CARTESIAN AND LAGRANGIAN MOMENTUM

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Introduction

What we now call “momentum” has theological roots, in Descartes’ *Principia philosophiæ* (*Pars secunda*):

> 36. *Deum esse primariam motus causam: et eandem semper motus quantitatem in universo conservare*.
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> [...] Et generalem quod attinet, manifestum mihi videtur illam non aliam esse, quam Deum ipsum, qui materiam simul cum motu et quiete in principio creavit, jamque, per solum suum concursum ordinarium, tantumdem motus et quietis in ea tota quantum tunc posuit conservat. Nam quamvis ille motus nihil aliud sit in materia mota quam ejus modus; certam tamen et determinatam habet quantitatem, quam facile intelligimus eandem semper in tota rerum universitate esse posse, quamvis in singulis ejus partibus mutetur. Ita scilicet ut putemus, cum una pars materiæ duplo celerius movetur quam altera, et hæc altera duplo major est quam prior, tantundem motus esse in minore quam in majore; ac quanto motus unius partis lentior fit, tanto motum alicujus alterius ipsiæ qualis fieri celeriorem.

The conservation of motion follows from the perfection of the Creator; for how could the world have more or less motion than He first put into it. The amount of motion depends on *size* and *celerity*: a piece of matter has as much motion as another that moves twice as fast but is half as big. Tradition has, rightly or wrongly, turned these two ingredients into *mass* and *velocity*, and preferred the term *momentum*; as we see no reason to depart from it (too much at any rate), *Cartesian momentum* will be taken to be something like *the product of mass and velocity*—which is the form assumed in Newton’s *Definitio II*:

> *Quantitas motus est mensura ejusdem orta ex velocitate et quantitate materiæ conjunctim.*
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> Motus totius est summa motuum in partibus singulis; ideoque in corpore duplo majore, æquali cum velocitate, duplus est, & dupla cum velocitate quadruplus.

where, from *Definitio I*, “*Quantitas materiæ est mensura ejusdem orta ex illius densitate et magnitudine conjunctim.*”

In Lagrange’s *Mécanique analytique* we find $\delta\Phi/\delta d\xi$, nameless and buried in larger expressions; Φ is a function of position and velocity and $d\xi$ a velocity. Hamilton will write $\varpi_i = \delta T/\delta\eta_i'$ or $m_i\,dx_i/dt = \delta V/\delta x_i$ and call it *momentum*, where T is kinetic energy, η_i' a velocity and V the principal function. But as the expression first appeared in Lagrange, the term “Lagrangian momentum” seems preferable.

The two momenta

We have seen that momentum is closely related to velocity (or to something like it at any rate). We can try to characterize the relationship in more modern terms by writing $p=\mu^\flat\dot{q}.$ But what is $\mu^\flat$? Velocity $\dot{q}$ is a vector, belonging to the linear space T_qQ tangent at q to the configuration space $Q,$ momentum p a covector[1] in the corresponding cotangent space $T_q^*Q;$ so $\mu^\flat,$ whatever it is, turns velocities in T_qQ into momenta in $T_q^*Q:$ $\mu^\flat : T_qQ \to T_q^*Q.$ The inverse[2] $\mu^\sharp = (\mu^\flat)^{-1}$ goes the other way: $\mu^\sharp : T_q^*Q \to T_qQ.$

But it will be best to begin with simpler cases. For a single free particle of unit mass moving in a Euclidean space $\mathbb{E},$ $\mu^\flat$ can be the (partially evaluated) twice-covariant metric tensor $e^\flat : T_q\mathbb{E} \to T_q^*\mathbb{E}$ characterizing the geometry of $\mathbb{E}.$ The squared length $\langle \dot{q}^\flat, \dot{q}\rangle = \langle e^\flat\dot{q}, \dot{q}\rangle$ will be twice the kinetic energy $K,$ where $\langle r, \dot{s}\rangle$ is the value of the covector r at the vector $\dot{s}.$ Even if a little more than mere multiplication by (unit) mass is involved, it seems reasonable to view $e^\flat\dot{q}$ as the Cartesian momentum. The Lagrangian momentum will be $dL_q(\dot{q}),$ where L_q is the restriction $L_q = L|_{T_q\mathbb{E}}$ of the Lagrangian $L=K$ to the tangent space $T_q\mathbb{E},$ and the differential $dL_q(\dot{q}) : T_q\mathbb{E} \to \mathbb{R}$ approximates the Lagrangian linearly around $\dot{q}.$

To go the other way $T_q^*\mathbb{E} \to T_q\mathbb{E}$ we need the *energy*

$$E(q,\dot{q}) = \langle \dot{q}^\flat, \dot{q}\rangle - L(q,\dot{q}) : T\mathbb{E} \to \mathbb{R},$$

defined on the tangent bundle

$$T\mathbb{E} = \bigcup_{q\in\mathbb{E}} T_q\mathbb{E}.$$

For the energy provides the Hamiltonian $H(q,p) = \langle p, p^\sharp\rangle - L(q,p) : T^*\mathbb{E} \to \mathbb{R},$ which to every $(q,\dot{q}^\flat) \in T\mathbb{E}$ assigns the same value $H(q,\dot{q}^\flat) = E(q,\dot{q})$ that $E(q,\dot{q})$ assigns to $(q,\dot{q}) \in T\mathbb{E},$ where $p=\dot{q}^\flat,$ $p^\sharp = \dot{q},$ and the cotangent bundle

$$T^*\mathbb{E} = \bigcup_{q\in\mathbb{E}} T_q^*\mathbb{E}.$$

The differential dH_q of the restriction $H_q = H|_{T_q^*\mathbb{E}}$ takes us back to the tangent space: $dH_q(\cdot) : T_q^*\mathbb{E} \to T_q\mathbb{E}.$ The *particular value* $dH_q(p) : T_q^*\mathbb{E} \to \mathbb{R}$ acts as a linear functional.

Here $\mu^\flat = m^\flat = dL_q,$ so both momenta are the same, and we can write $p = \dot{q}^\flat = e^\flat\dot{q} = dL_q(\dot{q}).$

For a free particle of mass m we have

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[1] If Hamilton's η_i' are the components of a vector, the ϖ_i transform like a covector. And furthermore the $\delta V/\delta x_i$ are the components of the differential $dV,$ which is a covector.

[2] Whose existence we assume.

$$m^\flat = me^\flat : T_q\mathbb{E} \to T_q^*\mathbb{E}$$

$$m^\sharp = \frac{1}{m}e^\sharp = (m^\flat)^{-1} : T_q^*\mathbb{E} \to T_q\mathbb{E}$$

and the kinetic energy $K = \langle m^\flat \dot{q}, \dot{q}\rangle / 2 = m\langle e^\flat \dot{q}, \dot{q}\rangle / 2$. Both momenta still coincide: $p = \dot{q}^\flat = dL_q(\dot{q}) = me^\flat \dot{q}$.

For N particles with masses $m_1, \dots, m_N$ moving freely in the Euclidean spaces $\mathbb{E}_1, \dots, \mathbb{E}_N$ we have

$$K = \frac{1}{2}\sum_{k=1}^{N} m_k \langle e_k^\flat \dot{q}_k, \dot{q}_k\rangle,$$

where $e_k^\flat : T_q\mathbb{E}_k \to T_q^*\mathbb{E}_k$ and $\dot{q}_k \in T_q\mathbb{E}_k$. We can also write $K = \langle m^\flat \dot{q}, \dot{q}\rangle / 2$, where

$$m^\flat = \sum_{n=1}^{N} m_n e_n^\flat : T_q\mathbb{E} \to T_q^*\mathbb{E},$$

$$\mathbb{E} = \mathop{\times}_{n=1}^{N} \mathbb{E}_k \quad \text{and} \quad \dot{q} = \bigoplus_{n=1}^{N} \dot{q}_k.$$

The inverse of $m^\flat$ is

$$m^\sharp = \sum_{k=1}^{N} \frac{1}{m_k} e_k^\sharp : T_q^*\mathbb{E} \to T_q\mathbb{E}.$$

Here the isolagrangian surfaces are spherical with respect to $m^\flat$ but ellipsoidal when referred to

$$e^\flat = \sum_{k=1}^{N} e_k^\flat : T_q\mathbb{E} \to T_q^*\mathbb{E},$$

where the ratios of the masses are also those of the principal axes. The momentum $m^\flat \dot{q}$ remains a fairly direct descendant of what Descartes was groping for in *Principia philosophiæ*, and is still the same as the Lagrangian momentum $dL_q(\dot{q}) = dK_q(\dot{q})$.

Suppose holonomic, scleronomic constraints determine a Riemannian configuration space $Q \subset \mathbb{E}$. At each point $q \in Q$, the tangent space $T_qQ \subset T_q\mathbb{E}$. The mappings $\underline{e}^\flat : T_qQ \to T_q^*Q$ and $e^\flat : T_q\mathbb{E} \to T_q^*\mathbb{E}$ agree on all vectors in T_qQ (but of course some velocities of $T_q\mathbb{E}$ are excluded by the constraints), so $\underline{e}^\flat = e^\flat\big|_{T_qQ}$. With masses we can write $\underline{m}^\flat = m^\flat\big|_{T_qQ}$.

So far there has been no potential, and $L = K$. A potential $U(q)$ depending on position alone will not contribute to the geometry, for with a Lagrangian $L(q,\dot{q}) = K(q,\dot{q}) - U(q)$ the differential $dL_q = dK_q - dU_q$ will be equal to dK_q. But if the potential $U(q,\dot{q})$ depends on velocity as well, $dL_q \neq dK_q$, and there may be no tensor $\underline{m}^\flat : T_qQ \to T_q^*Q$ with the same effect as dL_q. In other words if $L(q,\dot{q}) = K(q,\dot{q}) - U(q,\dot{q})$, the isolagrangian surfaces may not even be ellipsoidal with respect to $\underline{e}^\flat$. Forces with an anisotropic dependence on velocity will cause the Cartesian and Lagrangian momenta to differ.

We now have three kinds of mechanical ingredients—constraints, masses, forces (potential)—and can introduce a corresponding nomenclature.

GEOMETRY	**TRANSFORMATION**
Empty	$e^\flat = dK_q = dL_q$
Background (constraints)	$\underline{e}^\flat = dK_q = dL_q$
Kinematical (constraints, masses)	$\underline{m}^\flat = dK_q = dL_q$
Dynamical (constraints, masses, forces)	$dL_q\ (\neq dK_q)$

The masses here are kinematical as they are seen as calibrating distance.

With a potential depending on position alone, dK_q and dL_q are the same, and hence the kinematical and dynamical geometries coincide. But a velocity-dependent potential can produce a dynamical geometry that differs from the kinematical.

Attitude to geometry can be conditioned by ontological prejudice, which can, by favouring constraints and masses over forces, lead one to view geometry as being fundamentally kinematical. But since forces can be geometrically significant, the dynamical geometry is a proper generalization of the kinematical.

Hamilton-Jacobi theory

A central feature of Hamilton-Jacobi theory is the relationship, determined by a transformation $T_qQ \to T_q^*Q,$ between a vector field $\dot{q}(q)$ tangent to a flow on the configuration space $Q,$ and the differential $dW(q) = p(q)$ of the characteristic function $W : Q \to \mathbb{R}.$ Here again the relationship can be Cartesian and merely kinematical or Lagrangian and dynamical, but there will only be a difference in the case of a velocity-dependent potential. Given a characteristic function $W,$ a metric tensor $m^\sharp$ will turn the momentum dW into the "Cartesian" velocity $m^\sharp(dW) \in T_qQ,$ which will be the same as the "Lagrangian" velocity $dH_q(dW) \in T_qQ$ as long as the Hamiltonian is of the form $H = T + U(q),$ in which case we can write $\dot{q} = (dW)^\sharp = m^\sharp(dW) = dH_q(dW).$

Statements like "The orthogonality of the light rays to the wave surfaces does not hold in crystal optics. Nor is a mechanical path always perpendicular to the surfaces $S = \text{const.}$ An electron moving in a magnetic field does not cross the surfaces $S = \text{const.}$

perpendicularly”[3] can be understood by mixing the Cartesian and Lagrangian pictures. The level surface σ of W at q determines a ray $\sigma_q \subset T_q^* Q,$ and any vector $p^\sharp$ dual to a covector $p \in \sigma_q$ will be “orthogonal” to the level surface. But as this orthogonality can be Cartesian or Lagrangian, the vector $dH_q(p)$ is necessarily orthogonal in the Cartesian sense neither to the ray σ_q nor to the surface $\sigma,$ for $m^\sharp(p)$ is.

References

Lanczos, C. (1970) *The variational principles of mechanics*, University of Toronto Press

[3] Lanczos (1970) p.267.