PHY 444. Quantum Mechanics — Fall 2021
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The Levi-Civita Symbol

A more compact form for the cross product is obtained by introducing the completely antisymmetric symbol, ϵ_{ijk}.[1] This symbol is defined by the relations[2]

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1,$$

and

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1,$$

and all other combinations, like ϵ_{113}, vanish. Note that all indices must differ. Also, if the order is a cyclic permutation of $\{1, 2, 3\}$, then the value is $+1$. For this reason ϵ_{ijk} is also called the permutation symbol or the Levi-Civita permutation symbol. We can also indicate the index permutation more generally using the following identities:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}.$$

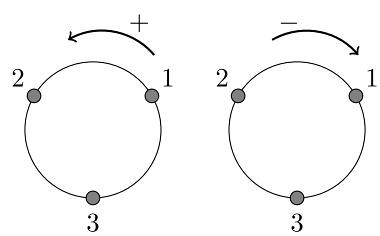

Figure 1: The sign for the permutation symbol can be determined from a simple cyclic diagram similar to that for the cross product. Arrange the numbers from 1 to 3 on a circle. If the needed computation goes counterclockwise, then the sign is positive, otherwise it is negative.

Returning to the cross product, we can introduce the standard basis $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$. With this notation, we have that

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk} \mathbf{e}_k. \tag{1}$$

Example 1. *Compute the cross product of the basis vectors $\mathbf{e}_2 \times \mathbf{e}_1$ using the permutation symbol. A straight*

[1] The completely antisymmetric symbol, or permutation symbol, ϵ_{ijk}. This is also called the Levi-Civita symbol, named after the Italian mathematician Tullio Levi-Civita (1873-1941). He is known for work in tensor calculus and was the doctoral student of the inventor of tensor calculus, Gregorio Ricci-Curbastro (1853-1925).

[2] Here we are only interested in the three-dimensional case. For example, in two dimensions, we have $\epsilon_{12} = -\epsilon_{21} = 1$ and in four dimensions, $\epsilon_{1234} = -\epsilon_{2134} = 1$, etc

forward application of the definition of the cross product,

$$\begin{aligned}
\mathbf{e}_2 \times \mathbf{e}_1 &= \sum_{k=1}^{3} \epsilon_{21k} \mathbf{e}_k \\
&= \epsilon_{211} \mathbf{e}_1 + \epsilon_{212} \mathbf{e}_2 + \epsilon_{213} \mathbf{e}_3 \\
&= -\mathbf{e}_3.
\end{aligned} \tag{2}$$

It is helpful to write out enough terms in these sums until you get familiar with manipulating the indices. Note that the first two terms vanished because of repeated indices. In the last term we used $\epsilon_{213} = -1$.

Starting with the component form of the cross product in Equation (1), we now write out the general cross product as

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \mathbf{e}_i \times \mathbf{e}_j \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \left(\sum_{k=1}^{3} \epsilon_{ijk} \mathbf{e}_k \right) \\
&= \sum_{i,j,k=1}^{3} \epsilon_{ijk} u_i v_j \mathbf{e}_k.
\end{aligned} \tag{3}$$

Note that the last sum is a triple sum over the indices i, j, and k.

Example 2. *Let* $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ *and* $\mathbf{v} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$. *Compute* $\mathbf{u} \times \mathbf{v}$. *We can compute this easily using determinants.*

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 1 & 5 & 4 \end{vmatrix} \\
&= \begin{vmatrix} -3 & 0 \\ 5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} \mathbf{k} \\
&= -12\mathbf{i} - 8\mathbf{j} + 13\mathbf{k}.
\end{aligned} \tag{4}$$

Using the permutation symbol to compute this cross product, we have

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \epsilon_{123} u_1 v_2 \mathbf{k} + \epsilon_{231} u_2 v_3 \mathbf{i} + \epsilon_{312} u_3 v_1 \mathbf{j} \\
&\quad + \epsilon_{213} u_2 v_1 \mathbf{k} + \epsilon_{132} u_1 v_3 \mathbf{j} + \epsilon_{321} u_3 v_2 \mathbf{i} \\
&= 2(5)\mathbf{k} + (-3)4\mathbf{i} + (0)1\mathbf{j} - (-3)1\mathbf{k} - (2)4\mathbf{j} - (0)5\mathbf{i} \\
&= -12\mathbf{i} - 8\mathbf{j} + 13\mathbf{k}.
\end{aligned} \tag{5}$$

Sometimes it is useful to note from Equation (3) that the kth component of the cross product is given by

$$\boxed{(\mathbf{u} \times \mathbf{v})_k = \sum_{i,j=1}^{3} \epsilon_{ijk} u_i v_j.}$$

In more advanced texts, or in the case of relativistic computations with tensors, the summation symbol is suppressed. For this case, one writes

$$(\mathbf{u} \times \mathbf{v})_k = \epsilon_{ijk} u^i v^j,$$

where it is understood that summation is performed over repeated indices and vector components are defined by $\mathbf{u} = (u^1, u^2, u^3)$. This is called the Einstein summation convention.[3]

The cross product can be written as a determinant,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix} \\ &= \begin{vmatrix} u^2 & u^3 \\ v^2 & v^3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u^1 & u^3 \\ v^1 & v^3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u^1 & u^2 \\ v^1 & v^2 \end{vmatrix} \mathbf{k} \\ &= \epsilon_{ij1} u^i v^j \mathbf{i} + \epsilon_{ij2} u^i v^j \mathbf{j} + \epsilon_{ij3} u^i v^j \mathbf{k}. \end{aligned} \tag{6}$$

Writing the unit vectors as $\mathbf{e}^k$, $k = 1, 2, 3$, we can write the cross product using the full permutation symbol,

$$\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u^i v^j \mathbf{e}^k.$$

Note that the Einstein summation convention has suppressed a triple sum,

$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} u^i v^j \mathbf{e}^k.$$

Once again, we can read off the k-the component as

$$(\mathbf{u} \times \mathbf{v})_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ijk} u^i v^j.$$

We can also write a 3×3 determinant using the Levi-Civita symbol. We start with the determinant in Equation (6) and replace the entries using

$$\begin{aligned} \mathbf{a}_1 &= (\mathbf{i}, \mathbf{j}, \mathbf{k}) \\ \mathbf{a}_2 &= \mathbf{u} \\ \mathbf{a}_3 &= \mathbf{v}. \end{aligned} \tag{7}$$

This gives the determinant in terms of the Levi-Civita symbol.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k}. \tag{8}$$

Here we included the triple sum in order to emphasize the hidden summations.

[3]The Einstein summation convention is used to suppress summation notation. In general relativity, one employs raised indices, so that vector components are written in the form u^i. The convention then requires that one only sums over a combination of one lower and one upper index. Thus, we would write $\epsilon_{ijk} u^i v^j$.

$$\begin{vmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 4 & 0 & -3 \\ 2 & 4 & -1 & 2 & 4 \end{vmatrix}$$

Figure 2: Diagram for computing determinants.

Example 3. *Compute the determinant* $\begin{vmatrix} 1 & 0 & 2 \\ 0 & -3 & 4 \\ 2 & 4 & -1 \end{vmatrix}$.

We insert the components of each row into the expression for the determinant:

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 2 \\ 0 & -3 & 4 \\ 2 & 4 & -1 \end{vmatrix} &= \epsilon_{123}a_{11}a_{22}a_{33} + \epsilon_{231}a_{12}a_{23}a_{31} + \epsilon_{312}a_{13}a_{21}a_{32} \\ &\quad + \epsilon_{213}a_{12}a_{21}a_{33} + \epsilon_{132}a_{11}a_{23}a_{32} + \epsilon_{321}a_{13}a_{22}a_{31} \\ &= \epsilon_{123}(1)(-3)(-1) + \epsilon_{231}(0)(4)(2) + \epsilon_{312}(2)(0)(4) \\ &\quad + \epsilon_{213}(0)(0)(-1) + \epsilon_{132}(1)(4)(4) + \epsilon_{321}(2)(-3)(2) \\ &= 3 + 0 + 0 - 0 - 16 - (-12) \\ &= -1. \end{aligned} \tag{9}$$

Note that if one adds copies of the first two columns, as shown in Figure 3, then the products of the first three diagonals, downward to the right (dashed), give the positive terms in the determinant computation and the products of the last three diagonals, downward to the left (dotted), give the negative terms.

One useful identity is[4]

$$\boxed{\epsilon_{jki}\epsilon_{j\ell m} = \delta_{k\ell}\delta_{im} - \delta_{km}\delta_{i\ell},}$$

where δ_{ij} is the Kronecker delta. Note that the Einstein summation convention is used in this identity; i.e., summing over j is understood. So, the left side is really a sum of three terms:

$$\epsilon_{jki}\epsilon_{j\ell m} = \epsilon_{1ki}\epsilon_{1\ell m} + \epsilon_{2ki}\epsilon_{2\ell m} + \epsilon_{3ki}\epsilon_{3\ell m}.$$

This identity is simple to understand. For nonzero values of the Levi-Civita symbol, we have to require that all indices differ for each factor on the left side of the equation: $j \neq k \neq i$ and $j \neq \ell \neq m$. Since the first two slots are the same j, and the indices only take values $1, 2,$ or 3, then either $k = \ell$ or $k = m$. This will give terms with factors of $\delta_{k\ell}$ or δ_{km}. If the former is true, then there is only one possibility for the third slot, $i = m$. Thus, we have a term $\delta_{k\ell}\delta_{im}$. Similarly, the other case yields the second term on the right side

[4]Technically, we should be using upper and lower indices to indicate summation. Thus,

$$\sum_{i,j,k=1}^{3} \epsilon_{jki}\epsilon^{j\ell m} = \epsilon_{jki}\epsilon^{j\ell m} = \delta_k^\ell \delta_i^m - \delta_k^m \delta_i^\ell,$$

of the identity. We just need to get the signs right. Obviously, changing the order of ℓ and m will introduce a minus sign. A little care will show that the identity gives the correct ordering.

Other identities involving the permutation symbol are

$$\epsilon_{mjk}\epsilon_{njk} = 2\delta_{mn},$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6.$$

We will end this section by recalling triple products. There are only two ways to construct triple products. Starting with the cross product $\mathbf{b} \times \mathbf{c}$, which is a vector, we can multiply the cross product by another vector, $\mathbf{a}$, to either obtain a scalar or a vector.

In the first case we have the triple scalar product, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Actually, we do not need the parentheses. Writing $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ could only mean one thing. If we computed $\mathbf{a} \cdot \mathbf{b}$ first, we would get a scalar. Then, the result would be a multiple of $\mathbf{c}$, which is not a scalar. So, dropping the parentheses would mean that we want the triple scalar product by convention.

Let's consider the component form of this product. We will use the Einstein summation convention and the fact that the permutation symbol is cyclic in ijk. Using $\epsilon_{jki} = \epsilon_{ijk}$,

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{a}_i (\mathbf{b} \times \mathbf{c})_i \\
&= \epsilon_{jki} a_i b_j c_k \\
&= \epsilon_{ijk} a_i b_j c_k \\
&= (\mathbf{a} \times \mathbf{b})_k c_k \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \qquad (10)
\end{aligned}$$

In order to appreciate the summation convention, here is the same computation with the explicit sums shown.

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \sum_{i=1}^{3} \mathbf{a}_i (\mathbf{b} \times \mathbf{c})_i \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{jki} a_i b_j c_k \\
&= \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ijk} a_i b_j c_k \\
&= \sum_{k=1}^{3} (\mathbf{a} \times \mathbf{b})_k c_k \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \qquad (11)
\end{aligned}$$

We have proven that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Now, imagine how much writing would be involved if we had expanded everything out in terms of all of the components. Well, this might be a good time to show how one proves identities using the components

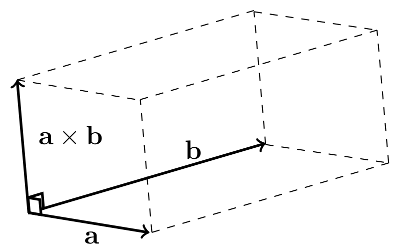

Figure 3: Three non-coplanar vectors define a parallelepiped. The volume is given by the triple scalar product, $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$.

without the simplifying notation. So, we expand the scalar triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ as

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{a}\cdot[(b_2c_3-b_3c_2)\mathbf{i}+(b_3c_1-b_1c_3)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k}]\\ &= a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\\ &= a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1.\end{aligned}$$

Now we stare at the expression to figure out how we can regroup the terms to obtain $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. You could just expand the terms for this expression and see if you get the same terms, or we could regroup the terms based on the components of $\mathbf{c}$. Thus,

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (a_2b_3-a_3b_2)c_1+(a_3b_1-a_1b_3)c_2+(a_1b_2-a_2b_1)c_3).$$

Now, we can see that this is indeed $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

Note that this result suggests that the triple scalar product can be computed by just computing a determinant. In particular, the third equation in (10) gives

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \epsilon_{ijk}a_ib_jc_k\\ &= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.\end{aligned} \tag{12}$$

There is a geometric interpretation of the scalar triple product. Consider the three vectors in Figure 3. If they do not all lie in a plane, then they form the sides of a parallelepiped. The cross product $\mathbf{a}\times\mathbf{b}$ gives the area of the base as we had seen earlier. The cross product is perpendicular to this base. The dot product of $\mathbf{c}$ with this cross product gives the height of the parallelepiped. So, the volume of the parallelepiped is the height times the base, or the triple scalar product. In general, one gets a signed volume, as the cross product could be pointing below the base.

The second type of triple product is the triple cross product,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \epsilon_{mnj}\epsilon_{ijk}a_ib_mc_n\mathbf{e}_k.$$

In this case we cannot drop the parentheses as this would lead to a real ambiguity. Lets think a little about this product. The vector $\mathbf{b}\times\mathbf{c}$ is a vector that is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$. Computing the triple cross product would then produce a vector perpendicular to $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$. But the later vector is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ already. Therefore, the triple cross product must lie in the plane spanned by these vectors. In fact, there is an identity that tells us exactly the right combination of vectors $\mathbf{b}$ and $\mathbf{c}$. It is given by

$$\boxed{\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \qquad (13)}$$

This rule is called the BAC-CAB rule because of the order of the vectors on the right side of this equation. We prove this identity later in the chapter using components. In the next example we prove it using the Levi-Civita symbol.

Example 4. *Prove that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ *using the Levi-Civita symbol.*

We can prove the BAC-CAB rule using the permutation symbol and some identities. We first note that vectors can be expanded in a basis as $\mathbf{a} = a_i\mathbf{e}_i$ *and* $\mathbf{b}\times\mathbf{c} = (\mathbf{b}\times\mathbf{c})_j\mathbf{e}_j$. *We will also need the cross products* $\mathbf{e}_i\times\mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$ *and* $\mathbf{b}\times\mathbf{c} = \epsilon_{mnj}b_mc_n\mathbf{e}_j$. *Then, the computation proceeds as follows:*

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= (a_i\mathbf{e}_i)\times((\mathbf{b}\times\mathbf{c})_j\mathbf{e}_j) \\
&= a_i(\mathbf{b}\times\mathbf{c})_j(\mathbf{e}_i\times\mathbf{e}_j) \\
&= a_i(\mathbf{b}\times\mathbf{c})_j\epsilon_{ijk}\mathbf{e}_k \\
&= \epsilon_{mnj}\epsilon_{ijk}a_ib_mc_n\mathbf{e}_k \qquad (14)
\end{aligned}$$

In order to evaluate the product of the Levi-Civita symbols, we use the identity

$$\epsilon_{mnj}\epsilon_{ijk} = \delta_{mk}\delta_{ni} - \delta_{mi}\delta_{nk}$$

and the properties of the Kronecker delta functions. Thus, we obtain

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \epsilon_{mnj}\epsilon_{ijk}a_ib_mc_n\mathbf{e}_k \\
&= a_ib_mc_n\left(\delta_{mk}\delta_{ni} - \delta_{mi}\delta_{nk}\right)\mathbf{e}_k \\
&= a_nb_mc_n\mathbf{e}_m - a_mb_mc_n\mathbf{e}_n \\
&= (b_m\mathbf{e}_m)(c_na_n) - (c_n\mathbf{e}_n)(a_mb_m) \\
&= \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \qquad (15)
\end{aligned}$$

Line 3 is obtained by noting that for the first term, $i = n$ *and* $k = m$. *For the second term in line 3,* $i = m$ *and* $k = n$. *We then group the results by noting that*

$$b_m\mathbf{e}_m = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = \mathbf{b},$$

and

$$c_na_n = a_1c_1 + a_2c_2 + a_3c_3 = \mathbf{a}\cdot\mathbf{c}.$$

A similar computation simplifies the second term.

Example 5. *Show that*

$$\hat{\mathbf{L}}^2 = (\hat{\mathbf{r}}\times\hat{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}}) = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 + i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2.$$

$$
\begin{aligned}
\hat{\mathbf{L}}^2 &= \sum_{i=1}^{3} (\hat{\mathbf{r}} \times \hat{\mathbf{p}})_i \cdot (\hat{\mathbf{r}} \times \hat{\mathbf{p}})_i \\
&= \sum_{i=1}^{3} \left(\sum_{j,k=1}^{3} \epsilon_{ijk} \hat{x}_j \hat{p}_k \right) \left(\sum_{\ell,m=1}^{3} \epsilon_{i\ell m} \hat{x}_\ell \hat{p}_m \right) \\
&= \sum_{j,k=1}^{3} \sum_{\ell,m=1}^{3} \left[\sum_{i=1}^{3} \epsilon_{ijk} \epsilon_{i\ell m} \right] \hat{x}_j \hat{p}_k \hat{x}_\ell \hat{p}_m \\
&= \sum_{j,k=1}^{3} \sum_{\ell,m=1}^{3} \left[\delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell} \right] \hat{x}_j \hat{p}_k \hat{x}_\ell \hat{p}_m \\
&= \sum_{j,k=1}^{3} \left[\hat{x}_j \hat{p}_k \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_k \hat{x}_k \hat{p}_j \right] \\
&= \sum_{j,k=1}^{3} \left[\hat{x}_j (\hat{x}_j \hat{p}_k - i\hbar \delta_{jk}) \hat{p}_k - \hat{x}_j \hat{p}_k (\hat{p}_j \hat{x}_k + i\hbar \delta_{jk}) \right] \\
&= \sum_{j,k=1}^{3} \left[\hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - 2i\hbar \delta_{jk} \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_k \hat{p}_j \hat{x}_k \right] \\
&= \sum_{j,k=1}^{3} \left[\hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - 2i\hbar \delta_{jk} \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_j (\hat{x}_k \hat{p}_k - i\hbar \delta_{kk}) \right] \\
&= \sum_{j,k=1}^{3} \left[\hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k + i\hbar \delta_{jk} \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_j \hat{x}_k \hat{p}_k \right] \\
&= \hat{\mathbf{r}}^2 \hat{\mathbf{p}}^2 + i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2. \qquad (16)
\end{aligned}
$$

Here we used $[x_j, p_k] = i\hbar\delta_{jk}$, *or* $x_j p_k = p_k x_j + i\hbar\delta_{jk}$. *Also,* $\sum_{k=1}^{3} \delta_{kk} = 3$.

Example 6. *Show that*

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k.$$

Noting that $\hat{L}_i = \sum_{k,\ell} \epsilon_{ikl} \hat{x}_k \hat{p}_\ell$,

$$[\hat{L}_i, \hat{L}_j] = \sum_{k,\ell} \sum_{m,n} \epsilon_{ikl} \epsilon_{ikl} [\hat{x}_k \hat{p}_\ell, \hat{x}_k \hat{p}_\ell]. \qquad (17)$$

The commutator can be expanded as

$$
\begin{aligned}
[\hat{x}_k \hat{p}_\ell, \hat{x}_k \hat{p}_\ell] &= \hat{x}_k [\hat{p}_\ell, \hat{x}_m \hat{p}_n] + [\hat{x}_k, \hat{x}_m \hat{p}_n] \hat{p}_\ell \\
&= \hat{x}_k [\hat{p}_\ell, \hat{x}_m] \hat{p}_n + \hat{x}_m [\hat{x}_k, \hat{p}_n] \hat{p}_\ell \\
&= -i\hbar \delta_{\ell m} \hat{x}_k \hat{p}_n + i\hbar \delta_{kn} \hat{x}_m \hat{p}_\ell. \qquad (18)
\end{aligned}
$$

Then, we have

$$\begin{aligned}
[\hat{L}_i, \hat{L}_j] &= \sum_{k,\ell}\sum_{m,n} \epsilon_{ikl}\epsilon_{ikl}[\hat{x}_k\hat{p}_\ell, \hat{x}_k\hat{p}_\ell] \\
&= \sum_{k,\ell}\sum_{m,n} \epsilon_{ikl}\epsilon_{ikl}\left(-i\hbar\delta_{\ell m}\hat{x}_k\hat{p}_n + i\hbar\delta_{kn}\hat{x}_m\hat{p}_\ell\right) \\
&= i\hbar\sum_{k}\sum_{m,n} \epsilon_{mik}\epsilon_{mjn}\hat{x}_k\hat{p}_n - i\hbar\sum_{k,\ell}\sum_{m} \epsilon_{ki\ell}\epsilon_{kjm}\hat{x}_m\hat{p}_\ell \\
&= i\hbar\sum_{k}\left(\delta_{ij}\hat{x}_k\hat{p}_k\right) - i\hbar\hat{x}_j\hat{p}_i - i\hbar\sum_{\ell}\left(\delta_{ij}\hat{x}_\ell\hat{p}_\ell\right) + i\hbar\hat{x}_i\hat{p}_j \\
&= i\hbar(\hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i) \\
&= i\hbar\sum_{k} \epsilon_{ijk}\hat{L}_k. \qquad (19)
\end{aligned}$$

Here we have noted that

$$\begin{aligned}
\sum_{k} \epsilon_{ijk}\hat{L}_k &= \sum_{k}\sum_{\ell,m} \epsilon_{ijk}\epsilon_{klm}\hat{x}_\ell\hat{p}_m \\
&= \sum_{\ell,m}(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\hat{x}_\ell\hat{p}_m \\
&= \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i. \qquad (20)
\end{aligned}$$