

An Empirical Comparison of Alternative Models of the Short-Term Interest Rate

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ABSTRACT

We estimate and compare a variety of continuous-time models of the short-term riskless rate using the Generalized Method of Moments. We find that the most successful models in capturing the dynamics of the short-term interest rate are those that allow the volatility of interest rate changes to be highly sensitive to the level of the riskless rate. A number of well-known models perform poorly in the comparisons because of their implicit restrictions on term structure volatility. We show that these results have important implications for the use of different term structure models in valuing interest rate contingent claims and in hedging interest rate risk.

THE SHORT-TERM RISKLESS interest rate is one of the most fundamental and important prices determined in financial markets. More models have been put forward to explain its behavior than for any other issue in finance. Many of the more popular models currently used by academic researchers and practitioners have been developed in a continuous-time setting, which provides a rich framework for specifying the dynamic behavior of the short-term riskless rate. A partial listing of these interest rate models includes those by Merton (1973), Brennan and Schwartz (1977, 1979, 1980), Vasicek (1977), Dothan (1978), Cox, Ingersoll, and Ross (1980, 1985), Constantinides and Ingersoll (1984), Schaefer and Schwartz (1984), Sundaresan (1984), Feldman (1989), Longstaff (1989a), Hull and White (1990), Black and Karasinski (1991), and Longstaff and Schwartz (1992).

Despite a bewildering array of models, relatively little is known about how these models compare in terms of their ability to capture the actual behavior

* All authors are from the College of Business, The Ohio State University. We are grateful for the comments and suggestions of Warren Bailey, Emilio Barone, Fischer Black, Tim Bollerslev, Stephen Buser, John Campbell, Jennifer Conrad, George Constantinides, Ken Dunn, Margaret Forster, Campbell Harvey, Patric Hendershott, David Mayers, Huston McCulloch, Daniel Nelson, David Shimko, René Stulz, Stuart Turnbull, Curt Wells, and Finance Workshop participants at the Commodity Futures Trading Commission, the University of Iowa, the Kansallis Research Foundation, The Ohio State University, Purdue University, and participants at the 1991 Western Finance Association meetings, the 1991 European Finance Association meetings, the 1992 American Finance Association meetings, and the 1992 Federal Reserve Bank of Atlanta's Conference on Financial Market Issues. All errors are our responsibility.

of the short-term riskless rate. The primary reason for this has probably been the lack of a common framework in which different models could be nested and their performance benchmarked. Without a common framework, it is difficult to evaluate relative performance in a consistent way.¹ The issue of how these models compare with each other is particularly important, however, since each model differs fundamentally in its implications for valuing contingent claims and hedging interest rate risk.

This paper uses a simple econometric framework to compare the performance of a wide variety of well-known models in capturing the stochastic behavior of the short-term rate. Our approach exploits the fact that many term structure models—both single-factor and multifactor—imply dynamics for the short-term riskless rate r that can be nested within the following stochastic differential equation:

$$dr = (\alpha + \beta r)dt + \sigma r^\gamma dZ. \quad (1)$$

These dynamics imply that the conditional mean and variance of changes in the short-term rate depend on the level of r . We estimate the parameters of this process in discrete time using the Generalized Method of Moments technique of Hansen (1982). As in Marsh and Rosenfeld (1983), we test the restrictions imposed by the alternative short-term interest rate models nested within equation (1). In addition, we compare the ability of each model to capture the volatility of the term structure. This property is of primary importance since the volatility of the riskless rate is a key variable governing the value of contingent claims such as interest rate options. In addition, optimal hedging strategies for risk-averse investors depend critically on the level of term structure volatility.

The empirical analysis provides a number of important results. Using one-month Treasury bill yields, we find that the value of γ is the most important feature differentiating interest rate models. In particular, we show that models which allow $\gamma \geq 1$ capture the dynamics of the short-term interest rate better than those which require $\gamma < 1$. This is because the volatility of the process is highly sensitive to the level of r ; the unconstrained estimate of γ is 1.50. We also show that the models differ significantly in their ability to capture the volatility of the short-term interest rate. We find no evidence of a structural shift in the interest rate process in October 1979 for the models that allow $\gamma \geq 1$.

We show that these interest rate models differ significantly in their implications for valuing interest-rate-contingent securities. Using the estimated parameters for these models from the 1964 to 1989 sample period, we employ numerical procedures to value call options on long-term coupon bonds under

¹ Because of this problem, empirical work in this area has tended to focus on specific models instead of comparisons across models. See, for example, Brennan and Schwartz (1982), Brown and Dybvig (1986), Gibbons and Ramaswamy (1986), Pearson and Sun (1989) and Barone, Cuoco, and Zautzik (1991).

different economic conditions. Our findings demonstrate that the range of possible call values varies significantly across the various models.

The remainder of the paper is organized as follows. Section I describes the short-term interest rate models examined in the paper. Section II discusses the econometric approach. Section III describes the data. Section IV presents the empirical results from comparing the models. Section V contrasts the models' implications for valuing options on long-term bonds. Section VI summarizes the paper and makes concluding remarks.

I. The Interest Rate Models

The stochastic differential equation given in (1) defines a broad class of interest rate processes which includes many well-known interest rate models. These models can be obtained from (1) by simply placing the appropriate restrictions on the four parameters α , β , σ , and γ . In this paper, we focus on eight different specifications of the dynamics of the short-term riskless rate that have appeared in the literature. These specifications are listed below and the corresponding parameter restrictions are summarized in Table I:

1. Merton	$dr = \alpha dt + \sigma dZ$
2. Vasicek	$dr = (\alpha + \beta r)dt + \sigma dZ$
3. CIR SR	$dr = (\alpha + \beta r)dt + \sigma r^{1/2}dZ$
4. Dothan	$dr = \sigma r dZ$
5. GBM	$dr = \beta r dt + \sigma r dZ$
6. Brennan-Schwartz	$dr = (\alpha + \beta r)dt + \sigma r dZ$
7. CIR VR	$dr = \sigma r^{3/2}dZ$
8. CEV	$dr = \beta r dt + \sigma r^\gamma dZ$

Model 1 is used in Merton (1973), footnote 34, to derive a model of discount bond prices. This stochastic process for the riskless rate is simply a Brownian motion with drift. Model 2 is the Ornstein-Uhlenbeck process used by Vasicek (1977) in deriving an equilibrium model of discount bond prices. This Gaussian process has been used extensively by others in valuing bond options, futures, futures options, and other types of contingent claims. Examples include Jamshidian (1989) and Gibson and Schwartz (1990). The Merton model can be nested within the Vasicek model by the parameter restriction $\beta = 0$. Both of these models imply that the conditional volatility of changes in the riskless rate is constant.

Model 3 is the square root (SR) process which appears in the Cox, Ingersoll, and Ross (CIR) (1985) single-factor general-equilibrium term structure model. This model has also been used extensively in developing valuation models for interest-rate-sensitive contingent claims. Examples include the mortgage-backed security valuation model in Dunn and McConnell (1981), the discount bond option model in CIR (1985), the futures and futures option pricing models in Ramaswamy and Sundaresan (1986), the swap pricing model in

Table I
Parameter Restrictions Imposed
by Alternative Models of
Short-Term Interest Rate

Alternative models of the short-term riskless rate of interest r can be nested with appropriate parameter restrictions within the unrestricted model

$$dr = (\alpha + \beta r)dt + \sigma r^\gamma dZ$$

Model	α	β	σ^2	γ
Merton		0		0
Vasicek				0
CIR SR				$\frac{1}{2}$
Dothan	0	0		1
GBM	0			1
Brennan-Schwartz				1
CIR VR	0	0		$\frac{3}{2}$
CEV	0			

Sundaresan (1989), and the yield option valuation model in Longstaff (1990b). The CIR SR model implies that the conditional volatility of changes in r is proportional to r .

Model 4 is used by Dothan (1978) in valuing discount bonds and has also been used by Brennan and Schwartz (1977) in developing numerical models of savings, retractable, and callable bonds. Model 5 is the familiar geometric Brownian motion (GBM) process of Black and Scholes (1973). Geometric Brownian motion is also one of the interest rate models considered by Marsh and Rosenfeld (1983). Model 6 is used by Brennan and Schwartz (1980) in deriving a numerical model for convertible bond prices. This process is also used by Courtadon (1982) in developing a model of discount bond option prices. The GBM model is nested within the Brennan-Schwartz model by the parameter restriction $\alpha = 0$. In turn, the Dothan model is nested within the GBM model by the parameter restriction $\beta = 0$. All three of these models imply that the conditional volatility of changes in the riskless rate is proportional to r^2 .

Model 7 is introduced by CIR (1980) in their study of variable-rate (VR) securities. A similar model is also used by Constantinides and Ingersoll (1984) to value bonds in the presence of taxes. Finally, Model 8 is the constant elasticity of variance (CEV) process introduced by Cox (1975) and by Cox and Ross (1976). The application of this process to interest rates is discussed in Marsh and Rosenfeld (1983), footnote 4. Table I shows that the CEV model nests the Dothan, Brennan-Schwartz, and CIR VR models.

Although the majority of these interest rate processes were introduced in the context of a single-factor model of the term structure, it is important to

note that our analysis is not limited to single-factor term structure models. By comparing different models of the short-term interest rate, our analysis provides insights into the properties of any economic model in which assumptions about interest rate dynamics are made. For example, our results are applicable to any multifactor term structure model in which assumptions about the dynamic behavior of r are embedded.

Finally, we note that our framework has some features in common with Marsh and Rosenfeld (1983), Fischer and Zechner (1984), and Melino and Turnbull (1986). For example, Marsh and Rosenfeld use a general stochastic process similar to (1) in estimating the parameters of several continuous-time interest rate models. Their framework, however, nests only three interest rate processes. A comparison of their model with (1) shows that two of these three interest rate processes are nested within (1). These nested models correspond to the CIR SR and GBM models in our framework.

II. The Econometric Approach

In this section, we describe the econometric approach used in estimating the parameters of the interest rate models and in examining their explanatory power for the dynamic behavior of short-term interest rates. To illustrate the approach clearly, we focus first on the unrestricted process given in equation (1). The same approach can then be used for the nested models after imposing the appropriate parameter restrictions.

Following Brennan and Schwartz (1982), Dietrich-Campbell and Schwartz (1986), Sanders and Unal (1988), and others, we estimate the parameters of the continuous-time model using a discrete-time econometric specification

$$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{t+1}, \quad (2)$$

$$E[\varepsilon_{t+1}] = 0, \quad E[\varepsilon_{t+1}^2] = \sigma^2 r_t^{2\gamma}. \quad (3)$$

This discrete-time model has the advantage of allowing the variance of interest rate changes to depend directly on the level of the interest rate in a way consistent with the continuous-time model.

It is important to acknowledge that the discretized process in (2) and (3) is only an approximation of the continuous-time specification. The reason for this is that in measuring changes in r over discrete intervals of time, integrals appear on the right side of (1). This is the temporal aggregation issue described by Grossman, Melino, and Shiller (1987), Breeden, Gibbons, and Litzenberger (1989), and Longstaff (1989b, 1990a). Given the continuity of the interest rate process, however, the amount of approximation error introduced can be shown to be of second-order importance if changes in r are measured over short periods of time.²

² See also Campbell (1986).

Our econometric approach is to test (2) and (3) as a set of overidentifying restrictions on a system of moment equations using the Generalized Method of Moments (GMM) of Hansen (1982). This technique has a number of important advantages that make it an intuitive and logical choice for the estimation of the continuous-time interest rate processes. First, the GMM approach does not require that the distribution of interest rate changes be normal; the asymptotic justification for the GMM procedure requires only that the distribution of interest rate changes be stationary and ergodic and that the relevant expectations exist. This is of particular importance in testing the continuous-time term structure models since each implies a different distribution for changes in r . For example, the Vasicek and Merton models assume that interest rate changes are normal, whereas CIR SR assumes that they are proportional to a noncentral χ^2 variate. Second, the GMM estimators and their standard errors are consistent even if the disturbances, ε_{t+1} , are conditionally heteroskedastic. Since the temporal aggregation problem that arises from estimation of a continuous-time process with discrete-time data is likely to influence the distribution of the disturbances, the GMM approach should further alleviate the impact of this approximation error on the parameter estimates. For example, even though the CIR SR continuous-time model assumes that changes in r are distributed as a random variable proportional to a noncentral χ^2 , the discrete-time version of the model may not. Finally, the GMM technique has also been used in other empirical tests of interest rate models by Gibbons and Ramaswamy (1986), Harvey (1988), and Longstaff (1989a).

Define θ to be the parameter vector with elements α , β , σ^2 and γ . Given $\varepsilon_{t+1} = r_{t+1} - r_t - \alpha - \beta r_t$, let the vector $f_t(\theta)$ be

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \\ (\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}) r_t \end{bmatrix}. \quad (4)$$

Under the null hypothesis that the restrictions implied by (2) and (3) are true, $E[f_t(\theta)] = 0$. The GMM procedure consists of replacing $E[f_t(\theta)]$ with its sample counterpart, $g_T(\theta)$, using the T observations where

$$g_T(\theta) = \frac{1}{T} \sum_{t=1}^T f_t(\theta), \quad (5)$$

and then choosing parameter estimates that minimize the quadratic form,

$$J_T(\theta) = g_T'(\theta) W_T(\theta) g_T(\theta), \quad (6)$$

where $W_T(\theta)$ is a positive-definite symmetric weighting matrix. Matrix differentiation shows that minimizing $J_T(\theta)$ with respect to θ is equivalent to

solving the homogeneous system of equations (orthogonality conditions),

$$D'(\theta)W_T(\theta)g_T(\theta) = 0, \tag{7}$$

where $D(\theta)$ is the Jacobian matrix of $g_T(\theta)$ with respect to θ .

For the unrestricted model, the parameters are just identified and $J_T(\theta)$ attains zero for all choices of $W_T(\theta)$. For the nested interest rate models, the GMM estimates of the overidentified parameter subvector of θ do depend on the choice of W_T . Hansen (1982) shows that choosing $W_T(\theta) = S^{-1}(\theta)$, where

$$S(\theta) = E[f_t(\theta)f_t'(\theta)], \tag{8}$$

results in the GMM estimator of θ with the smallest asymptotic covariance matrix. Designating an estimator of this covariance matrix as $S_0(\theta)$, the asymptotic covariance matrix for the GMM estimate of θ is

$$\frac{1}{T}(D_0'(\theta)S_0^{-1}(\theta)D_0(\theta))^{-1}, \tag{9}$$

where $D_0(\theta)$ is the Jacobian evaluated at the estimated parameters. This covariance matrix is used to test the significance of the individual parameters.

The minimized value of the quadratic form in (6) is distributed χ^2 under the null hypothesis that the model is true with degrees of freedom equal to the number of orthogonality conditions net of the number of parameters to be estimated. This χ^2 measure provides a goodness-of-fit test for the model. A high value of this statistic means that the model is misspecified.³

We also use the hypothesis-testing methods developed by Newey and West (1987) in order to evaluate the restrictions imposed by the various models on the unrestricted model. They show that for a general null hypothesis of the form, $H_0 : a(\theta) = 0$, where $a(\theta)$ is a vector of order k , each element representing a model restriction, the test statistic,

$$R = T[J_T(\tilde{\theta}) - J_T(\hat{\theta})], \tag{10}$$

is asymptotically distributed χ^2 with k degrees of freedom. This test statistic is the normalized difference of the restricted ($J_T(\tilde{\theta})$) and unrestricted ($J_T(\hat{\theta})$) objective functions for the efficient GMM estimator (both using the same weighting matrix from the unrestricted model) and is analogous to the likelihood ratio test. We employ these tests for a number of the pairwise comparisons of performance among the various models.

In addition to these statistical tests, we also examine the economic importance of differences between the interest rate models. In doing this, our metric is the ability of the model to capture the volatility of changes in the riskless rate. We focus on volatility since it plays a central role in two of the most important applications of term structure models: valuing contingent

³ Newey (1985) examines the asymptotic power properties of such tests against general model misspecification.

Table II**Summary Statistics**

Means, standard deviations, and autocorrelations of monthly Treasury bill yields and yield changes are computed from June 1964 through December 1989. The variable r_t denotes the yield on Treasury bills maturing in one month and $r_{t+1} - r_t$ is the associated monthly yield change. ρ_j denotes the autocorrelation coefficient of order j . N represents the number of observations used.

Variables	N	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
r_t	307	0.06715	0.02675	0.95	0.91	0.86	0.82	0.80	0.78
$r_{t+1} - r_t$	306	0.00009	0.00821	-0.08	0.07	-0.12	-0.14	-0.03	-0.04

claims and hedging interest rate risk. For example, the volatility of interest rates is a fundamental determinant of the value of interest rate options. In addition, optimal hedging strategies for risk-averse investors can be very sensitive to changes in expected interest rate volatility. The ability of a term structure model to capture interest rate volatility is a direct measure of its hedging usefulness.

III. The Data

The Treasury bill yield data for our study were obtained from the data set originally constructed by Fama (1984) and subsequently updated by the Center for Research in Security Prices (CRSP). The one-month yields are based on the average of bid and ask prices for Treasury bills and are normalized to reflect a standard month of 30.4 days. The data are monthly and cover the period from June 1964 to December 1989, providing 307 observations in total. All yields are expressed in annualized form.

Table II shows the means, standard deviations, and first six autocorrelations of the one-month yield and the monthly changes in the one-month yield. The unconditional average level of the one-month yield is 6.715% with a standard deviation of 2.675%. Although the autocorrelations in the interest rate levels decay slowly, those of the month-to-month changes are generally small and are not consistently positive or negative. This offers some evidence that interest rates are stationary.

IV. The Empirical Results

In this section, we present our empirical results. We begin by estimating the unrestricted and the eight restricted interest rate processes. We compare the explanatory power of the nested models with that of the unrestricted model and the nested models with each other using the methods of Newey and West (1987) outlined in Section II. We also compare the models in terms of their explanatory power for an ex post measure of interest rate volatility.

Finally, we test whether the change in monetary policy in October 1979 resulted in a structural break in the individual models.

A. Estimation Results and Model Comparisons

Table III reports the parameter estimates, asymptotic t -statistics, and GMM minimized criterion (χ^2) values for the unrestricted model and for each of the eight nested models. As shown, the models vary in their explanatory power for interest rate changes. The χ^2 tests for goodness-of-fit suggest that the Merton, Vasicek, and CIR SR models are misspecified. All three models have χ^2 values in excess of 6 and can be rejected at the 95% confidence level.⁴ These are followed by the Dothan, GBM, Brennan-Schwartz, CIR VR, and CEV models, all of which have low χ^2 values. Except for the CEV model, these latter models cannot be rejected at even the 90% confidence level.

An important property of this ranking is that it can be basically classified by γ values, that is, those models which assume $\gamma < 1$ are rejected and those which assume $\gamma \geq 1$ are not rejected. Furthermore, differences in the minimized GMM criterion values between models with the same value of γ are generally much smaller than differences in models with different values of γ . These results suggest that the relation between interest rate volatility and the level of r is the most-important feature of any dynamic model of the short-term riskless rate. This is significant since much of the debate about the relative merits of the various models has focused on other issues. For example, the Vasicek and Merton models are often criticized for allowing negative interest rates. Our results indicate that a far more serious drawback of these models is their implication that interest rate changes are homoskedastic.

The estimates of the unrestricted model provide a number of interesting insights about the dynamics of the short-term interest rate. First, there appears to be only weak evidence of mean reversion in the short-term rate; the parameter β is insignificant in the unrestricted model.⁵ This is important since it is the mean reversion feature which makes many term structure models so complex; the additional generality obtained by allowing the short-term interest rate process to be mean reverting may not justify the additional complexity. We also find that the conditional volatility of the process is highly sensitive to the level of the short-term yield; the unconstrained estimate of γ is 1.499. This result is important since this is much higher than the values used in most of the models.⁶ In particular, six of the eight nested models

⁴ Recall that since the unrestricted model represents an exactly identified system, the minimized GMM criterion value is exactly zero.

⁵ Note that by using a longer sample period or a more powerful test methodology (Dickey and Fuller (1979) and (1981), for example), it may be possible to reject the hypothesis that β equals zero for the unrestricted model.

⁶ Similar results are reported in Melino and Turnbull (1986) for LIBOR rates and by Chan, Karolyi, Longstaff, and Sanders (1992) for the Japanese Gensaki interest rate.

Table III
Estimates of Alternative Models for the Short-Term Interest Rate

The estimation horizon for r_t , the annualized one-month U.S. Treasury bill yield, is from June, 1964 to December, 1989 (306 observations). The parameters are estimated by the Generalized Method of Moments with t -statistics in parentheses. The R_j^2 statistics are computed as the proportion of the total variation of the actual yield changes ($j = 1$) and their volatility (squared yield changes) ($j = 2$) explained by the respective predictive values for each model. Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The χ^2 test statistics are reported with p -values in parentheses and associated degrees of freedom (d.f.). The parameters are estimated from the following discrete-time system of equations:

$$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{t+1}$$

$$E[\varepsilon_{t+1}] = 0, \quad E[\varepsilon_{t+1}^2] = \sigma^2, \quad E[\varepsilon_{t+1}^2 \gamma] = \sigma^2 \gamma$$

Model	α	β	σ^2	γ	χ^2 Test (p -value)	d.f.	R_1^2	R_2^2
Unrestricted	0.0408 (1.85)	-0.5921 (-1.55)	1.6704 (0.77)	1.4999 (5.95)			0.0259	0.2046
Merton	0.0055 (1.44)	0.0	0.0004 (7.27)	0.0	6.7579 (0.0341)	2	0.0000	0.0000
Vasicek	0.0154 (0.76)	-0.1779 (-0.50)	0.0004 (6.53)	0.0	8.8467 (0.0029)	1	0.0132	0.0000
CIR SR	0.0189 (0.94)	-0.2339 (-0.66)	0.0073 (7.55)	0.5	6.1512 (0.0131)	1	0.0164	0.0546
Dothan	0.0	0.0	0.1172 (7.99)	1.0	5.6018 (0.1327)	3	0.0000	0.1340
GEM	0.0	0.1101 (1.50)	0.1185 (8.04)	1.0	3.1541 (0.2066)	2	-0.0096	0.1329
Brennan-Schwartz	0.0242 (1.24)	-0.3142 (-0.92)	0.1185 (8.09)	1.0	2.2172 (0.1364)	1	0.0202	0.1395
CIR VR	0.0	0.0	1.5778 (8.00)	1.5	6.2067 (0.1019)	3	0.0000	0.2049
CEV	0.0	0.1026 (1.52)	0.5207 (0.62)	1.2795 (4.15)	3.0801 (0.0793)	1	-0.0098	0.1801

imply $0 \leq \gamma \leq 1$. The t -statistic for γ is 5.95, which is highly significant. Note that the estimate for γ is nearly two standard deviations above 1.00.⁷

As described in Section I, a number of the models are subnested within other models. The additional restrictions imposed by these subnested models can also be tested using the Newey-West tests. The associated p -values for the test statistics also provide us with a simple way of evaluating how a subnested model performs relative to the model in which it is nested.⁸ Table IV gives the results of the pairwise comparisons. As shown, the Merton model cannot be rejected against the alternative of the Vasicek model; the p -value of the Merton model against the Vasicek model is 0.5759. Similarly, the Dothan model cannot be rejected against the alternative of the GBM model. Neither the Dothan nor GBM model is rejected against the alternative of the Brennan-Schwartz or CEV models. It appears that no rejections can be observed in pairwise comparisons of models that make similar assumptions about the dependence of the conditional volatility on the level of the interest rate. These tests further illustrate that the primary distinguishing feature of these models is their ability to capture the time-varying volatility of the short-term interest rate.

B. An Alternative Measure of Model Performance

In order to gauge further the relative performance of the alternative nested models, we test their forecast power for interest rate changes. In addition, we test their forecast power for squared interest rate changes, which provide simple ex post measures of interest rate volatility. This is done by first computing the time series of conditional expected-yield changes and conditional variances for each model using the fitted values of (2) and (3). We then compute the proportion of the total variation in the ex post yield changes or squared yield changes that can be explained by the conditional expected-yield changes and conditional volatility measures, respectively. We refer to this as the coefficient of determination, or R^2 . These R^2 values provide information about how well each model is able to forecast the future level and volatility of the short-term rate. We propose these measures as alternative metrics for model comparison since they provide an intuitive way of evaluating the economic significance of differences between the interest rate models.

The results are presented in the last two columns of Table III. The first R^2 measure describes the fit of the various models for the actual yield changes. Except for the Dothan, Merton, and CIR VR models, which have no explana-

⁷ Because of the concern about the approximation error introduced by using discrete monthly intervals to estimate the continuous-time parameters of equation (1), we estimated the unrestricted model using daily data on three-month Treasury bill yields from 1978 to 1984 obtained from the Federal Reserve Bank of New York. Our estimate of γ is 1.6060, indistinguishable from the estimate when we used monthly data. The other parameter estimates were $\alpha = 0.1252$, $\beta = -0.9590$, and $\sigma^2 = 1.1189$.

⁸ Unlike the general specification tests in Table III, these various pairwise comparisons involve two overidentified models. As a result, they employ the weighting matrix of the unrestricted alternative model for the test statistic in each case.

Table IV
Pairwise Comparisons of Alternative Nested Models for the Short-Term Interest Rate

Models for the short-term interest rate r_t , the annualized one-month Treasury bill yields, are estimated using the Generalized Method of Moments from June 1964 to December 1989 (306 observations). The χ^2 test statistics are computed for hypothesis tests of the restrictions imposed by the nested alternative model following methods outlined in Newey and West (1987).

Alternative Unrestricted Model	Restricted Nested Model	χ^2 Test Statistic	p -value
Vasicek	Merton	$\chi_1^2 = 0.313$	0.5759
GBM	Dothan	$\chi_1^2 = 2.262$	0.1326
Brennan-Schwartz	Dothan	$\chi_2^2 = 3.795$	0.1499
Brennan-Schwartz	GBM	$\chi_1^2 = 1.595$	0.2066
CEV	Dothan	$\chi_2^2 = 2.856$	0.2398
CEV	GBM	$\chi_1^2 = 0.6649$	0.4148
CEV	CIR VR	$\chi_1^2 = 2.995$	0.0835

tory power for interest rate changes, the models appear similar in their forecast ability. Specifically, the remaining models explain only 1 to 3% of the total variation in yield changes.⁹ This is not the case for the volatility of interest rate changes, however. The proportion of the total variation in volatility captured by the various models ranges from 5.46% for the CIR SR model to 20.49% for the CIR VR model. Note that the R^2 s for the Merton and Vasicek models are zero since these models imply that the volatility of interest rate changes is constant. Interestingly, the ranking of the models based on their predictive power for the volatility of interest rate changes is closely aligned to the ranking implied by the minimized GMM criterion values. Figure 1 presents the time series plot of the absolute value of the interest rate changes and the estimated conditional volatility estimates from the unrestricted model.

C. Structural Breaks

Many empirical studies of the term structure have concluded that the shift in Federal Reserve monetary policy in October 1979 resulted in a structural break in the interest rate process.¹⁰ Our framework allows us to test for a

⁹ Since we compute the coefficients of determination also for models that have no intercept, such as CEV and GBM, negative R^2 values are possible.

¹⁰ For example, see Huizinga and Mishkin (1984), Clarida and Friedman (1984), Campbell (1987), and Sanders and Unal (1988). Antoncic (1986) has shown that the shift toward higher volatility of the real rate of interest occurred in April 1980 and not October 1979.

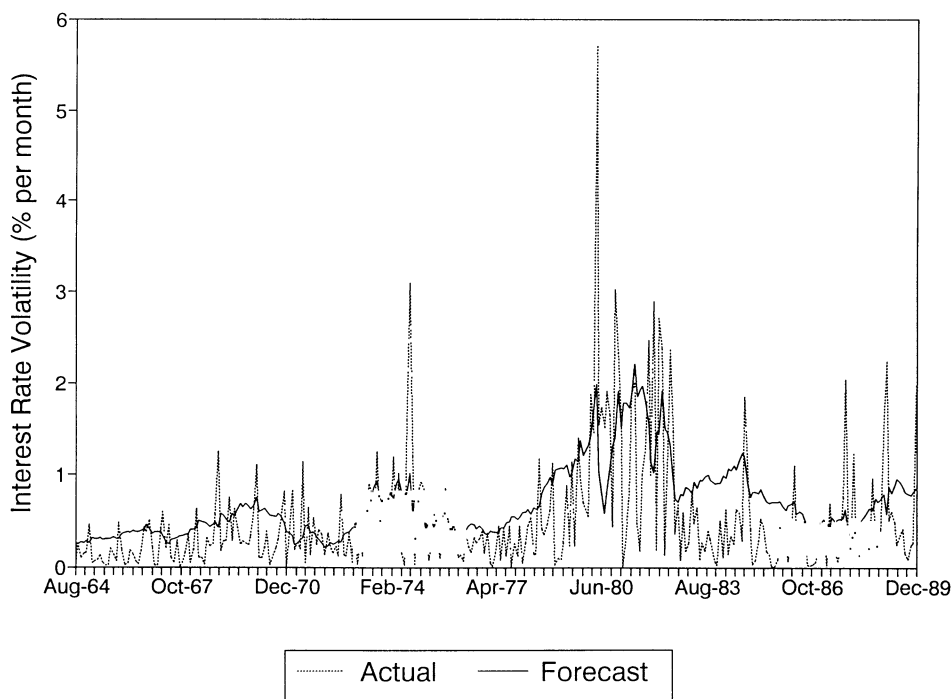


Figure 1. Forecasts of annualized ex post volatility of the short-term interest rate using the unrestricted model. The ex post volatility (*dashed line*) is measured as the absolute value of the month-to-month change in the one-month Treasury bill yield from August 1964 to December 1989. The forecast (*solid line*) is the square root of the conditional variance implied by the estimates of the unrestricted model.

structural break by introducing a dummy variable, D_t , that equals unity for all monthly observations following October 1979 and zero otherwise. We allow for a dummy shift for each of the parameters that are estimated in the respective models (ranging, therefore, from one to four additional parameters). Specifically, our model takes the form,

$$r_{t+1} - r_t = (\alpha + D_t \delta_1) + (\beta + D_t \delta_2)r_t + \varepsilon_{t+1} \tag{11}$$

$$E[\varepsilon_{t+1}] = 0, \quad E[\varepsilon_{t+1}^2] = (\sigma^2 + D_t \delta_3)r_t^{2(\gamma + D_t \delta_4)}, \tag{12}$$

where the δ parameters are those associated with the dummy shift variables. Because we introduce four more parameters into the system of equations for the unrestricted model, four more orthogonality conditions must also be established. These are given by requiring that ε_{t+1} and $\varepsilon_{t+1}^2 - (\sigma^2 + D_t \delta_3)r_t^{2(\gamma + D_t \delta_4)}$ be orthogonal to the instrument vector, $[1, r_t, D_t, D_t r_t]$. In order to test the null hypothesis of a structural break, we compute the minimized GMM criterion values associated with these expanded models and

compare them with those of the models in Table III.¹¹ These tests are reported in Table V.

The empirical results are striking. The χ^2 test statistic for the unrestricted, CIR VR, Brennan-Schwartz, and CEV models are not significant at the 95% confidence level. Thus, there is no evidence of a structural break in October 1979 for the models that capture the dependence of the conditional volatility on the level of the interest rate. These results are reassuring since they suggest that current interest rate models may be rich enough to capture the change in interest rate behavior evident in the post-1979 period. These results also raise the possibility that previous tests for structural breaks may be misspecified because of their failure to model the conditional heteroskedasticity in interest rate changes correctly.

V. Bond Option Valuation: An Example

We have shown that the models differ significantly in their ability to capture the dynamics of the short-term interest rate. Another important issue, however, is whether the models differ significantly in terms of their implications for valuing interest-rate-contingent claims. In this section, we compare the prices of bond options implied by each of the different interest rate processes. A key feature of this comparison is that we use the parameter estimates for each model that represent the best fit to a common time series of Treasury bill yields. In this way, we provide the same benchmark for comparison across models.

In this comparison, we focus on a 2-year call option on a default-free 30-year coupon bond with varying degrees of moneyness. For each interest rate model, we use the parameter estimates presented in Table III computed from the time series of monthly Treasury bill yields from 1964 to 1989. To ensure comparability across models, we assume that the local expectations hypothesis holds, so that the expected return on all interest-rate-sensitive contingent claims is the riskless rate. Using these parameter estimates and the different assumptions about the evolution of the short-term rate, we use standard option-pricing methodology to determine the value of the call option by solving the appropriate partial differential equation subject to the boundary conditions using numerical techniques. The computational methodology is discussed in Buser, Hendershott, and Sanders (1990).

The results of the experiment are presented in Table VI for the Vasicek, CIR SR, Brennan-Schwartz, and unrestricted models. We compute values for call options with strike prices of 95, 100, and 105, given the par value of the coupon bond is 100. Our findings indicate that the long-run mean of the short-term interest rate decreases and the implied slope of the term structure increases for models that allow greater sensitivity of the conditional volatility

¹¹ Again, following the test procedures outlined in Newey and West (1987), we use the weighting matrices associated with the expanded models in computing the test statistics in equation (10).

Table V
Tests for Structural Shifts in Alternative Models for the Short-Term Interest Rate

We estimate the parameters for different processes for r_t , the annualized one-month U.S. Treasury bill yields, across monetary regimes by the Generalized Method of Moments with t -statistics in parentheses. An exclusion test is evaluated for the joint significance of the set of dummy variable coefficients for each model associated with dummy variable D_t , equal to unity after the change in Federal Reserve Monetary policy in October, 1979, and zero otherwise. The χ^2 test statistics are computed following the methods outlined in Newey and West (1987) with p -values in parentheses and associated degrees of freedom (d.f.). The number of dummy variables k differs depending on the number of parameter constraints that are imposed by the respective models. The critical values for χ_k^2 are 9.49 at 5% significance for $k = 4$; 9.35 at 5% for $k = 3$; 5.99 at 5% for $k = 2$; and 3.84 at 5% for $k = 1$. The parameters are estimated from the following discrete-time system of equations:

$$r_{t+1} - r_t = (\alpha + D_t \delta_1) + (\beta + D_t \delta_2)r_t + \varepsilon_{t+1}$$

$$E[\varepsilon_{t+1}] = 0, \quad E[\varepsilon_{t+1}^2] = (\sigma^2 + D_t \delta_3)r_t^{2(\gamma + D_t \delta_4)}$$

Model	System Parameters				Dummy Parameters				χ^2 Test (p -value) d.f.	
	α	β	σ^2	γ	δ_1	δ_2	δ_3	δ_4		
Unrestricted	0.0174 (0.87)	-0.2213 (-0.53)	1.3846 (0.45)	1.4808 (3.83)	0.0608 (1.20)	-0.0751 (-1.01)	-0.2082 (-0.06)	-0.0641 (-0.12)	2.2939 (0.6818)	4
Merton	0.0069 (1.93)	0.0	0.0002 (6.83)	1.0	-0.0021 (-0.21)	0.0	0.0006 (3.66)	0.0	13.813 (0.0010)	2
Vasicek	-0.0009 (-0.05)	0.1612 (0.42)	0.0002 (6.83)	0.0	0.0374 (0.79)	-0.6019 (-0.87)	0.0006 (3.53)	0.0	14.380 (0.0024)	2
CIR SR	0.0020 (0.11)	0.1025 (0.27)	0.0041 (7.63)	0.5	0.0421 (0.90)	-0.6336 (-0.92)	0.0074 (3.34)	0.0	12.642 (0.0054)	3
Dothan	0.0	0.0	0.0778 (7.81)	1.0	0.0	0.0	0.0666 (2.46)	0.0	6.0575 (0.0138)	1
GEM	0.0	0.1391 (1.89)	0.0819 (8.10)	1.0	0.0	-0.0862 (-0.60)	0.0644 (2.33)	0.0	6.6672 (0.0356)	2
Brennan-Schwartz	0.0078 (0.43)	-0.0174 (-0.05)	0.0825 (8.13)	1.0	0.0522 (1.12)	-0.7138 (-1.04)	0.0690 (2.47)	0.0	7.1759 (0.0665)	3
CIR VR	0.0	0.0	1.4390 (7.97)	1.5	0.0	0.0	0.1653 (0.49)	0.0	1.8559 (0.1731)	1
CEV	0.0	0.1311 (1.77)	0.4921 (0.43)	1.3059 (3.26)	0.0	-0.0796 (-0.56)	-0.1352 (-0.10)	-0.1244 (-0.21)	1.4090 (0.7034)	3

Table VI

**Values of a 2-Year Call Option on a 30-Year Coupon Bond with
Par Value of 100 for Alternative Models and Conditions**

The coupon on the 30-year coupon bond is set to initially price the bond at par. The parameter values α , β , and σ^2 for the alternative models are shown in Table III; the associated values for γ are presented below. The initial value for the short-term rate of interest is $r_0 = 0.06$. The long-run mean is given by μ and the "implied" slope of the term structure is expressed as the basis-point spread between the 30-year bond and the short-term rate of interest. Numerical procedures are used to value the option under the assumption that the local expectations hypothesis holds.

Model	Value of γ	Value of μ	Implied Slope	Exercise Price	Call Value
Vasicek	0.0	0.0866	168.34	95	6.02
				100	3.16
				105	1.47
CIR SR	0.5	0.0811	142.85	95	5.79
				100	2.56
				105	0.76
Brennan-Schwartz	1.0	0.0747	110.56	95	5.45
				100	1.74
				105	0.11
Unrestricted	1.5	0.0689	74.27	95	5.17
				100	1.03
				105	0.00

of interest rate changes to the level of the interest rate. More importantly, the results confirm that the call option values vary significantly across term structure models. Call option prices range from 6.02 for the Vasicek model to 5.17 for the unrestricted model when out-of-the-money and from 1.47 for the Vasicek model to 0.00 for the unrestricted model when in-the-money. Our findings suggest that these interest rate models have very different implications not only for capturing the dynamics of the short-term rate but also for valuing interest-rate-contingent claims.

VI. Conclusion

In this paper, we compare eight competing models of short-term interest rate dynamics in order to determine which model best fits the short-term Treasury bill yield data. All of the models are nested within a simple framework that allows us to compare the models directly to each other.

The results of the tests for one-month Treasury bills indicate that it is critical to model volatility correctly. The models that best describe the dynamics of interest rates over time are those that allow the conditional volatility of interest rate changes to be highly dependent on the level of the interest rate. Surprisingly, the most commonly used models (Vasicek (1977) and CIR SR (1985)) perform poorly relative to less well-known models such as Dothan (1978) and CIR VR (1980). We find that there is no evidence of a

structural shift in the interest rate process in October 1979 for the models which capture the conditional volatility of the interest rate process.

These results have important implications for current models of the term structure. We have shown that one of the most important features of the term structure is the dependence of its volatility on the level of the interest rate. Most commonly used term structure models, however, fail to capture this dependence. Since interest rate volatility is of fundamental importance in valuing contingent claims and hedging interest rate risk, our results suggest that future models of the term structure should focus on this relation.

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