



Identities inspired by Ramanujan Notebooks (part 2)

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These are new findings related to the ones found in 1998. I had the idea that maybe the sums in the first note I wrote were looking in the bad direction. They are good yes but maybe there is better. Actually there is, instead of considering powers of $\exp(2\pi)$ I should consider powers of $\exp(k\pi)$.

After a few experiments with Maple (computer algebra system) and PSLQ (integer relations algorithm) here are the findings and remarks.

- I could go up to Zeta(39) and powers of $\exp(30\pi)$ after that the numbers are too close together for any computation, $1/\exp(30\pi)$ is of the order of 10^{-40} and even with 2000 decimal digits I get an error.

- The powers of $\exp(k\pi) \pm 1$ are related to the divisors of a number.

- The numbers that appear to be expressible are so far : $\log(A)$ where A is an integer or simple algebraic like $\operatorname{arccosh}(1/2)$ or $\operatorname{arctanh}(2/3)$, $\zeta(2n+1)$, π^{2n+1} , $\zeta(2n+1)$ and some formulas involving $\ln\Gamma(1/4)$ and $\ln(\pi)$ but not separately.

Formulas for odd powers of π and $\zeta(2n+1)$

$$\zeta(5) = \frac{694}{204813} \pi^5 - \frac{6280}{3251} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{4\pi n} - 1)} + \frac{296}{3251} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{5\pi n} - 1)} - \frac{1073}{6502} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{10\pi n} - 1)} + \frac{37}{6502} \sum_{n=1}^{\infty} \frac{1}{n^5 (e^{20\pi n} - 1)}$$

That one is the fastest converging series so far with 5.45 digits for each n.

$$\zeta(3) = \frac{13\pi^3\sqrt{3}}{45} + 2 \sum_{n=1}^{\infty} \frac{1}{n^3 (e^{\sqrt{3}\pi n} - 1)} - \frac{9}{2} \sum_{n=1}^{\infty} \frac{1}{n^3 (e^{2\sqrt{3}\pi n} - 1)} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^3 (e^{4\sqrt{3}\pi n} - 1)}$$

$$\begin{aligned}
\zeta(5) &= \frac{11\pi^5\sqrt{3}}{5670} + 2\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\sqrt{3}\pi n} - 1)} - \frac{33}{8}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\sqrt{3}\pi n} - 1)} + \frac{1}{8}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\sqrt{3}\pi n} - 1)} \\
\zeta(5) &= \frac{5\pi^5\sqrt{7}}{3906} + \frac{64}{31}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\sqrt{7}\pi n} - 1)} + \frac{130}{31}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\sqrt{28}\pi n} - 1)} - \frac{4}{31}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\sqrt{112}\pi n} - 1)} \\
\pi &= 72\sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} - 96\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)} + 24\sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n} - 1)} \\
\pi^3 &= 720\sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n} - 1)} - 900\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)} + 180\sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n} - 1)} \\
\pi^5 &= 7056\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n} - 1)} - 6993\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} - 1)} + 63\sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n} - 1)} \\
\pi^7 &= \frac{907200}{13}\sum_{n=1}^{\infty} \frac{1}{n^7(e^{\pi n} - 1)} - 70875\sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n} - 1)} + \frac{14175}{13}\sum_{n=1}^{\infty} \frac{1}{n^7(e^{4\pi n} - 1)} \\
\pi^9 &= \frac{28226880}{41}\sum_{n=1}^{\infty} \frac{1}{n^9(e^{\pi n} - 1)} - \frac{112920885}{164}\sum_{n=1}^{\infty} \frac{1}{n^9(e^{2\pi n} - 1)} + \frac{13365}{164}\sum_{n=1}^{\infty} \frac{1}{n^9(e^{4\pi n} - 1)} \\
\pi^{11} &= \frac{27243216000}{4009}\sum_{n=1}^{\infty} \frac{1}{n^{11}(e^{\pi n} - 1)} - \frac{218158565625}{32072}\sum_{n=1}^{\infty} \frac{1}{n^{11}(e^{2\pi n} - 1)} + \frac{212837625}{32072}\sum_{n=1}^{\infty} \frac{1}{n^{11}(e^{4\pi n} - 1)} \\
\pi^{13} &= \frac{159687158400}{2381}\sum_{n=1}^{\infty} \frac{1}{n^{13}(e^{\pi n} - 1)} - \frac{5109979245525}{76192}\sum_{n=1}^{\infty} \frac{1}{n^{13}(e^{2\pi n} - 1)} + \frac{9823275}{76192}\sum_{n=1}^{\infty} \frac{1}{n^{13}(e^{4\pi n} - 1)} \\
\pi^{15} &= \frac{9094280832000}{13739}\sum_{n=1}^{\infty} \frac{1}{n^{15}(e^{\pi n} - 1)} - \frac{145517374445625}{219824}\sum_{n=1}^{\infty} \frac{1}{n^{15}(e^{2\pi n} - 1)} \\
&+ \frac{8881133625}{219824}\sum_{n=1}^{\infty} \frac{1}{n^{15}(e^{4\pi n} - 1)} \\
\pi^{17} &= \frac{19118853097344000}{2926507}\sum_{n=1}^{\infty} \frac{1}{n^{17}(e^{\pi n} - 1)} - \frac{1223606634625249875}{187296448}\sum_{n=1}^{\infty} \frac{1}{n^{17}(e^{2\pi n} - 1)} \\
&+ \frac{36395233875}{187296448}\sum_{n=1}^{\infty} \frac{1}{n^{17}(e^{4\pi n} - 1)} \\
\pi^{19} &= \frac{1688703016492800000}{26190337}\sum_{n=1}^{\infty} \frac{1}{n^{19}(e^{\pi n} - 1)} - \frac{216154810673098171875}{3352363136}\sum_{n=1}^{\infty} \frac{1}{n^{19}(e^{2\pi n} - 1)} \\
&+ \frac{824562019771875}{3352363136}\sum_{n=1}^{\infty} \frac{1}{n^{19}(e^{4\pi n} - 1)} \\
\zeta(3) &= 28\sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n} - 1)} - 37\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)} + 7\sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n} - 1)} \\
\zeta(5) &= 24\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n} - 1)} - \frac{259}{10}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} - 1)} - \frac{1}{10}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n} - 1)} \\
\zeta(7) &= \frac{304}{13}\sum_{n=1}^{\infty} \frac{1}{n^7(e^{\pi n} - 1)} - \frac{103}{4}\sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n} - 1)} + \frac{19}{52}\sum_{n=1}^{\infty} \frac{1}{n^7(e^{4\pi n} - 1)} \\
\zeta(9) &= \frac{20000}{861}\sum_{n=1}^{\infty} \frac{1}{n^9(e^{\pi n} - 1)} - \frac{1617613}{1640}\sum_{n=1}^{\infty} \frac{1}{n^9(e^{2\pi n} - 1)} + \frac{373}{34440}\sum_{n=1}^{\infty} \frac{1}{n^9(e^{4\pi n} - 1)} \\
\zeta(11) &= \frac{92992}{4009}\sum_{n=1}^{\infty} \frac{1}{n^{11}(e^{\pi n} - 1)} - \frac{1617613}{64144}\sum_{n=1}^{\infty} \frac{1}{n^{11}(e^{2\pi n} - 1)} + \frac{1453}{64144}\sum_{n=1}^{\infty} \frac{1}{n^{11}(e^{4\pi n} - 1)}
\end{aligned}$$

$$\begin{aligned} \zeta(13) &= \frac{717696}{30953} \sum_{n=1}^{\infty} \frac{1}{n^{13}(e^{\pi n} - 1)} - \frac{124738499}{4952480} \sum_{n=1}^{\infty} \frac{1}{n^{13}(e^{2\pi n} - 1)} + \frac{2179}{4952480} \sum_{n=1}^{\infty} \frac{1}{n^{13}(e^{4\pi n} - 1)} \\ \zeta(15) &= \frac{3503872}{151129} \sum_{n=1}^{\infty} \frac{1}{n^{15}(e^{\pi n} - 1)} - \frac{243606007}{9672256} \sum_{n=1}^{\infty} \frac{1}{n^{15}(e^{2\pi n} - 1)} + \frac{13687}{9672256} \sum_{n=1}^{\infty} \frac{1}{n^{15}(e^{4\pi n} - 1)} \\ \zeta(17) &= \frac{203545088}{8779521} \sum_{n=1}^{\infty} \frac{1}{n^{17}(e^{\pi n} - 1)} - \frac{801871973691}{31840396160} \sum_{n=1}^{\infty} \frac{1}{n^{17}(e^{2\pi n} - 1)} + \frac{2986673}{95521188480} \sum_{n=1}^{\infty} \frac{1}{n^{17}(e^{4\pi n} - 1)} \\ \zeta(19) &= \frac{7893541888}{340474381} \sum_{n=1}^{\infty} \frac{1}{n^{19}(e^{\pi n} - 1)} - \frac{168852101149}{6704726272} \sum_{n=1}^{\infty} \frac{1}{n^{19}(e^{2\pi n} - 1)} + \frac{7708537}{87161441536} \sum_{n=1}^{\infty} \frac{1}{n^{19}(e^{4\pi n} - 1)} \end{aligned}$$

Other formulas

$$\begin{aligned} \log(2) &= 10 \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} + 1)} + 6 \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} - 4 \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)} \\ \log(3) &= -9 \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} - \frac{49}{3} \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} + 1)} + \frac{14}{3} \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} + 1)} \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{n(e^{3\pi n} - 1)} + \frac{7}{3} \sum_{n=1}^{\infty} \frac{1}{n(e^{3\pi n} + 1)} - \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n(e^{6\pi n} + 1)} \\ \log(5) &= \frac{57}{4} \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} - 1)} + \frac{91}{4} \sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n} + 1)} - \frac{13}{2} \sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} + 1)} \\ &\quad + \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n(e^{5\pi n} - 1)} - \frac{7}{4} \sum_{n=1}^{\infty} \frac{1}{n(e^{5\pi n} + 1)} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(e^{10\pi n} + 1)} \\ \zeta(9) &= \frac{64}{3} \sum_{n=1}^{\infty} \frac{1}{n^9(e^{\pi n} - 1)} + \frac{441}{20} \sum_{n=1}^{\infty} \frac{1}{n^9(e^{2\pi n} - 1)} - 32 \sum_{n=1}^{\infty} \frac{1}{n^9(e^{3\pi n} - 1)} \\ &\quad - \frac{4763}{60} \sum_{n=1}^{\infty} \frac{1}{n^9(e^{4\pi n} - 1)} + \frac{529}{8} \sum_{n=1}^{\infty} \frac{1}{n^9(e^{6\pi n} - 1)} - \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n^9(e^{12\pi n} - 1)} \\ \zeta(5) &= \frac{-149}{43983} \pi^5 + \frac{785}{4344} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{12\pi n} - 1)} - \frac{22765}{4344} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{6\pi n} - 1)} + \frac{1570}{543} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{3\pi n} - 1)} \\ &\quad - \frac{61}{4344} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{20\pi n} - 1)} + \frac{1769}{4344} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{10\pi n} - 1)} - \frac{122}{543} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{5\pi n} - 1)} \end{aligned}$$

The last formula for Zeta(5) is interesting since the lowest term gives a rate of convergence of the order of $1/\exp(3\pi)$ which is 4.09 decimal digits.

Here is a table of coefficients :

Results with Zeta(z), Pi^z, S(z,1), S(z,2), S(z,4)
where S(k,m) is

$$\begin{array}{l} \text{infinity} \\ \text{-----} \\ \backslash \\) \\ / \\ \text{-----} \\ n = 1 \end{array} \frac{1}{n^k (\exp(m \text{ Pi } n) - 1)} = S(k,m).$$

The display is the integer relation that
make the vector [Zeta(z), Pi^z, S(z,1), S(z,2), S(z,4)] equal to 0.

3, [1, 0, -28, 37, -7]
5, [-10, 0, 240, -259, -1]
7, [-52, 0, 1216, -1339, 19]
9, [27930, -1, 39680, 16401, -221]
11, [-538116, 2, -1108992, 33843, -1083]
13, [-4616430, 2, -27095040, 17859891, 2289]
15, [52761192, -2, 100614144, 4902099, 6141]
17, [-2381720400, 8, 2954035200, -7717548915, 72915]
19, [39005722340, -14, -1613496320, 79624947155, -6155]
21, [-221481750994700, 6732, 850750073733120, -1293713997563111, 421840591]
23, [1226556695374, 9, -84963147907072, 87416281554613, -20256793]
25, [5203911059453925, -1921, -1567010009907200, 11974832749256745, -620441695]
27, [8148451902123610, -306, -1701430826106880, 17998334655707395, -25353295]
29, [-13964212268709884725, -140282, 1170802376844509184000, -1198730801485834158745,
103905205295]
31, [-54171522182777606620, 20123, 56675100822926786560, -165018145241264800615, 52782800815]
33, [-58909621176539933053170, 2545893, -131688610996976101621760, 13869368616464330113914,
27431905401506]

Results with Zeta(z), S(z,1), S(z,2), S(z,4).

The display is the integer relation that
make the vector [Zeta(z), S(z,1), S(z,2), S(z,4)] equal to 0.

3, [1, -28, 37, -7]
5, [-10, 240, -259, -1]
7, [-52, 1216, -1339, 19]
9, [34440, -800000, 869253, -373]
11, [-64144, 1487872, -1617613, 1453]
13, [-4952480, 114831360, -124738499, 2179]
15, [-9672256, 224247808, -243606007, 13687]
17, [-95521188480, 2214570557440, -2405615921073, 2986673]
19, [87161441536, -2020746723328, 2195077314937, -7708537]
21, [-511417392598515200, 11856658470320209920, -12879494224969023401, 969451783081]
23, [-809054768954368, 18757052482453504, -20375166492392041, 4472029801]
25, [-9402626177680619520, 217989612878618951680, -236794866356583343203, 1122603152483]
27, [-433559729330180096, 10051608326929645568, -10918727935370641697, 149780635937]
29, [70591765772239009938104320, -1636592903864970323479756800, 1777776435935195869731249349, -
525747526375283909]
31, [-520167571795410350915584, 12059516381224761076744192, -13099851536046881623355143,
11231299844779783]
33, [-10652244887519367873408902922240, 246960650058691978789414796001280, -
268265139838691529135407355836517, 4960814599174753990757]
35, [-2732942456837303169145438208, 63360282530610722654381932544, -68826167447973382833596090501,
3688053840923281541]
37, [-1424192973610401074580480646493634560, 33018356811066418402946396259121889280, -
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39, [31536188043819957788458947444736, -731132036585425088557545661923328,
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41, [12773377697620891851198465395561679146188800, -296136794882949433788107046141906375032176640,
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43, [233097371723978108279933298059247616, -5404107683343089621820512227764994048,
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45, [140600817767916545581693183361009315526861326909440, -
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47, [2254919587126878277155357568098096977543168, -52277844988066489224430209234308641738194944,
56787684162320988691600873634610990609888551, -742912859949264106154916607271]
49, [284524049801951474153643015128700095967174408724807680, -
6596378982132770562246518166744835627859463792649830400,
7165427081736675532219530224305149732457982689130518611, -
2021665726027302913912664170079031072851]
51, [1221343551464851306588720257503511850254336, -283155148492348076155312278372927288186503168,
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53, [130137454758409843860165531701040966161594385248587486330880, -
3017094589909565702616435024906390294964042473906645824962560,
3277369499426385448129400121915923378120436526114126848209189, -
57792634033607451150833205281710306050584869]
55, [-19787331441001612103854548589972560860344628543488,
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57, [-8229610565782363379309230843178332823243893220736032646620365455360,
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228417583703134726518153542940959974324117920332221]
59, [31760943205719479335823285328215456670357182181431056531456, -
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61, [11228750712595691857993899010018102022037871254939583265845661500073574400, -
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63, [-4420734755976561477819352219602229305413295023063050334765056,
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111331370981461787011628115392902787866895877725893259127542497,
22223954766213317384532039590736747648635617]
65, [-8683784727375452650614759128937062365289209832692753414899240041785839908290560,
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67, [-14974422008672519045548749269274479895709052562995131254761848832,
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377114444522290949135710157102120959683567879802427217434372529149,
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69, [-185314968362017705213459062217245680311133697058676732436818759208189611338841184010240,
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71, [61765379767975943763564298671677357518170611735643783882563340122256733896704, -
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73, [249380990069788456384931798640644120554751176319643422932932502097890221096955660601267322880,
-578162556938456821207869671616593770935223736916678380623777686489311432178114313091557621760,
628038754952414512484866593040473963109200116123894385538012001016336947709809541332345736249, -
10561695751368063026026567929159048369310331245073337783907038253468729]
75, [-390204487785772245552169373144049299002675650800864945178530088720281894912,
9046464380622959539308570407853288465993852509926942216097125384641855881216, -
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478915836659382129612763840358992819232085424547810249]
77,
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2720, -
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9760,
83205188948335899983821539249624415965891398580506684694869993499347264129163401786190661989396450
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085822873600, -
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Results with Pi^z , $S(z,1)$, $S(z,2)$, $S(z,4)$.
The display is the integer relation that
make the vector $[\text{Pi}^z, S(z,1), S(z,2), S(z,4)]$ equal to 0.

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