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"What Fermented in Me for Years": Cantor's Discovery of Transfinite Numbers

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Transfinite (ordinal) numbers were a crucial step in the development of Cantor's set theory. The new concept depended in various ways on previous problems and results, and especially on two questions that were at the center of Cantor's attention in September 1882, when he was about to make his discovery. First, the proof of the Cantor–Bendixson theorem motivated the introduction of transfinite numbers, and at the same time suggested the "principle of limitation," which is the key to the connection between transfinite numbers and infinite powers. Second, Dedekind's ideas, which Cantor discussed in September 1882, seem to have played an important heuristic role in his decision to consider the "symbols of infinity" that he had been using as true *numbers*, i.e., as autonomous objects; to this end Cantor introduced in his work, for the first time, ideas on (well) ordered sets. © 1995 Academic Press, Inc.

Los números (ordinales) transfinitos constituyeron un paso clave en el desarrollo de la teoría de conjuntos de Cantor. La nueva idea dependió en varias formas de resultados y problemas previos, pero especialmente de dos cuestiones que ocuparon a Cantor en septiembre de 1882, estando a punto de realizar su descubrimiento. En primer lugar, el teorema de Cantor–Bendixson, cuya demostración motivó la introducción de los números transfinitos, y a la vez sugirió el "principio de limitación" que constituye la clave de la conexión entre números transfinitos y potencias infinitas. En segundo lugar, las ideas de Dedekind, que Cantor discutió en septiembre de 1882, parecen haber desempeñado un importante papel heurístico en la decisión de considerar los "símbolos de infinitud" que Cantor venía empleando como verdaderos *números*, esto es, como objetos autónomos; para ello, Cantor introdujo en su obra por vez primera consideraciones sobre conjuntos (bien) ordenados. © 1995 Academic Press, Inc.

Die transfiniten (Ordnungs)Zahlen stellten einen wesentlichen Schritt in der Entwicklung der Cantorschen Mengenlehre dar. Der neue Begriff war von verschiedenen früheren Problemstellungen und Resultaten abhängig, darunter besonders zwei Themen, die im Oktober 1882 im Mittelpunkt von Cantors Interesse lagen, als er auf die neue Idee kam. Erstens motivierte der Beweis des Cantor–Bendixsonschen Satzes die Einführung der transfiniten Zahlen und legte gleichzeitig das sogenannte Hemmungsprinzip nahe, welches die Verbindung zwischen transfiniten Zahlen und unendliche Mächtigkeiten herstellte. Zweitens scheinen Dedekinds Ideen, welche Cantor im September 1882 kennenlernte, eine wichtige heuristische Rolle bei dem Schritt gespielt zu haben, Cantors "Unendlichkeitssymbole" als wirkliche *Zahlen*, d.h. als selbständige Objekte, zu behandeln. Cantor sah sich dadurch veranlasst, zum ersten Mal Ideen über (wohl)geordneten Mengen in seine Arbeiten einzuführen. © 1995 Academic Press, Inc.

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Although Cantor's transfinite numbers have been discussed by several historians (e.g., [10, 11, 18, 19, 25]), two aspects of their discovery have not received the attention they deserve.

First, it can be shown that the context of Cantor's work at the time had an intimate connection with his discovery. By late 1882, Cantor was trying to prove a part of the Cantor–Bendixson theorem, a part that I will call *Cantor's theorem*. The proof of this theorem motivated the introduction of transfinite numbers, and, most importantly, it led directly to a restricted version of the principle of limitation, which was crucial to the new theory. (The principle of limitation defines "natural sections" of the unlimited series of transfinite numbers, the number-classes; these classes became central to Cantor's theory of infinite cardinalities [5, 166–167, 196–197].)

Second, the introduction of transfinite numbers depended on a reconceptualization of Cantor's previous ideas in *ordinal* terms. Here, Cantor's interaction with Dedekind during September 1882 seems to have played an important heuristic role.

1. CANTOR'S THEOREM

The 1883 *Grundlagen,* where Cantor presented his new numbers, was part of a series of papers that he published in *Mathematische Annalen* between 1879 and 1884, under the title "Über unendliche, lineare Punktmannichfaltigkeiten" [On infinite, linear point-sets]. Most of the results in them deal with the powers of point-sets (subsets of $\mathbb{R}$ or $\mathbb{R}^n$) studied through an analysis of properties of their derived sets; quite clearly, Cantor's aim was to find a way to prove the continuum hypothesis (but, in spite of some progress, the series had to end with a new announcement of a proof never found [6, 244; cf. 24]). Among those results, one can be said to have structured Cantor's work from 1882 to 1884: the well-known Cantor–Bendixson theorem, first announced in a paper dated 1 September 1882 [4].

Derived sets had been introduced by Cantor in 1872, and they led him to "symbols of infinity" that he would later recognize as transfinite numbers.[1] The first derived set of a point-set P, denoted P', is the set of all limit points of P [2, 98]. Since P' will also, in general, have limit points, Cantor considered iterations of the derivation procedure: P^n is the set of all limit points of P^{n-1}. Most importantly, the first derived set of an infinite order was defined in 1880 as the intersection of all derived sets of finite order:

[1] Even granting this connection, and the fact that the "symbols of infinity" carried a germ of the arithmetic of transfinite numbers, it would be too optimistic to consider that early idea as representing a clear grasp of the ordinals. Cantor had seen the possibility of unlimited continuation of the process of set derivation, but all the crucial details that made the transfinite numbers a part of his set theory were lacking. Only in 1882 did Cantor come to regard the symbols as true numbers, making them correspond to well-ordered sets and defining rigorously the arithmetical operations. And only in 1882 did he see the possibility of establishing their crucial connection with infinite powers, through the "principle of limitation" which clustered them into number-classes.

$$P^\omega = \bigcap(P', P'', \ldots P^n, \ldots) = \bigcap(P^n, P^{n+1}, \ldots).$$

Sets of higher orders, P^α, can be defined similarly, either by reference to an immediately preceding set $P^{\alpha-1}$, or as the intersection of an infinite family.

The Cantor–Bendixson theorem establishes a (unique) decomposition valid for all derived sets, $P' = R \cup S$, where S is a *perfect* set ($S = S'$), and R is a denumerable set such that, for some transfinite number of the second class α, $R \cap R^\alpha = \phi$ (R is a *separated* set) [6, 221–225]. This result grew out of the generalization to nondenumerable sets P' of a previous result, which I will call *Cantor's theorem*. If P' is denumerable, $S = \phi$, and we obtain the part of the Cantor–Bendixson theorem that Cantor proved first:

Cantor's Theorem. If a point-set P is such that P' is denumerable, then there exists some index α such that $P^\alpha = \phi$; and conversely, if $P^\alpha = \phi$ for some index α, then P' and P are denumerable [4, 160; 5, 171].

In fact, the indices α turn out to be either natural numbers or transfinite numbers of the *second number-class* (the set of all denumerable ordinals). But when he first announced the theorem in the paper of 1 September 1882, Cantor still lacked basic results on the indices α, so that he really could not prove it yet. Those results were provided in his *Grundlagen* [5], and the theorem itself was finally proven in [6, 218–221].

While working on this theorem in 1882, Cantor was well aware of its importance, particularly in relation to Mittag-Leffler's Theorem for the construction of analytic functions with an infinite set of singular points. This line of research generalized Weierstrass' Product Theorem, and Mittag-Leffler himself emphasized in his letters to Cantor that Cantor's theorem would give a natural conclusion to his own work.[2] Therefore, in this case Cantor was aware of a very important application of one of his results to central mathematical problems of the time.

In his paper of September 1882, Cantor only proved a very restricted version of the above theorem. His strategy was based on the notion of an *isolated* set (a point set P is isolated iff $P \cap P' = \phi$), and exploited the fact that isolated sets are denumerable [4, 158–159]. This fact had far-reaching consequences. First, if a derived set P' is denumerable, then P itself is denumerable, because P is the union of the isolated set $P\backslash(P \cap P')$ and a subset of P', both denumerable. In this way, Cantor could reduce the proof of Cantor's theorem to the characteristics of P'. Second, since for derived sets it holds that $P^\alpha \subseteq P'$, Cantor was able to establish the following decomposition [4, 158]:

$$P' = (P'\backslash P'') \cup (P''\backslash P''') \cup \ldots \cup P^\omega.$$

[2] Cf. their correspondence of 17 and 22 October 1882, in [20, 88–89]. After seeing the result of this "conclusion" [21], Weierstrass wrote to its author (Mittag-Leffler) that the main problem of the theory of the representation of analytic functions, which seemed a matter for the future, had found its most general solution [22, 25].

The sets $P^n \backslash P^{n+1}$ that appear in this formula are all isolated, and therefore the "only if" part of Cantor's theorem is easy to establish in this particular case: whenever $P^\omega = \phi$ we can conclude that P' is denumerable [4, 160]. Moreover, the same kind of decomposition is easily carried further up to any derived set P^α:

$$P' = (P' \backslash P'') \cup (P'' \backslash P''') \cup \ldots \cup (P^\omega \backslash P^{\omega+1}) \cup \ldots \cup P^\alpha. \quad (1)$$

This opens the way to a general proof, but the step from $P^\alpha = \phi$ to the denumerability of P' can only be done after a condition is imposed on the decomposition.

This essential condition is that we deal with a denumerable family of sets, since the union of a denumerable family of denumerable sets is still denumerable, but the union of a nondenumerable family of nonvoid sets is nondenumerable. To warrant denumerability, therefore, it is necessary that the set of all indices that precede α in (1) be denumerable. Thus the proof of Cantor's theorem forced attention on sets of indices, and specifically it suggested the basic idea behind the principle of limitation that Cantor used in defining the transfinite numbers. The denumerability condition of Cantor's theorem, when transformed into a principle, yielded the definition of the second number class.

The foregoing has to do with proving the "only if" part of Cantor's theorem, but it is the proof of the "if" part, first announced in [5, 171], that was a real problem: given a denumerable set P', it required proving the *existence* of an index α such that $P^\alpha = \phi$. This required considering the second number-class as a whole and introducing the first nondenumerable ordinal (cf. [6, 220–221]). Thus, Cantor's theorem necessitated the development of a theory of the indices and indicated a move toward considering them as mathematical objects, so that we can talk about sets of them, about the *totality* of all them, and about existence results.

Cantor expressed his awareness of such a direct connection between transfinite numbers and his contemporary research when he began the *Grundlagen* with the following words:

> The preceding exposition of my research in the theory of manifolds has come to a point when its further development depends on an extension of the notion of true integer number [des realen ganzen Zahlbegriffs] beyond previous boundaries, an extension which goes in a direction that, to my knowledge, nobody has tried yet. My dependence on this extension of the notion of number is so great that without it it would be almost impossible for me to make freely the least step further in set theory. [5, 165]

As we have seen, these words were not mere rhetoric, but rather a true depiction of the situation of Cantor's research on point-sets.

On 17 October 1882 Cantor confessed to Mittag-Leffler that "the proof of [Cantor's] theorem is well concealed, so that I have searched for it in vain literally for years; but now it is possible to present it quite easily" [20, 88]. The advance was made possible by the above strategy of proof, based on decompositions involving isolated sets, and by the introduction of transfinite numbers—more specifically, by the definition of classes of transfinite numbers, through the principle of limitation.

It was easy for Cantor to see that all the indices he had previously introduced.

i.e., all that can be expressed in terms of ω, satisfied the denumerability condition crucial for the proof of (1): the set of all indices that precede any such index α, $\{\beta|\beta < \alpha\}$, is denumerable. Turning this fact into a principle, Cantor considered the class of all indices α that satisfy this condition, i.e., such that $\{\beta|\beta < \alpha\}$ is denumerable. This is the *second number-class* [5, 197], the first number-class being that of the finite or natural numbers. The full-fledged principle of limitation involved a further generalization, made possible by the fundamental connection that exists between number-classes and infinite powers (see Section 3 below).

In conclusion, it was the context of Cantor's theorem that led to the restricted principle of limitation, which defines the second number-class. This is further confirmed by the fact that it was precisely in the context of Cantor's theorem that Cantor communicated his discovery of transfinite numbers to Mittag-Leffler, on 25 October 1883 [20, 90–91]. Cantor's exposition to Mittag-Leffler also underscores the connection between the denumerability requirement involved in Cantor's theorem and the principle of limitation.

But once Cantor began considering the second number-class, he discovered that it was an example, not only of a new transfinite power, but of the power *immediately greater* than that of denumerable sets [5, 197–200]. This opened the way for a long-desired development of the theory of powers: until then Cantor "lacked a . . . simple and natural definition of the *higher* [infinite] powers" [5, 167]. The classes of transfinite numbers made it possible to define a "scale" of cardinalities, against which it should in principle be possible to measure the power of the continuum. Thus the transfinite numbers far outstripped Cantor's theorem in importance.

2. SEPTEMBER 1882: THE CONVERSATIONS WITH DEDEKIND

Cantor's discovery of transfinite numbers took place within a month of his September 1882 meetings with Dedekind, to whom he wrote:

> just since our last meetings in Harzburg and Eisenach, it has pleased Almighty God that I have attained the most remarkable and unexpected results in set theory and the theory of numbers, or rather, that I have found what fermented in me for years and what I have long been searching for. [Cantor to Dedekind, 5 November 1882, in [9, 55]]

The meetings took place in mid-September [15, 352ff], while Cantor's letters to Mittag-Leffler [20, 88–91] show that by mid-October he was already in possession of the new idea. In fact, the most interesting aspect of the discussion between Cantor and Dedekind is that it seems to have played a heuristic role in the former's process of discovery.

One of the crucial aspects of Cantor's discovery was the decision to call the former "symbols of infinity" numbers. This was not a trivial step, because thus far the symbols were just indices in the derivation process, and abstracted from point-sets they lacked any independent existence. The very name "symbols of infinity" implies that they had no objectivity in themselves. Therefore Cantor was not able to think about transfinite *numbers*, and to consider the second number-class on which Cantor's theorem depended, until he convinced himself that it was possible to regard the indices as true numbers.

Cantor's concern about this question is visible in all his early letters communicating the discovery: those of 1882 to Mittag-Leffler [20, 91] and Dedekind [9, 57], and also an 1884 letter to Kronecker [19, 240]. The same can be observed, of course, in the *Grundlagen* [5, *passim*, but especially 165–170]. As he explained to Kronecker, the opinion that the transfinite numbers "have to be conceived as *numbers* is based on the possibility of determining concretely the [arithmetical] relations among them, and on the fact that they can be conceived under a common viewpoint with the familiar finite numbers" [19, 240]. Showing that they could be seen from a common viewpoint with finite numbers involved two points: the use of numbers for numbering and counting, i.e., their ordinal and cardinal aspects [5, 168], and the possibility of a precise transfinite arithmetic [5, 169–170, 201–204].

2.1. *Cardinals and Ordinals*

In his reflections on the set $\mathbb{N}$ of natural numbers, Dedekind considered the ordinal aspect as the primitive one, and thought that the cardinal numbers should properly be derived from the ordinals [12, 337; 13, 3 : 488–489]. This conception was of course embodied in his theory of natural numbers: he defined natural numbers as the elements of an ordered set of a special kind [12, 359–360], leaving the finite cardinals to be derived only after a long series of theorems [12, Section 14, especially 387]. Although the 1872/1878 draft that he discussed with Cantor did not reach the topic of cardinal numbers, that conception was obviously presupposed in it [cf. 14, 293–309]. That view of natural numbers clashed with Cantor's conceptions.

Cantor always held the opinion that the cardinal aspect of numbers is the primitive one. Prior to the meetings (in March 1882) he expressed his view that cardinality is the "most general genuine factor" as regards sets, and the basis for the notion of number [3, 150]. And in 1884 he wrote that cardinality is the simplest and earliest notion of set theory, its "matrix idea" [17, 86]. Even in his mature exposition of set theory, the *Beiträge*, we find the same basic conception: the notion of "power" is the first one discussed [7, Secs. 1–6], and it is regarded as the "most natural, brief, and rigorous foundation" of finite numbers [7, 289].[3]

Given that distinct contrast of opinions, it is very unlikely that the question as to which of the two aspects of number is basic had not been touched upon in the Harzburg meeting. That they discussed it seems to be confirmed by a second draft written by Dedekind in 1887: after stating his opinion that cardinality is really a very complicated notion, and not a simple one, he writes: "contrary to Cantor" [10, 120–121; 1, III,1,II, p. 41].

It is thus tempting to speculate on the arguments presented by both mathematicians. Cantor probably objected that the cardinality of a set is more general than

[3] This order of presentation had its shortcomings: the theory of finite numbers presented by Cantor had obvious weaknesses, which, as Zermelo commented in [8, 352, note 4], only a treatment of cardinals on the basis of a previous development of the ordinals could have avoided.

anything else: the cardinality of a set is an essential, unchanging feature, apt to bring both continuous and discontinuous, ordered and unordered sets under a higher unity [cf. 3, 152; 17, 86; 7, 282, 297–298]. To this, Dedekind probably answered that if we analyze how we determine the cardinality of a set, it turns out to be by ordering its elements [12, 336 and *passim*]. Even Cantor's denumerability proofs could have served as an example of this: either they explicitly establish an ordering isomorphic to that of ℕ, or they take such an ordering as given in order to reason by *reductio ad absurdum*. In this view, it would appear that ordinality is prior to cardinality, since determinations of cardinality depend, sooner or later, on ordinal arguments.

2.2. *Chains and Well-Ordered Sets*

With this theory of natural numbers, Dedekind therefore drew attention to the importance of ordinal considerations for determining cardinals. But even more generally, he emphasized the importance of ordered sets. This is made clear by the so-called chain theory that formed the core of his 1872/1878 draft [14, 293–309]. Chain theory was Dedekind's main tool for defining ℕ and for establishing rigorously the properties of numbers, and of finite and infinite subsets of ℕ. An exposition of the basic ideas of the theory can be found in [15, 354–355].

Although Dedekind's notion of chain was a rather general one, for the specific purposes of his 1888 book he needed only to consider a particular kind of chain: chains of unitary sets, or as he expressed it—in a slightly ambiguous way—chains of a single element. These number-chains are nothing but infinite well-ordered sets of the simplest kind, i.e., of type ω in Cantor's symbolism.

Moreover, Dedekind was led to introduce the notions of "section" and "remainder" that Cantor would later use in his theory of well-ordered sets [7, 314]. The chain n_0 of any natural number n is simply the set of all numbers $m \geq n$, which can be called the remainder of n in ℕ; and the complements of remainders in ℕ are the "initial sections" for which he used the notation Z_n. Z_n is the set of all numbers $m < n$, and obviously $\mathbb{N} = Z_n \cup n_o$ [12, Sect. 7]. Dedekind made extensive use of initial sections; in fact, the sections Z_n are crucial for the definition of cardinal numbers at the end of the book [12, Sect. 14].

The dependence on chains of the whole theory of numbers and sets presented in [12] amounts to a primacy of ordinal notions in arithmetic and elementary set theory. The theory of chains forms the main content of Dedekind's draft, written between 1872 and 1878 [14, 293–309]; it was to the development of chain theory that he devoted his attention during that time. Thus it is natural to think that Dedekind's views should have had an important effect on Cantor, suggestive of a possible turn to ordered sets, and of their potential importance for cardinality problems.

2.3. *Orderings*

The letters written by Cantor after the Harzburg meeting reveal an increased awareness of orderings [9, 52–54]. In particular, Cantor expressed here in very clear terms, apparently for the first time, the idea that an abstract set can be given many different orderings, and that some of its properties will depend on the assumed ordering. Likewise, in letters of 15 September and 2 October, Cantor emphasized that it is only relative to some particular ordering that a set can be called a continuum [9, 52–54]. The attempt to give a general definition of the continuum that can be found here is Cantor's first elaboration of general ordinal notions; the natural result of these endeavors was his theory of order-types, first explained in the unpublished paper of 1884/1885 [17].

It thus seems that the stimulus coming from Dedekind's views helped Cantor to turn to ordinal considerations and to reconsider his previous problems in this new setting. The former "symbols of infinity," if considered as elements of sets, offered examples of well-ordered sets (i.e., totally ordered sets such that every subset has a least element), and so Cantor was able to abstract and define the notion of well-ordering. (It has to be emphasized that Dedekind had only considered the simplest kind of well-ordered set and possessed no general notion of well-ordering.) Since the "symbols of infinity" were apt to represent the different types of order characteristic of well-ordered sets, Cantor was finally in the position to consider them as numbers, as true ordinal numbers.

As Cantor said, the connection between well-ordered sets and the new numbers "establishes the reality of the latter, that I emphasize, even in case they are actually infinite" [5, 168]. It established the transfinite numbers as true objects, whose foundation was similar to that of the finite ordinal numbers—modulo Cantor's unrestricted acceptance of actual infinities. Well-ordered sets were also fundamental for the definition of arithmetic operations on transfinite numbers, and offered a new way of dealing with the continuum problem;[4] so they became a new, essential ingredient of Cantor's theory [cf. 5, 168–171].

3. CONCLUSION

Above all, the key aspect of transfinite numbers was the clear relation established between classes of transfinite numbers and infinite powers. The second number-class is just the set of all transfinite numbers that correspond to denumerable well-ordered sets. This class, in its turn, has and defines the second transfinite power $\aleph_1$ (in the 1895 aleph-notation). Now, the third number-class is that of the transfinite numbers which correspond to well-ordered sets of power $\aleph_1$, and that class has the immediately greater power $\aleph_2$. The process can then go on: number-classes depend on a cardinality condition, and cardinalities are defined through number-classes, in a circular but nonvicious process (one would be tempted to call it "helical") the result of which is Cantor's paradise of transfinite sets.

This is what made the ordinals invaluable for Cantor, forming the true cornerstone of his mature theory of transfinite sets. As we saw in Section 1, the question of proving Cantor's theorem suggested the crucial clue for this development: the

[4] Cantor was confident that it would be possible to establish a one-to-one mapping between the continuum and the transfinite numbers of the second number class [20, 91; 9, 59].

principle of limitation. But Cantor only considered the question of the power of the second number-class after having reconceptualized the "symbols of infinity" as ordinal numbers.

Since the number-classes established the connection between transfinite numbers and infinite cardinalities, Cantor could feel confident that all the essential aspects of number had been covered. It had been possible to conceive of transfinite numbers under a common viewpoint with the natural numbers, since these emerged as specializations of the former in the finite case [cf. 5, 181]. Both ordinal and cardinal aspects had been intertwined with the new notions [5, 168], and it had become possible to define a precise transfinite arithmetic [5, 169–170]. A new kind of number had been born.

Even if Dedekind's work stimulated Cantor in his turn to ordinal considerations, the process that we have reviewed was very complex; there is a great difference between such a stimulus and Cantor's developed theory. Only the suggestion of an intimate connection between cardinals and ordinals, and the emphasis on ordinal ideas (including the simplest example of well-ordered sets), could have come from Dedekind. The "symbols of infinity," the general notion of well-ordered set, and the number-classes defined through the principle of limitation—in a word, all the essential aspects of Cantor's theory of transfinite numbers—were original. With the transfinite numbers at his disposal, Cantor was able to step from his former theory of point-sets to an abstract theory. After his nondenumerability results, the discovery of transfinite numbers was the crucial breakthrough which allowed the creation of transfinite set theory.

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