Relativity and Electromagnetism

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Reading

The emperor's new mind, R.Penrose, pages 150-201, OUP (1989).

Notes on relativity and electromagnetism, N.M.J.Woodhouse, Institute Lecture Notes.

Introduction to special relativity, W. Rindler. OUP 1982.

1 Introduction

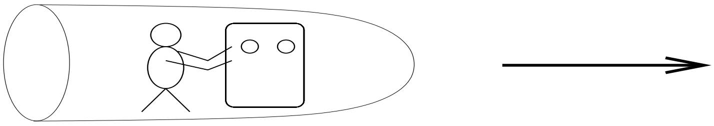
This course is concerned with the development of special relativity from the attempt to reconcile

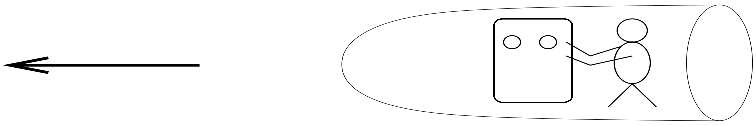
1. The principle of relativity (PR).
   In classical mechanics this is Galilean relativity: physics does not distinguish between frames with constant relative velocities.

2. Maxwell's theory of electromagnetism in which the speed of light is a fundamental constant.

The first two lectures will explain what these are and why they are in conflict. We then resolve the conflict, not by sacrificing (1) or (2), but by integrating space and time into a single geometric structure in which a 'special' principle of relativity operates, rather than the Galilean one.

1.1 Galileo's principle of relativity Two space ships pass in empty space. In each ship, the passengers imagine they are at rest and that the other ship is moving. Is there some physical test that will resolve the question?



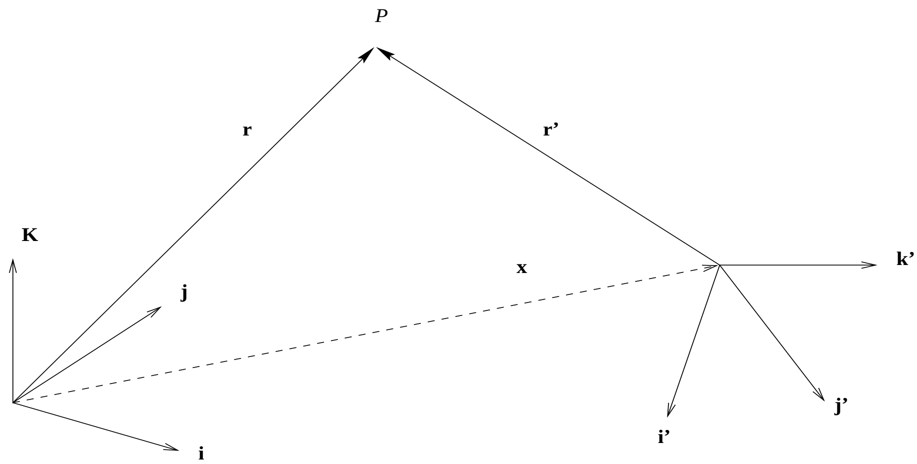


1. Yes, if one is accelerating and the other is not: one can 'feel' acceleration.

2. No if neither ship is accelerating: there is no experiment that will distinguish
   'A is at rest and B is moving at constant speed'
   from
   'B is at rest and B ... '

Galileo's principle: There is no absolute standard of rest; only relative motion is measureable.

1.2 Moving frames of reference Galileo's Principle follows from Newton's laws (it was originally Newton's 4th law). In Newtonian mechanics, we can make the statement more precise. Two frames of reference, R and R':



We have $\mathbf{r} = \mathbf{r}' + \mathbf{x}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and $\mathbf{r}' = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'$.

Suppose now that
(1) R' is not rotating relative to R and
(2) $\ddot{\mathbf{x}} = 0$ (uniform motion)
then $(\mathrm{d}^2\mathbf{r}/\mathrm{d}t^2)_R = \ddot{\mathbf{r}}_R = \ddot{\mathbf{r}}'_{R'}$ and the acceleration of P is measured to be the same in both frames.
Then: if Newton's laws hold in R they will hold in R'.

The two coordinate systems are then related by:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = H \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} t + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad (*)$$

where $\mathbf{x} = \mathbf{v}t + \mathbf{c}$, H is the rotation from the frame $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$, and H, $\mathbf{v}$ and $\mathbf{c}$ are constants.

Axiom of classical mechanics $\exists$ a preferred class of frames of reference, called *inertial frames*, which are related by transformations of the form (*) in which Newton's law holds (Newton IV).
(We only need one, as then the transformations above give a family.)

Principle of relativity: All inertial frames are on an equal footing.
(Einstein kept principle of relativity, but changed $(*)$).

Galilean transformations

Clearly we can also translate the time coordinate

$$t = t' + c_0\,, \qquad c_0 = \text{constant}$$

and write (*) together with this as

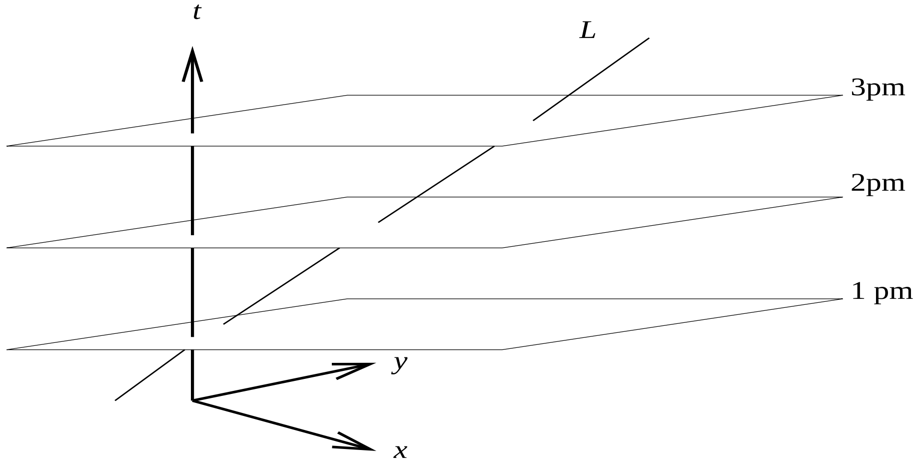
$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & H_{11} & H_{12} & H_{13} \\ v_2 & H_{21} & H_{22} & H_{23} \\ v_3 & H_{31} & H_{32} & H_{33} \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

This is a *Galilean transformation* and it is an affine linear transformation of space *and* time.

Galilean transformations make up the *symmetry group* of classical space-time. To see this we need to specify what structures the transformations are to preserve.

1.3 Classical space-time We must get accustomed to thinking of space and time as coordinates on a single four-dimensional space.

Space-time = { Events} = $\mathbb{R}^4$. An event is a particular place at a fixed time. With one space dimension suppressed, we picture space-time like this: (t-axis is always upwards)



This is a *Space-time diagram.* Coordinates label events. The straight line L is the history of a particle moving with constant velocity. Greater slope gives lower speed.

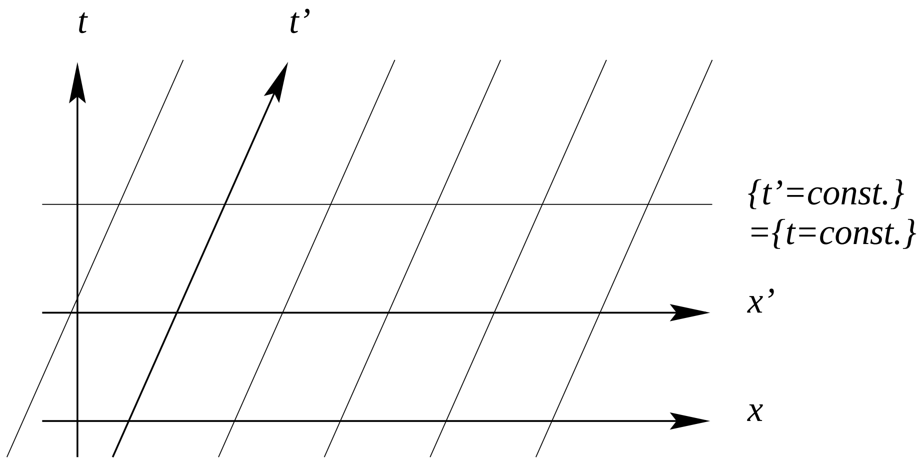
Terminology: A curve in space-time representing the history of a particle is called a *worldline.*

Invariant structures Galilean transformations are those transformations that preserve:

1. The time separation between two events.
2. The distance between *simultaneous* events.
3. Straight lines.

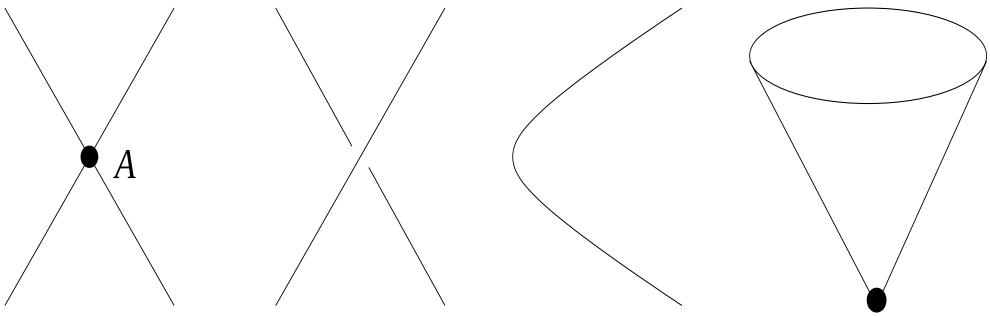
Let A and B be events. Consider the statements:
(1) A, B are simultaneous
(2) B happens time t after A
(3) A, B are simultaneous, distance D apart
(4) A, B happen in same place (at different times)
(5) A, B happen at different times and are separated by distance F.
1–3 are *invariant* i.e. true in every inertial frame if true in one.
Statements 4–5 are *not* invariant—they require a standard of rest (e.g. now &now are separated by 19 miles in a frame fixed with respect to the sun).

Galilean transformations in 1-space dimension Two frames of reference, R and R' related by a translation and 'boost' in the x direction:



$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

Examples of space-time diagrams



(i) (ii)

Two particles in rectilinear motion (i) meet at A (ii) dont meet.

Uniform Acceleration in straight line

Circular wave spreading across pond

Example: Four ghosts travel in straight lines at different constant speeds across a flat field. Five of the six pairs pass through each other at different times. Show that the 6th pair also pass through each other.

Solution: In 3-dim space-time, histories are straight lines. Let the ghosts be G_1, ... , G_4. Suppose the 5 pairs are G_1G_2, G_1G_3, G_1G_3, G_2G_3 and G_2G_4. Then $G_1G_2G_3$ must be coplanar with G_1G_2, G_2G_3 and G_1G_3, so $G4$ lies in this plane as it intersect G_1 and $G2$ so must intersect G_3 as its not parallel (different speeds).

2 Introduction to Maxwell's equations (Preview)

Maxwell's theory was a radical departure from previous physical theories.

The gravitational field $\mathbf{g}(t, \mathbf{r})$ is the force per unit mass of a test particle at the event $(t, \mathbf{r})$, and it can be obtained from a potential ϕ with $\mathbf{g} = -\nabla\phi$. These were thought of as mathematical devices to encode the $1/r^2$ action at a distance force law from a general distribution of masses.

The electric field is the force per unit charge experienced by a charged particle. It was also originally conceived of as a way of encoding the inverse square force law for all electric charges present (the magnetic field could be defined analogously). This works for electrostatics and magnetostatics.

However Faraday conjectured and Maxwell and Hertz proved that they can be non-trivial in the absence of charges and so must be real in their own right. Indeed they showed that electromagnetic waves exist and are light.

The objects of electromagnetic theory

1. **Charged particles.** Charge is denoted by e: it is an intrinsic property analagous to mass except that we can have $e > 0$, $e = 0$ or $e < 0$.
   In continuous media we define
   $$\text{charge density}: \quad \rho(t, \mathbf{r}) = \lim_{\delta V \to 0} \frac{\sum e}{\delta V}$$
   $$\text{current density}: \quad \mathbf{j}(t, \mathbf{r}) = \lim_{\delta V \to 0} \frac{\sum e\mathbf{v}}{\delta V}$$
   where $\mathbf{v}$ is the velocity of the charge e and the volumes δV are balls centred at $\mathbf{r}$ and the sum/integral is over charges contained in δV.

2. **The electric and magnetic fields:** $\mathbf{E}(t, \mathbf{r})$ and $\mathbf{B}(t, \mathbf{r})$, which are vector valued functions of position and time.

The electric and magnetic fields are measured using the force law below.

The equations of electromagnetism These interact according to the following equations.

(A) The charge and current densities generate electric and magnetic fields by Maxwell's equations:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\varepsilon_0 & (1)\\ \nabla \cdot \mathbf{B} &= 0 & (2)\\ \nabla \wedge \mathbf{B} - \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{j} & (3)\\ \nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 & (4) \end{aligned}$$

where ε_0, μ_0 are constants.

(B) Electric and magnetic fields generate forces on charges

$$\mathbf{f} = e(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}), \quad \mathbf{v} = \text{velocity} \tag{5}$$

We shall see where these laws come from later.

Comparison with gravity In the static case, (4) implies that $\mathbf{E} = \nabla\psi$ for some scalar ψ. Equation (1) then gives Poisson's equation $\nabla^2\psi = \rho/\varepsilon_0$.

For gravity $\mathbf{g} = -\nabla\phi$ the gravitational acceleration field is determined by

$$\nabla \cdot \mathbf{g} = -\nabla^2\phi = -4\pi G\rho$$

where ρ is the matter density. In Gravity Poisson's equation is this 'differential form' of the inverse square law. In (1) the reversal of the sign gives a repulsive inverse-square forces between like charges.

Units

We use mks units with charge e measured in Coulombs, $\mathbf{B}$ in Teslas, $\mathbf{E}$ in volts per metre.

$$\varepsilon_0 = 8.9 \times 10^{-12}, \qquad \mu_0 = 4\pi \times 10^{-7} \quad \text{(exact)}$$

The electron charge is $e = -1.6 \times 10^{-19}$ Coulombs
The earth's magnetic field is 4×10^{-5} Teslas
Neutron star's magnetic field is 10^8 Teslas.

The source-free equations

In empty space $\rho = 0 = \mathbf{j}$.

Put $c = 1/\sqrt{\varepsilon_0 \mu_0} = 3 \times 10^8$ metres per second. Then

$$\nabla \cdot \mathbf{E} = 0 \tag{6}$$

$$\nabla \cdot c\mathbf{B} = 0 \tag{7}$$

$$\nabla \wedge c\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 \tag{8}$$

$$\nabla \wedge \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{9}$$

What we are going to see is that these equations admit wave solutions, light.
We note first two features of the equations:

- A) Symmetry under *duality*:
  $\mathbf{E} \mapsto c\mathbf{B}$, and $c\mathbf{B} \mapsto -\mathbf{E}$.
- B) Consistency: We have *eight* equations in *six* unknowns, so they might be inconsistent. But the divergences of (8) and (9) give the time derivatives of (6) and (7) respectively so no inconsistencies arise (see exercises).

Electromagnetic waves Recall that: $\nabla \wedge (\nabla \wedge \mathbf{X}) = \nabla(\nabla \cdot \mathbf{X}) - \nabla^2 \mathbf{X}$.
Thus curl (9) gives

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \wedge c\mathbf{B}) = 0$$

so that using (6) and (8) we have

$$-\nabla^2 \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and also

$$-\nabla^2 c\mathbf{B} + \frac{1}{c^2}\frac{\partial^2 c\mathbf{B}}{\partial t^2} = 0$$

from duality.

So, in empty space, the components of $\mathbf{E}$ and $c\mathbf{B}$ satisfy the *wave equation*:

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{10}$$

Plane waves

This is the 3-d version of the 1-d wave equation

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

which has solutions $\phi = f(ct \pm x)$. These generalize to give plane wave solutions in 3-d: put $v = ct - \mathbf{e} \cdot \mathbf{r}$ where $\mathbf{e}$ is a constant unit vector, $\mathbf{e} \cdot \mathbf{e} = 1$.

Proposition 1 *For any constant unit vector* $\mathbf{e}$ *and any function* f *of one variable* $\phi = f(v)$ *satisfies (10); The motion of the wave is directed along* $\mathbf{e}$*.*

Proof: $\nabla\phi = -f'\mathbf{e}$, so

$$\nabla^2\phi = -\mathbf{e}\cdot\nabla f' = \mathbf{e}\cdot\mathbf{e}f''\phi = f''.$$

Whereas $\frac{\partial\phi}{\partial t} = cf'$, so that $\frac{\partial^2\phi}{\partial t^2} = c^2 f''$. □

Example: The *harmonic wave* is

$$\phi = A\cos(\omega v/c + \varepsilon)\,.$$

It has frequency ω, amplitude A, phase ε and travels in direction $\mathbf{e}$ with speed c.

Proposition 2 *For any constant* $\mathbf{k}$ *with* $\mathbf{k}\cdot\mathbf{e} = 0$*, and any function* f *of one variable*

$$\mathbf{E} = f(v)\mathbf{k}\,, \quad \mathbf{B} = \frac{f(v)}{c}\mathbf{e}\wedge\mathbf{k}$$

satsifies the source free Maxwell equations.

Proof: We have $\nabla v = \mathbf{e}$ so that

$$\nabla\cdot\mathbf{E} = -f'(v)\mathbf{e}\cdot\mathbf{k} = 0\,.$$

$$\nabla\wedge\mathbf{E} = -f'(v)\mathbf{e}\wedge\mathbf{k}\,, \quad \frac{\partial\mathbf{E}}{\partial t} = cf'(v)\mathbf{k}$$

and

$$\frac{\partial\mathbf{B}}{\partial t} = \frac{f'(v)}{c}\mathbf{e}\wedge\mathbf{k}$$

and using $\mathbf{e}\wedge(\mathbf{e}\wedge\mathbf{k}) = -\mathbf{k}$ we find

$$\nabla\wedge\mathbf{B} = -\frac{f'(v)}{c^2}\mathbf{e}\wedge(\mathbf{e}\wedge\mathbf{k}) = \frac{f'(v)}{c^2}\mathbf{k}$$

so Maxwell's equations have (plane) wave solutions travelling with speed c in any direction.

3 Maxwell's equations and relativity

Maxwell's theory describes all the forces apart from gravity that we see in ordinary mechanics. It also predicts the existence of electromagnetic waves which travel with velocity $c = 1/\sqrt{\varepsilon_0\mu_0}$ which can be identified with light and radio waves etc..

Question: Does Galileo's principle of relativity extend from mechanics to electromagnetism?

At the time there were two problems with this.
(1) It is difficult to believe that the velocity of light can be the same in all directions for a moving observer, if it is for a stationary observer.
(2) It was difficult to conceive of fields as being a fundamental object in its own right. It was thought that a field must be an approximation to a system of many small objects just as we imagine fluids to be composed of molecules of, say, water. Thus the waves are ripples in a medium, *the ether* which has its own preferred rest frame.

If one attempts to derive a relativity principle, one can make it work to first order in v (using the force law), but fails to order v^2/c^2.

3.1 The Michelson-Morley experiment According to this point of view, Maxwell's equations should only hold exactly in the rest frame of the ether. The Michelson-Morley experiment tested this part of the theory (although it was merely intended to distinguish different theories of the ether).

The earth must be moving through the ether as it rotates about the sun. The light travel time is compared in two orthogonal directions.

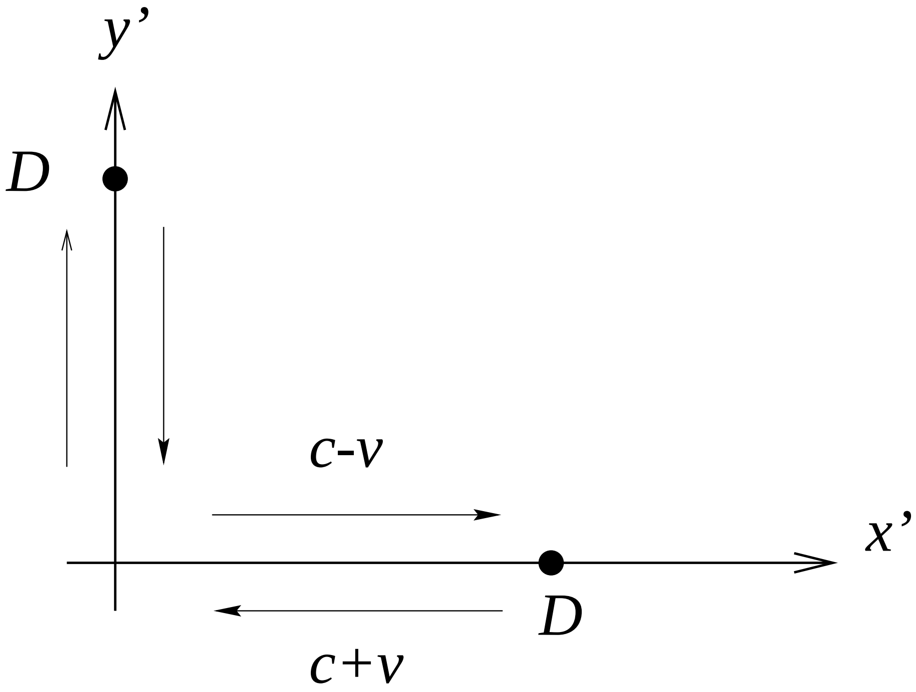
A photon P has velocity $\mathbf{u}$, $\mathbf{u} \cdot \mathbf{u} = c^2$, relative to the ether.

Make a Galilean transformation to a moving frame $x = x' + vt\,, \quad y = y'\,, z = z'$ so that $(u'_1, u'_2, u'_3) = (u_1 - v, u_2, u_3)$.

If $u_1 = V$, $u_3 = 0$, then $u'_1 = 0$ and P travels with speed $u_2 = u'_2$ in y' direction where $c^2 = u_1^2 + u_2^2 + u_3^2 = u_2^2 + V^2 = c^2$ so that

$$u_2 = \pm\sqrt{c^2 - V^2}\,.$$

Whereas, if $u_2 = u_3 = 0$, $\mathbf{u}' = (\pm c - v, 0, 0)$.



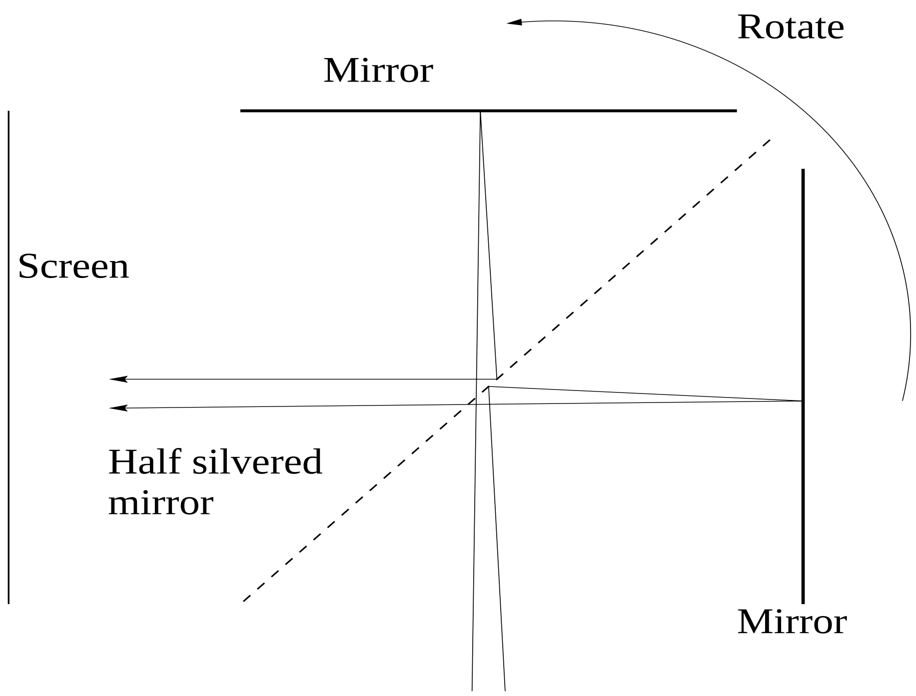
Thus the time $t_{x'}$ to go out and back along the x'-axis is

$$t_{x'} = \frac{D}{c-v} + \frac{D}{c+v} = \frac{2D}{c}(1 + \frac{v^2}{c^2}) + O(\frac{v^4}{c^4})$$

and the time t'_y to go out and back along the y'-axis is

$$t_{y'} = \frac{2D}{\sqrt{c^2 - v^2}} = \frac{2D}{c}(1 + \frac{v^2}{2c^2}) + O(\frac{v^4}{c^4})$$

Michelson and Morley tested this picture:



if the apparatus moves,

$$\text{Fringe Shift} = \frac{c(t_{x'} - t_{y'})}{\lambda} = \frac{v^2 D}{c^2 \lambda}$$

where λ is the wavelength of the light. Note the cancellation at $O(v/c)$.

No shift was observed!

An ether theory could be preserved if it was swept along by the earth (Stokes).

But, more profoundly, Lorentz and Fitzgerald observed that E-M implies a 'Lorentz contraction' in the direction of motion by factor of $\sqrt{1 - v^2/c^2}$. E-M is responsible for all forces and hence lengths of rods; the E-M field of a nucleus and hence length of an atom is squashed by a factor of $\sqrt{1 - v^2/c^2}$.

Poincaré proposed a conspiracy theory: cancellations at all orders in all experiments prevent the measurement of the velocity of the ether. (This indeed follows from Lorentz's transformations that show how E-M fields of moving bodies can be obtained from those of stationary bodies).

Problems: (A) It is meaningless to define length without reference to a procedure: If rulers contract when you move relative to the ether, what does it mean to say that a length should have different value than that measured? (B) What does it mean to be at rest wrt ether if measurements cannot distinguish rest frame?

3.2 Revolution: special relativity

Einstein saw that the way forward was to

(1) regard Maxwell's equations and the constancy of the velocity of light as fundamental,
(2) keep the principle of relativity, and
(3) introduce the principle that lengths etc. must be defined operationally using light travel times.

Assumptions: We make two basic physical assumptions:

1. Observers carry clocks (quantum mechanical oscillations). For inertial observers time must depend linearly on the coordinates.

2. Light rays are given as histories of the form

$$\mathbf{r}(t) = \mathbf{v}t + \mathbf{c}\,, \quad \mathbf{v}\cdot\mathbf{v} = c^2\,, \quad \mathbf{v} \text{ and } \mathbf{c} \text{ constant}$$

   with c^2 fixed. Two events separated by $(c\Delta t, \Delta x, \Delta y, \Delta z)$ are connected by a light ray iff

$$(c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = 0\,.$$

Operational definition of space and time

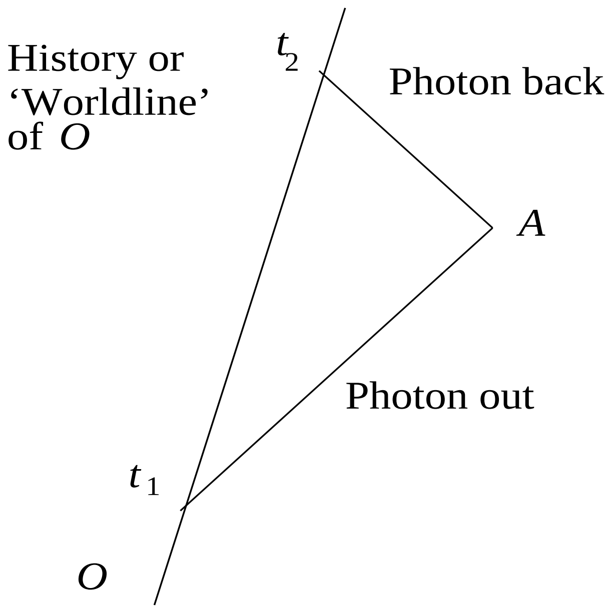
We define distance and time of an event A according to an inertial observer, O, by the following radar method:

The observer O sends out a photon at time t_1 which gets reflected back at the event A to reach O at time t_2. Define

The time of A is $t_A := \frac{1}{2}(t_1 + t_2)$

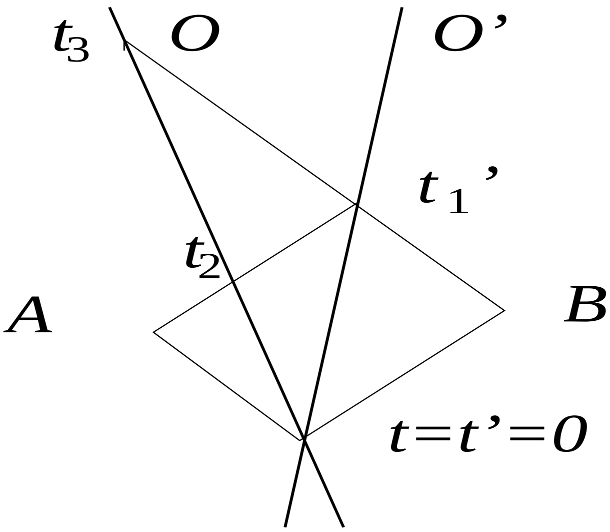
The distance of A to be $d_A := \frac{c}{2}(t_2 - t_1)$,

Thus light is used to assign times and distance to events that are not on $O's$ worldline automatically building in the light travel time.



Relativity of simultaneity

The first surprising sacrifice is the concept of absolute simultaneity



Observer O' assigns $t'_A = t'_B = \frac{1}{2}t'_1$: A and B are simultaneous.
However, O assigns $t_A = \frac{1}{2}t_3 > t_B = \frac{1}{2}t_2$: A happens after B according to O.

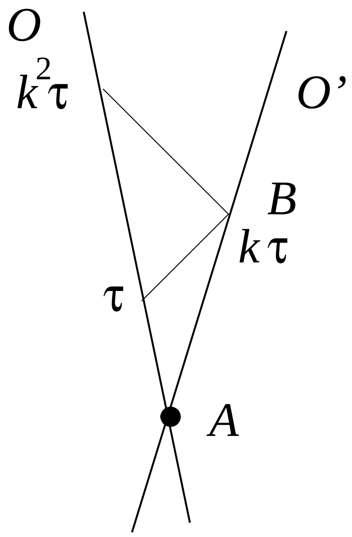
Bondi's k-calculus

We can deduce elementary aspects about S-R in 1 spatial dimension as follows.

Consider space-ships O and O', O' moving directly away from O. Each sends the other a stream of radio messages.

A signal sent by O at τ (on O clock) arrives at time $k\tau$ (on O' clock). Assume

1. k constant (indept of τ, since non-accelerating)
2. Principle of relativity: k depends only on *relative* motion (Bondi's k-factor).



Then signal from B arrives at O at $k^2\tau$ (according to O). Thus O measures:

$$t_B = \frac{1}{2}(k^2+1)\tau \quad \text{and} \quad d_B = \frac{c}{2}(k^2-1)\tau.$$

Thus velocity of O' measured by O is

$$u = \frac{d_B}{t_B} = c\frac{k^2-1}{k^2+1} < c$$

Hence

$$k = \sqrt{\frac{c+u}{c-u}} > 1\,.$$

Time dilation: Define the time dilation γ by

$$\gamma := \frac{\text{time from } A \text{ to } B \text{ meausured by } O}{\text{time from } A \text{ to } B \text{ meausured by } O'}$$

Thus dilation (dilatation) is

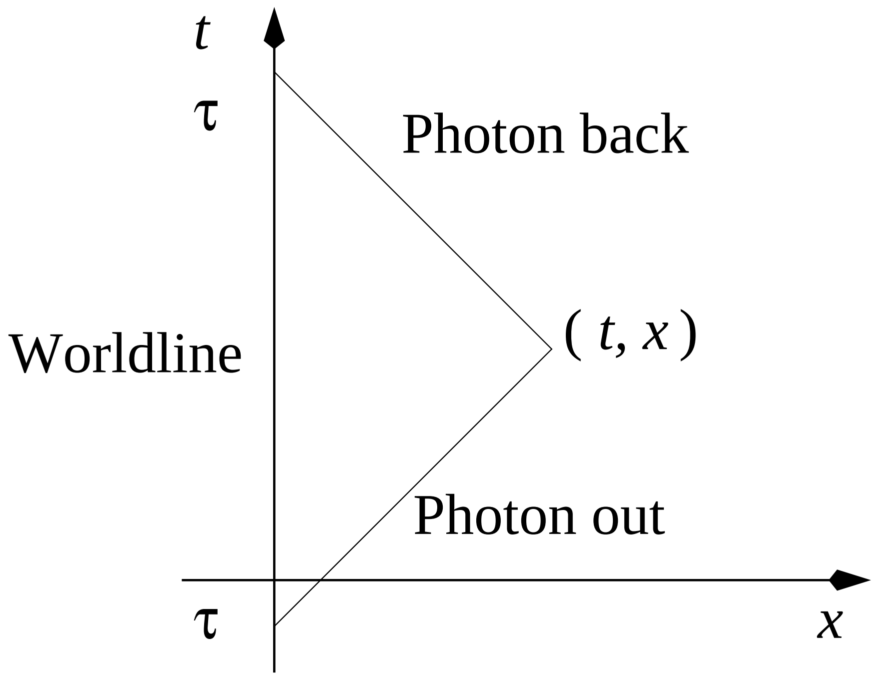
$$\gamma = \frac{t_B}{k\tau} = \frac{1}{2}\frac{(k^2+1)\tau}{k\tau} = \frac{1}{\sqrt{1-u^2/c^2}}$$

e.g. for $u = \sqrt{3}c/2$, $\gamma = 2$.

4 The Lorentz transformation

We shall first consider just one space dimension in this lecture. We consider non-accelerating particles and observers moving on a line.

Derivation We have seen how an observer O can measure the distance and time of distant events. He/she can therefore set up a coordinate system (t, x) on space-time, taking his own location as $x = 0$.



Definition: We call this an *inertial coordinate system* (ICS).

How are inertial coordinate systems (ICS's) related?

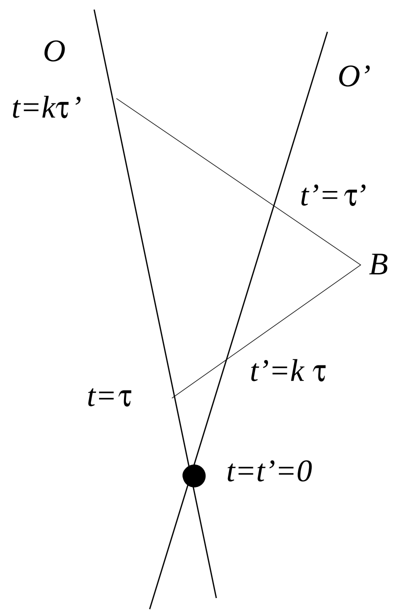
Two observers O, O', pass at event A which they take as their common origin. Let O O' assign coords (t, x), (t', x') to B respectively.

Proposition 3 *The coordinates are related by the* Lorentz transformation*:*

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}, \quad \gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}},$$

and $u =$ speed of O' measured by O.

Proof:



The coords of B measured by O are

$$t = \frac{1}{2}(k\tau' + \tau)\,, \quad x = \frac{c}{2}(k\tau' - \tau) \tag{11}$$

and the coords of B measured by O' are

$$t' = \frac{1}{2}(\tau' + k\tau)\,, \quad x' = \frac{c}{2}(\tau' - k\tau)\,. \tag{12}$$

$$\text{Eqn (11) implies} \quad \begin{pmatrix} ct \\ x \end{pmatrix} = \frac{c}{2}\begin{pmatrix} 1 & k \\ -1 & k \end{pmatrix}\begin{pmatrix} \tau \\ \tau' \end{pmatrix}$$

and (12) implies

$$\begin{aligned} \begin{pmatrix} ct \\ x \end{pmatrix} &= \frac{1}{2k}\begin{pmatrix} 1 & k \\ -1 & k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ k & k \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} k+k^{-1} & k-k^{-1} \\ k-k^{-1} & k+k^{-1} \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix} \end{aligned}$$

But $k = \sqrt{\frac{c+u}{c-u}}$ so $k + k^{-1} = 2c/\sqrt{c^2 - u^2}$ and $k - k^{-1} = 2u/\sqrt{c^2 - u^2}$ so

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u)\begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix}\begin{pmatrix} ct' \\ x' \end{pmatrix}\,.$$

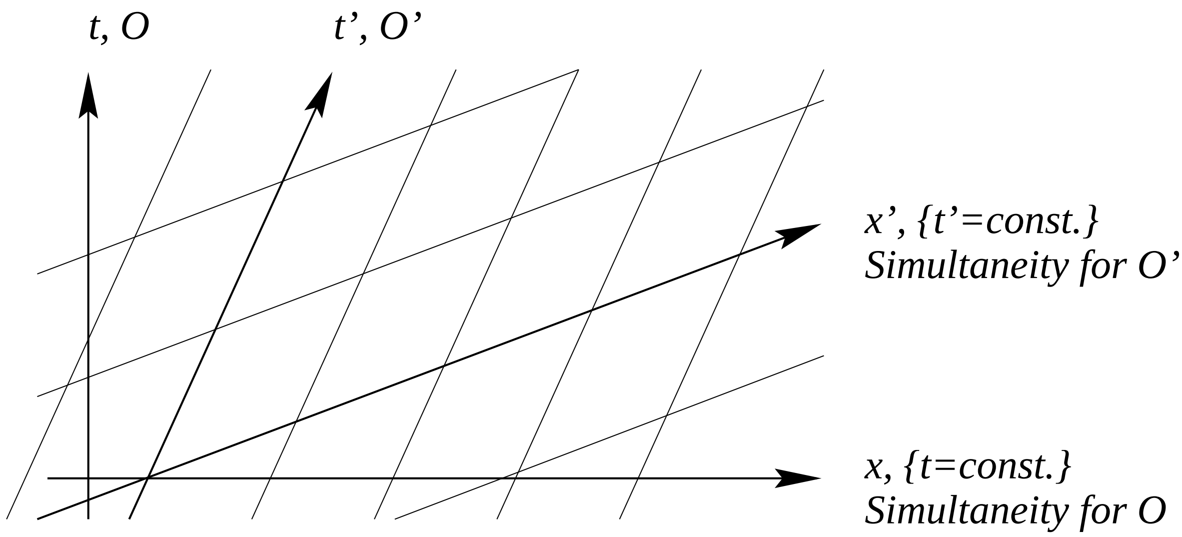
Properties of Lorentz transformations

1. It reduces to GT when $u \ll c$:

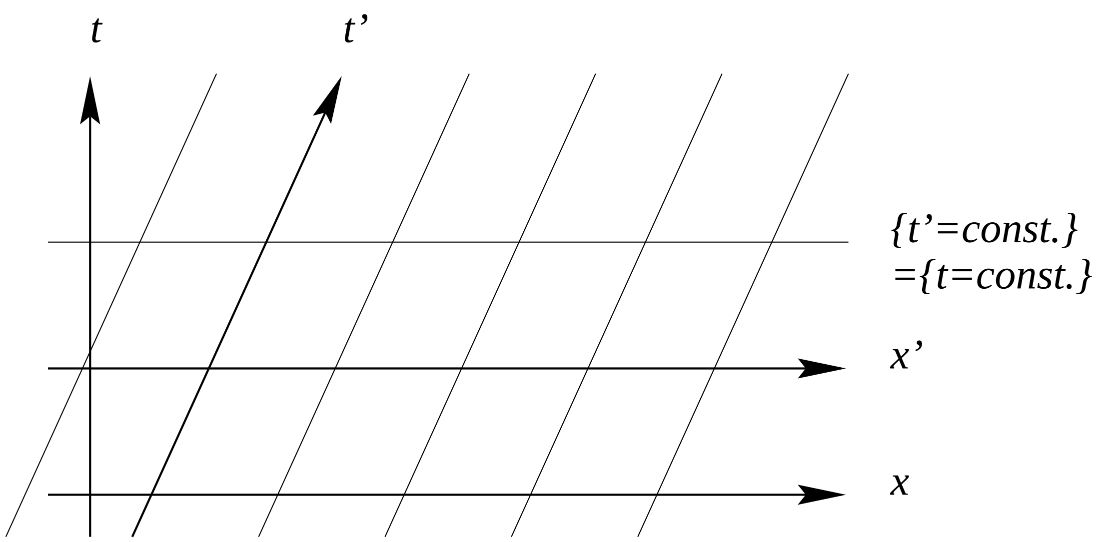
$$\begin{pmatrix} t \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & \frac{u}{c^2} \\ u & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

if we ignore u^2/c^2 (since $\gamma \sim 1$).

2. Diagrammatic relationship between coord systems:
$x' = 0$ corresponds to $x = ut$ and $t' = 0$ corresponds to $x = c^2 t/u$.



Compare with Galilean Tranformation



3. The inverse transformation is

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & -\frac{u}{c} \\ -\frac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

which is a LT with u replaced by $-u$.

4. **Time dilation revisited** Note that if $x' = 0$ (events on O' worldline) then $t = \gamma(u)t'$, which is again the time dilation formula. Note asymmetry arises from the fact that we are looking at events on worldline of O', not O.

Transformation of velocity Consider a nonaccelerating particle moving with speed v relative to O in negative x-direction, i.e. $x = -vt + a$ (a is constant). In the O' coordinates

$$t' = \gamma t - \frac{\gamma u}{c^2}(-vt + a), \quad x' - -\gamma ut + \gamma(-vt + a)$$

so speed relative to O' is

$$w = -\frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\gamma(v + u)}{\gamma(1 + \frac{uv}{c^2})} = \frac{u + v}{1 + \frac{uv}{c^2}}$$

Remarks:
1) This differs from the classical formula $w' = v + u$.
2) if $v = c$ then $w = c$.
3) If $|v| < c$, $|u| < c$ then $|w'| < c$.
Proof:

$$(c - u)(c - v) > 0 \implies u + v < c(1 + uv/c^2)$$
$$(c + u)(c + v) > 0 \implies u + v > -c(1 + uv/c^2)$$

Composition of Lorentz transformations Suppose that O, O' and O'' set up coordinates (t, x), (t', x') and (t'', x'') and that O' has velocity u relative to O and O'' has velocity v relative to O'.

Then $$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

and $$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix}$$

So $$\begin{aligned} \begin{pmatrix} ct \\ x \end{pmatrix} &= \gamma(u)\gamma(v) \begin{pmatrix} 1 & \frac{u}{c} \\ \frac{u}{c} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct'' \\ x'' \end{pmatrix} \\ &= \gamma(w) \begin{pmatrix} 1 & \frac{w}{c} \\ \frac{w}{c} & 1 \end{pmatrix} \end{aligned}$$

where $w = \frac{u+v}{1+uv/c^2}$. To see this use

$$\gamma(u)\gamma(v)(1 + uv/c^2) = \gamma(w)\,.$$

Rapidity

Definition: The *rapidity* is $\phi(u) = \tanh^{-1}(u/c)$.

With this definition, the Lorentz transformation becomes

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

An LT is thus a 'pseudo' or 'hyperbolic' rotation.

Note that $\gamma(u) = \cosh\phi$ and $k = \exp\phi$.

Under the composition of Lorentz transformations, the rapidities add, $\phi(w) = \phi(u) + \phi(v)$.