Dirichlet.

It is easy to check the identity

$$F(x) = \sum_{n=1}^{\infty} \mu(n) \frac{x}{n^2} e^{-\frac{x}{n^2}}$$

where $\mu(n)$ denotes the function of Mobius.

5°. We will add some remarks on the distribution of the zeros of the function $F(x)$. One has, for x real and positive, $\lim_{x\to\infty} \frac{e^{-x}F(-x)}{x} = -1$. In addition, for x arbitrary variable $\neq 0$, $F(x)$ satisfies an inequality $|F(x)| < |x|e^{|x|}$. I.E.(internal excitation) X I. From these two facts it results according to known theorems from POINCARE[5] and HADAMARD that $F(x)$ is of type (kind) one.

Now let us denote by γ_ν the zeros of $F(x)$, $(\nu = 1, 2, \ldots)$. From inequality above, one concludes immediately with the help of the theorem in question from Mr HADAMARD that (which?) the series $\sum_{\nu=1}^{\infty} \frac{1}{|\gamma_\nu|^{1+\epsilon}}$ is convergent, ϵ indicating a positive number arbitrarily small. We will show that the series $\sum_{\nu=1}^{\infty} \frac{1}{|\gamma_\nu|}$ is divergent.

Indeed, if it were convergent one would have according to the preceding facts and the quoted theorem of POINCARE $F(x) = xe^{-x}\prod_1^\infty \left(1 - \frac{x}{\gamma_\nu}\right)$. Let us observe now that the relation $F(x) = \mathcal{O}(x^{\frac{1}{4}-\delta})$ $(\delta > O)$, x tending towards the infinity on the positive real axis, is inadmissible. It would involve the obviously inaccurate result indeed that the function $\zeta(s)$ does not have the imaginary zeros. But if $F(x)$ were of the form above, it would result immediately from the known as theorem of POINCARE $F(x) = \mathcal{O}(e^{-x(1-\epsilon)}) = \mathcal{O}(x^{\frac{1}{4}-\delta})$. We thus showed that the series $\sum_{\nu=1}^{\infty} \frac{1}{|\gamma_\nu|}$ is divergent.

Thus we obtained for $F(x)$ the following canonical form:

$$F(x) = xe^{\gamma x} \prod_{\nu=1}^{\infty} \left(1 - \frac{x}{\gamma_\nu}\right) e^{\frac{x}{\gamma_\nu}}. \qquad (7)$$

The number γ is obviously real there.

We now will show that $F(x)$ has an infinity of imaginary zeros. This will result from an important theorem of Mr. POLYA which it had the friendship to communicate to me during a maintenance has Göttingen the summer 1913.

[5] Of the convergence of $\sum_{\nu=1}^{\infty} \frac{1}{|\gamma_\nu|}$ it follows $\left|\prod_1^\infty \left(1 - \frac{x}{\gamma_\nu}\right)\right| = \mathcal{O}(e^{\epsilon|x|})$, ϵ denoting arbitrarily small and positive number.

[6]

The theorem in question can be stated in the following way: *Let us suppose that the integral function* $\Phi(x) = \sum_{n=1}^{\infty} a_n x^n$ *admits a representation of the form*

$$\Phi(x) = cx^r e^{-\gamma x^2+\delta x} \prod_{\mu=1}^{\infty}(1+\beta_\mu x)(1+\overline{\beta_\mu} x) \prod_{\nu=1}^{\infty}(1+\delta_\nu x)e^{-\delta_\nu x}, \qquad (8)$$

where γ_ν *are real* $\geq 0, \beta_\mu$ *and* $\overline{\beta_\mu}$ *indicating combined imaginary quantities,* c, δ *and* δ_ν *are the real quantities, and* $\sum_{\nu=1}^{\infty} \delta_\nu^2$ *being convergent. Then, so that the function* $\Phi(x)$ *does not have at the same time a representation of the form*

$$\Phi(x) = cx^r e^{\gamma x} \prod_{\mu=1}^{\infty}(1+\beta_\mu x)(1+\overline{\beta_\mu} x) \prod_{\nu=1}^{\infty}(1+\delta_\nu x), \qquad (9)$$

γ *and all the* δ_ν *being real numbers of the same sign and the product* $\prod_{\nu=1}^{\infty}(1+\delta_\nu x)$ *being absolutely convergent, it is necessary and it is enough that each of the two sequences* $a_0, a_1, \ldots, a_n, \ldots$ *and* $a_0, -a_l \ldots, (-1)^n a_n, \ldots$ *have an infinity of changes of sign.*

With regard to our function $F(x)$, it is manifest, according to what we have just shown, that it cannot admit a representation of the form (9). In addition, it results from the formula (7) which if the function $F(x)$ had only finite number of imaginary zeros, it would admit a representation of the form (8). The sequence $\frac{1}{C_1\Gamma(1)}, \frac{1}{C_1\Gamma(1)}, \ldots$ not presenting any change of sign, it would follow according to the theorem of Mr. POLYA that $F(x)$ admits at the same time a representation of the form (9). This being impossible, it is established that $F(x)$ has really an infinity of imaginary zeros.

In addition, from the relation

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n^2}\right) = xe^{-x}$$

and the inadmissibility of the relation $F(x) = \mathcal{O}(x^{\frac{1}{4}-\delta})$, it results immediately that $F(x)$ changes sign at least once on the positive real axis.

Translated by Marek Wolf, July 2009

[6] the theorem is established in the following work of Mr. POLYA, published during the impression of work present: Algebraische Untersuchungen ber ganze Funktionen vom Null Geschlechte und Eins. J. F. Maths (145) 1915; cf. p. 247-249.