"Theorie der hypercomplexen Größen," Sitz. Kön. Preuss. Akad. Wiss. (1903), 504-537; Gesammelte Abhandlungen, art. 70, pp. 284-317.

Theory of hypercomplex quantities.

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The methods that I developed in my paper "Über die Primfactoren der Gruppendeterminante," Sitzungsberichte 1896 (cited as *Gr.* in the sequel) in the course of researching the properties of the determinant of a finite group, I also arrive at the examination of an arbitrary system of hypercomplex quantities in n basis numbers. MOLIEN concerned himself with such quantities in his ground-breaking treatise "Über Systeme höhere complexer Zahlen." Math. Ann, Bd. 41 (cited as MOL. in the sequel).

An understanding of his paper is impeded somewhat by the fact that he passed over the possibility of doing the calculations and sought to present the content of the proof abstractly. Thus, it is not clear to me whether the tools that he developed succeed in obtaining a rigorous proof of theorem 25. Conversely, in § 5, he introduced a fallacy that appeared in a Notice in volume 42 of Math. Ann. in the course of a truly circuitous calculation. When I employed a linear combination $S(x) + T(y)$ of the matrices in different variables from the two antistrophic groups, in place of the equation that he called the KILLING equation, I arrive at precisely this proof, although simplified substantially.

However, as I have briefly remarked (on pp. 408 of this volume), in addition, there was a not-inessential gap in his proof of theorem 19. Nonetheless, if one ignores this minor shortcoming then his trailblazing, thought-provoking paper, which he followed through on with unrelenting persistence and pervasive ingenuity, in spite of a rather incomplete skill set, defines one of the most important advances in the domain of algebra that one refers to as group theory. On the basis of the comments that I just made, I regard it as appropriate to take up the investigation anew with the tools that I, in fact, developed in my paper "Über vertauschbare Matrizen," Sitzungsberichte, 1896.

The methods of LIE will not be used at all in this purely algebraic study. By comparison, my presentation has many points of contact with that of DEDEKIND in his treatise "Zur Theorie der us n Haupteinheiten gebildeten complexen Grössen," Göttinger Nachrichten, 1885 (cited as DED. in the sequel).

The basis for my investigation is defined by formula (2), § 5. With its help, I show directly in § 6 that the variables that appear in the various prime factors of the group determinant are all mutually independent, and thus circumvent theorems 3, 4, and 5 of MOLIEN, as well as the proof of theorem 25. With the same formula, I prove in § 7 that the elementary parts of the determinant of a simple group are all linear. The entire development rests upon these two results.

For the classification of groups, for which only inadequate Ansätze have been made heretofore, nothing less than the exponents and degrees of the elementary parts into which the determinants of the two antistrophic groups decompose prove to be the most suitable invariants, and moreover, they are the most suitable numbers for the – up till now scarcely noticed – parastrophic matrix $R(\xi)$, especially the rank of $R(\sigma)$ (§ 9). However, for non-commutative groups, above all, the elementary invariants relate to the matrix $uR(\xi) + vR'(\eta)$, and the matrices $S(x) + T(y)$ and $R(\xi) + R'(\eta)$, which depend upon $2n$ variables, and indeed these invariants are meaningful when perhaps the determinants of $R(x)$ or $R(\xi) + R'(\eta)$ vanish identically.

It was only after completing this examination that the excellent treatise of CARTAN, "Sur les groupes bilinéaires er les systèmes de nombres complexes," Ann. de Toulouse, tome XII, 1898, was brought to my attention, in which he derived the result of MOLIEN, but without knowing of that paper, it seems. The path that was followed by CARTAN has very little in common with that method or the one employed here. The transformation of the basis, which is the starting point and final objective of his investigation, is, in my opinion, to be avoided as much as possible (§ 9). For him, the properties that are invariant of any representation of the group that I begin with only take on the meaning of a normal form for the group at the conclusion that is obtained by a long series of conversions whose purpose is first made clear at the end of the argument. The difference between the two methods is thus the same as the one between the procedures of WEIERSTRASS and KRONECKER in the theory of families of bilinear forms. I have obtained (§ 12, (5)) an especially noteworthy formula of CARTAN (§ 65, (37)) that was not found by MOLIEN in the simplest way by the decomposition of the determinant $|\,S(x) + T(y)\,|$ into prime factors.

§ 1.

The *coordinates* $x_1, x_2, \ldots, x_n$ of the of the *hypercomplex quantities:*

$$x = \varepsilon_1\, x_1 + \varepsilon_2\, x_2 + \ldots + \varepsilon_n\, x_n\,, \tag{1}$$

which are formed from the *n basic numbers* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, can all assume real or complex values. The totality of these quantities, which reproduces the addition and multiplication of *ordinary* quantities, I call a *group* (ε), when, in addition, the product of any two of them again belongs to the system, and the associative law $(xy)z = x(yz)$ is valid for any three of them.

The basic numbers can be coupled to each other by linear relations. I call the number of independent ones among them the *order* of the group. One can easily reduce this case to the one where the basic numbers are independent. I therefore assume that n is the order of the group (ε) considered.

If the multiplication of basic numbers results from the rule:

(2) $$\varepsilon_\beta\,\varepsilon_\gamma = \sum_\alpha a_{\alpha\beta\gamma}\varepsilon_\alpha$$

then this implies the following relations between the coefficients that appear:

(3) $$\sum_\kappa a_{\kappa\alpha\beta}a_{\gamma\kappa\delta}\,\mathrm{v} = \sum_\kappa a_{\kappa\beta}a_{\gamma\alpha\kappa}\,,$$

which follow from the associative principle and the independence of the basic numbers.

Then, if $x = yz$ is the product of the two quantities:

$$y = \sum \varepsilon_\lambda y_\lambda\,, \qquad z = \sum \varepsilon_\lambda z_\lambda\,,$$

then one has:

(4) $$x_\alpha = \sum_{\beta,\gamma} a_{\alpha\beta\gamma}y_\beta z_\gamma\,,$$

or, when $\xi_1, \xi_2, \ldots, \xi_n$ are variable, whose system I denote by x and call a *parameter:*

(5) $$\xi(x) = x(\xi) = \sum_\alpha \xi_\alpha x_\alpha = F(\xi, y, z) = \sum_{\alpha,\beta,\gamma} a_{\alpha\beta\gamma}\xi_\alpha y_\beta z_\gamma\,.$$

If one sets:

(6) $$r_{\alpha\beta}(\xi) = \sum_\kappa a_{\kappa\alpha\beta}\xi_\kappa\,, \qquad s_{\alpha\beta}(y) = \sum_\kappa a_{\alpha\kappa\beta}y_\kappa\,, \qquad t_{\alpha\beta}(z) = \sum_\kappa a_{\alpha\beta\kappa}z_\kappa\,,$$

so:

$$s_{\alpha\beta}(y) = \frac{\partial x_\alpha}{\partial z_\beta}\,, \qquad t_{\alpha\beta}(z) = \frac{\partial x_\alpha}{\partial y_\beta}\,,$$

then the trilinear form becomes:

(7) $$F(\xi, y, z) = \sum r_{\beta\gamma}(\xi)y_\beta z_\gamma = \sum s_{\alpha\gamma}(y)\xi_\alpha z_\gamma = \sum t_{\alpha\beta}(z)\xi_\alpha y_\beta\,.$$

If denote the matrices of n^{th} degree that are defined by any three systems of linear functions by:

$$R(\xi) = R_\xi = (r_{\alpha\beta}(\xi)), \quad S(y) = S_y = (s_{\alpha\beta}(y)), \quad T(z) = T_z = (t_{\alpha\beta}(z)).$$

By $f(x)$, I always understand this to mean a function of the *coordinates* $x_1, x_2, \ldots, x_n$ of the quantity x, and by $S(x)$ or S_x, a matrix whose elements are (linear) functions of these coordinates. When I am dealing with a function of the hypercomplex quantity x itself, I will employ the symbol $f((x))$.

The conditions (3) for the associative principle are more conveniently discussed when one combines them in such a way that one multiplies and adds them with variables. If x and y are two arbitrary quantities, one then obtains (DED. (36)):

(8) $$\sum_\lambda s_{\alpha\lambda}(x) t_{\lambda\beta}(y) = \sum_\lambda t_{\alpha\lambda}(y) s_{\lambda\beta}(x),$$

or, more simply:

(9) $$S_x\, T_y = T_y\, S_x,$$

and:

(10) $$\sum_\lambda r_{\alpha\lambda}(\xi) s_{\lambda\beta}(x) = \sum_\lambda t_{\lambda\alpha}(x) r_{\lambda\beta}(\xi),$$

or, more simply:

(11) $$R_\xi\, S_x = T'_x\, R_\xi,$$

where T' is the matrix *conjugate* to T.

I call $S(x)$ and $|\,S(x)\,|$ the *group matrix* and *group determinant*, resp., $T(x)$ and $|\,T(x)\,|$, the *antistrophic group matrix and group determinant*, resp., and $R(\xi)$ and $|\,R(\xi)\,|$, the *parastrophic matrix and determinant*, resp. The theorem that is included in formula (9), along with its converse, may be expressed in the following way:

The group matrix $S(x)$ commutes with the antistrophic matrix $T(y)$. If the parastrophic determinant does not vanish identically then one may bring it into the form $T(y)$ $(S(y))$ for any value of x that makes $S(x)$ $(T(y))$ commute.

Therefore, let the parameter x be chosen in such a way that the determinant of the matrix $R(\xi) = R = (r_{\alpha\beta})$ does not vanish and let $U = (u_{\alpha\beta})$ be any matrix of n^{th} degree that is independent of x, and which commutes $S(x)$ with x. One can then determine y in such a way that for $\lambda = 1, 2, \ldots, n$, one has (from (7)):

$$\sum_\kappa \xi_\kappa u_{\kappa\lambda} = \sum_\beta r_{\lambda\beta} y_\beta = \sum \xi_\alpha t_{\alpha\lambda}(y).$$

Now, one has $SU = US$, so:

$$\sum_\lambda s_{\alpha\lambda}(x) u_{\chi\beta} = \sum_\lambda u_{\alpha\lambda} s_{\lambda\beta}(x),$$

and with this:

$$\sum_{\kappa,\lambda} \xi_\kappa s_{\kappa\lambda}(x) u_{\lambda\beta} = \sum \xi_\kappa\, u_{\kappa\lambda}\, s_{\lambda\beta}(x) = \sum \xi_\kappa\, t_{\kappa\lambda}(y)\, s_{\lambda\beta}(x),$$

so from (8):

$$\sum_{\kappa,\lambda} \xi_\kappa s_{\kappa\lambda}(x) u_{\lambda\beta} = \sum \xi_\kappa\, s_{\kappa\lambda}(x)\, t_{\lambda\beta}(y).$$

Now, from (7), however:

$$\sum_{\kappa,\lambda} \xi_\kappa s_{\kappa\lambda}(x) = \sum_\alpha r_{\alpha\lambda} x_\alpha.$$

If one inserts this then by comparing the coefficients of x_α this yields:

$$\sum_\lambda r_{\alpha\lambda} u_{\lambda\beta} = \sum_\lambda r_{\alpha\lambda} t_{\lambda\beta}(y),$$

and with this, because $| R |$ is non-zero $u_{\alpha\beta} = t_{\alpha\beta}(y)$, $U = T(y)$.

In particular, if $U = E = (e_{\alpha\beta})$ is the *principal matrix* then one can determine a quantity $y = e$ such that $T(e) = E$ (DED. (44)). This quantity:

$$(12) \qquad e = \varepsilon_1 e_1 + \varepsilon_2 e_2 + \dots + \varepsilon_n e_n ,$$

is called the *principal unit*. Its coordinates satisfy the conditions:

$$(13) \qquad s_{\alpha\beta}(e) = t_{\alpha\beta}(e) = e_{\alpha\beta} ,$$

or:

$$(14) \qquad \sum_\lambda a_{\alpha\lambda\beta} e_\lambda = e_{\alpha\beta}, \qquad \sum_\kappa a_{\alpha\beta\kappa} e_\kappa = e_{\alpha\beta} .$$

Thus, if $S(x) = S(y)$ or $T(x) = T(y)$ or $S(x) = T(y)$ then one has $x = y$; if $R(\xi) = R(\eta)$ or $R(\xi) = R(\eta)$ then $\xi = \eta$.

§ 2.

If $x = yz$, so:

$$(1) \qquad x_\kappa = \sum_{\lambda,\mu} a_{\mu\lambda\kappa} y_\lambda z_\mu = \sum_\kappa s_{\alpha\kappa}(y) z_\kappa = \sum_\kappa t_{\alpha\kappa}(z) y_\kappa ,$$

then one has:

$$s_{\alpha\beta}(x) = \sum_\kappa a_{\alpha\kappa\beta} x_\kappa = \sum_{\kappa,\lambda,\mu} a_{\alpha\kappa\beta} a_{\kappa\lambda\mu} y_\lambda z_\mu = \sum_{\kappa,\lambda,\mu} a_{\alpha\lambda\kappa} a_{\kappa\mu\beta} y_\lambda z_\mu = \sum_\kappa s_{\alpha\kappa}(y) s_{\kappa\beta}(z)$$

and:

$$t_{\alpha\beta}(x) = \sum_\kappa a_{\alpha\beta\kappa} x_\kappa = \sum_{\kappa,\lambda,\mu} a_{\alpha\beta\lambda} a_{\kappa\lambda\mu} y_\lambda z_\mu = \sum_{\kappa,\lambda,\mu} a_{\alpha\kappa\mu} a_{\kappa\beta\lambda} y_\lambda z_\mu = \sum_\kappa t_{\alpha\kappa}(z) t_{\kappa\beta}(y) .$$

I denote by $f(yz)$ or $S(yz)$ the result of substituting the coordinates (1) of the product yz for the n variables x_κ in $f(x)$ or $S(x)$. One then obtains:

$$(2) \qquad S(yz) = S(y)\, S(z), \qquad T(yz) = T(z)\, T(y).$$

If $S(x) = x_1 E_1 + x_2 E_2 + \dots + x_n E_n$ then it follows that:

$$(3) \qquad E_\beta E_\gamma = \sum_\alpha a_{\alpha\beta\gamma} E_\alpha ,$$

and since $S(x) = 0$ only when $x = 0$, the n constant matrices $E_1, E_2, \dots, E_n$ are linearly independent. They therefore define a *representation* (cf., § 16) of the group (ε) and from this it follows that the assumption of the independence of the basic numbers is consistent with the quadratic equations (2), § 1, whose coefficients satisfy equations (3), § 1 and the inequalities (1), § 3 or that no linear relations between the basic numbers can come out of these conditions. It is first on the basis of this certainty that proofs are admissible, such as for formula (3), § 3, in which the hypercomplex numbers themselves are employed.

Furthermore, if $T(x) = x_1 F_1 + x_2 F_2 + \ldots + x_n F_n$ then one has:

(4)
$$F_\beta F_\gamma = \sum_\alpha a_{\alpha\gamma\beta} F_\alpha .$$

Since the associative law is valid, the matrices $F_1, F_2, \ldots, F_n$ define the *basis* for a group – the *antistrophic group* (ε'). Another representation of (ε) is given by the matrix T' that is conjugate to T. One can thus also refer to T' as the matrix of the group (ε) and S' as that of the antistrophic group (ε').

If z is a parameter, and one sets:

(5′)
$$\zeta_\alpha = \sum_{\kappa,\lambda} a_{\kappa\lambda\alpha} \xi_\kappa y_\lambda = \sum_\lambda r_{\lambda\alpha}(\xi) y_\lambda = \sum_\kappa s_{\kappa\alpha}(y) \xi_\kappa ,$$

then one will have:

$$r_{\alpha\beta}(\zeta) = \sum_\mu a_{\mu\alpha\beta} \zeta_\mu = \sum_{\kappa,\mu} \xi_\kappa s_{\kappa\mu}(y) a_{\mu\alpha\beta} .$$

This is the coefficient of z_β in:

$$\sum_\beta r_{\alpha\beta}(\zeta) z_\beta = \sum_{\kappa,\mu} \xi_\kappa s_{\kappa\mu}(y) t_{\mu\alpha}(z) = \sum_{\kappa,\mu} \xi_\kappa t_{\kappa\mu}(z) s_{\mu\alpha}(y) = \sum_{\mu,\beta} s_{\mu\alpha}(y) r_{\mu\beta}(\xi) z_\beta .$$

Therefore:

(5)
$$r_{\alpha\beta}(\zeta) = \sum_\mu s_{\mu\alpha}(y) r_{\mu\beta}(\xi) ,$$

or:

(5)
$$R(\zeta) = S'(y)\, R(\xi).$$

Finally, if one sets:

(6*)
$$\eta_\lambda = \sum_{\kappa,\mu} a_{\kappa\lambda\mu} \xi_\kappa z_\mu = \sum_\mu r_{\lambda\mu}(\xi) z_\mu = \sum_\mu t_{\mu\lambda}(z) \xi_\mu$$

then one will have:

$$r_{\alpha\beta}(\eta) = \sum_\lambda a_{\lambda\alpha\beta} \eta_\lambda = \sum_{\kappa,\lambda} \xi_\kappa t_{\kappa\lambda}(z) a_{\lambda\alpha\beta}$$

and therefore:

$$\sum_\alpha r_{\alpha\beta}(\eta) y_\alpha = \sum_{\kappa,\lambda} \xi_\kappa t_{\kappa\lambda}(z) s_{\lambda\beta}(y) = \sum_{\mu,\lambda} \xi_\mu s_{\mu\lambda}(y) t_{\lambda\beta}(z) = \sum_{\alpha,\lambda} r_{\alpha\lambda}(\xi) t_{\lambda\beta}(z) y_\alpha ,$$

from which:

(6)
$$r_{\alpha\beta}(\eta) = \sum_\lambda r_{\alpha\lambda}(\xi) t_{\lambda\beta}(z)$$

or:

(6)
$$R(\eta) = R(\xi)\, T(z).$$

With a suitable change in the notation, this implies:

If the parastrophic determinant does not vanish identically and the determinant of the matrix $R_\zeta = R = (r_{\alpha\beta})$ *is non-zero for the value* ζ *of the parameter then* R_ξ *goes to* RT_x *under the substitution* $\xi_\alpha = \sum_\beta r_{\alpha\beta} x_\beta$ *, and to* $S'_x R$ *under the conjugate substitution* $\xi_\beta = \sum_\alpha r_{\alpha\beta} x_\alpha$.

Under any condition, the group determinant of the antistrophic group determinant is identically equal, and up to the factor $[R]$, *it is also equal to the function in which the parastrophic determinant goes to under either of the two conjugate substitutions; all three determinants coincide in their elementary parts.*

If ξ, η, ζ, τ are arbitrary parameters then it follows from (9), § 2 that $R_\tau^{-1} R_\xi$ commutes with $R_\tau'^{-1} R'_\zeta$ or that the matrix $R'_\zeta R_\tau^{-1} R_\eta R_\xi'^{-1}$ is independent of ζ.

§ 3.

If $| R(x) |$ does not vanish identically then neither of the two determinants $| S(x) |$ or $| T(x) |$ vanishes identically, either, and there is a number e for which one has $S(e) = T(e) = E$. However, when $| R(x) | = 0$ identically I add the further assumption, that neither of the two determinants:

(1) $$| S(x) |, \qquad | T(x) |$$

is identically zero. If one chooses z such that $| S(z) |$ and $| T(z) |$ are both non-zero then one can (cf., e.g., MOL., theorem 1), because $| S(z) |$ is non-zero, determine a number e that satisfies the equations $ze = z$. Furthermore, if x is an arbitrary quantity then one can, because $| T(z) |$ is non-zero, determine y such that $zy = x$. One then also has $xe = x$ and $zex = zx$, so since $| S(z) |$ is non-zero, $ex = x$. Since, furthermore, from (2), § 2, $S(z) = S(z)\, S(e)$ and $T(z) = T(e)\, T(z)$, one has $S(e) = T(e) = E$.

The theorem of § 1 remains correct when either $| S(x) |$ or $| T(x) |$ vanishes identically, and also when $| R(\xi) | = 0$ identically. If one then sets $\sum_\mu u_{\alpha\mu} e_\mu = y_\alpha$ then one has:

$$\sum_{\beta,\mu} u_{\alpha\beta} s_{\beta\mu}(x) e_\mu = \sum_{\gamma,\mu} s_{\alpha\gamma}(x) u_{\gamma\mu} e_\mu ,$$

so:

$$\sum_\beta u_{\alpha\beta} x_\beta = \sum_\gamma s_{\alpha\gamma}(x) y_\gamma = \sum_\beta t_{\alpha\beta}(y) x_\beta ,$$

and therefore $u_{\alpha\beta} = t_{\alpha\beta}(y)$.

If $| S(x) |$ is non-zero for a certain quantity x then one can determine y in such a way that $xy = e$. Then, from (2), § 2, $T(y)T(x) = T(e) = E$. Thus, $| S(x) |$ is non-zero for the value x and so is $| T(x) |$, and conversely. Therefore, if $| S(x) | = 0$ then so is $| T(x) | = 0$. Any prime factor $\Phi(x)$ of the group determinant $| S(x) |$ is therefore also included in the antistrophic determinant $| T(x) |$, and conversely. Precisely the same factors $\Phi(x)$ appear in the two decompositions:

(2) $$| S(x) | = \Pi\ \Phi(x)^s, \qquad | T(x) | = \Pi\ \Phi(x)^t,$$

so the exponents s and t can be different, but only when $| R(\xi) | = 0$ identically.

The simplest example of this possibility, as MOLIEN has communicated to me, is provided by the only non-commutative group of order $n = 3$, for which one has:

$$x_1 = y_1\, z_1, \qquad x_2 = y_2\, z_2, \qquad x_3 = y_1\, z_3 + y_3\, z_2\,.$$

Therefore:

$$R(\xi) = \begin{pmatrix} \xi_1 & 0 & \xi_3 \\ 0 & \xi_2 & 0 \\ 0 & \xi_3 & 0 \end{pmatrix}, \quad S(y) = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & y_3 & y_1 \end{pmatrix}, \quad T(z) = \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ z_3 & 0 & z_2 \end{pmatrix},$$

so, when decomposed into elementary parts:

$$| S(x) | = x_1\, x_1\, x_2\,, \quad | T(x) | = x_1\, x_2\, x_2\,, \quad | u\, R(\xi) + vR'(\xi) | = -\, (\xi_1 + \xi_2)\ \xi_3^2\ u\, v\, (u + v).$$

From equation (2), § 2 it then follows by a repeated application:

$$S_x^\kappa = S(x^\kappa), \qquad T_x^\kappa = T(x^k).$$

By u, v, w, I intend this to always mean ordinary (not hypercomplex) quantities here. If $g(u)$ is an entire function of the variables u, so $g((x))$ is an entire function of the hypercomplex quantity x itself then one must have:

(3) $$g(S_x) = S\big(g((x))\big), \qquad g(T_x) = T\big(g((x))\big).$$

Therefore, if one of the expressions $g((x))$, $g(S_x)$, or $g(T_x)$ vanishes for all x then the other two vanish, as well (MOL., § 4). The equation $g(S) = 0$ of the lowest degree that a matrix $S_x = S$ satisfies will be called its *reduced equation*. One obtains the function $g(u)$ when one divides the *characteristic determinant* of S, which is the determinant of the matrix:

(4) $$u\, E - S(x) = S(ue - x),$$

by the greatest common divisor of its subdeterminants of $(n - 1)^{\text{th}}$ degree. From (3), S and T satisfy the same reduced equation $g(S) = 0$ and $g(T) = 0$, and this is likewise the equation $g((x)) = 0$ of lowest degree that a variable quantity x satisfies. Thus, while the determinants of the two antistrophic groups can be different, their first elementary parts always coincide.

The reduced function $g(u)$ vanishes for any of the *characteristic roots* of S, which are roots of the *characteristic equation* $| uE - S | = 0$. We have thus proved once more that the two antistrophic group determinants include precisely the same prime factors.

§ 4.

If $\Theta(x) = |S(x)|$ is the group determinant then from (2), § 2, $\Theta(yz) = \Theta(y)\,\Theta(z)$. In any prime factor $\Phi(x)$ of $\Theta(x)$, I imagine that the constant factor has been chosen in such a way that $\Phi(e) = 1$. Then, if $\Phi(x)$ is such a factor or a product of several of them then one also has:

$$\Phi(yz) = \Phi(y)\,\Phi(z). \tag{1}$$

Conversely, any homogeneous function $\Phi(x)$ that possesses this property is a product of prime factors of $\Theta(x)$ (*Gr.*, § 1). If one then determines y such that $xy = \Theta(x)\,e$ then y_1, y_2, …, y_n will be found from n linear equations with the determinant $\Theta(x)$, and are thus *entire* functions of $x_1, x_2, \ldots, x_n$. If r is the degree of $\Phi(x)$ then, from (1), one has:

$$\Phi(x)\,\Phi(y) = \Theta(x)^r.$$

I call the roots $u_1, u_2, \ldots, u_n$ of the equation $\Phi(ue - x) = 0$ the *characteristic roots* of $\Phi(x)$. One then has:

$$\Phi(ue - x) = u^r - \Phi_1(x)\,u^{r-1} + \Phi_2(x)\,u^{r-2} - \ldots \pm \Phi_r(x) - (u - u_1)\,(u - u_2)\,\ldots\,(u - u_r). \tag{2}$$

I call the sum of r roots, $\Phi_1(x)$ or:

$$u_1 + u_2 + \ldots + u_r = \chi(x) = \sum_\alpha \chi_\alpha x_\alpha\,, \tag{3}$$

the *trace* of $\Phi(x)$. The function $\Phi(x)$ is determined completely by its system of n coefficients, which I likewise denote by *ca* and call the *characteristic parameters* of $\Phi(x)$ (MOL., § 3, *Gr.*, § 3). One then has that:

$$g(u) = a(u + v_1)\,(u + v_2)\,\ldots\,(u + v_m)$$

is an entire function of u, so:

$$g = g\big((x)\big) = a(x + v_1\,e)\,(x + v_2\,e)\,\ldots\,(x + v_m\,e)$$

is, as well, and from (1), so is:

$$\Phi(y) = a^r\,\Phi(x + v_1\,e)\,\Phi(x + v_2\,e)\,\ldots\,\Phi(x + v_m\,e).$$

If one sets:

$$\Phi(x + v\,e) = (u_1 + v)\,(u_2 + v)\,\ldots\,(u_r + v)$$

then one obtains:

$$\Phi(y) = g(u_1)\,g(u_2)\,\ldots\,g(u_r).$$

If one now replaces $g(u)$ with $g(u) - v$ then one sees that $g(u_1), g(u_2), \ldots g(u_r)$ are the characteristic roots of $\Phi(y)$. Therefore:

(4) $$\chi\big(g((x))\big) = g(u_1) + g(u_2) + \dots + g(u_r)$$
and especially:
(5) $$\chi(x^\kappa) = u_1^\kappa + u_2^\kappa + \cdots + u_r^\kappa ,$$
so for $\chi = 0$:
(6) $$r = \chi(e) = \sum_\alpha \chi_\alpha e_\alpha .$$

From the well-known relations between the coefficients of an equation and the power series of its roots then the prime functions of first degree are then:

(7) $$\chi(x)\,\chi(y) - \chi(xy) = 0, \qquad \chi_\beta\,\chi_\gamma = \sum_\alpha a_{\alpha\beta\gamma}\chi_\alpha$$

identically, and for second degree they are:

(8) $$\chi(x)\chi(y)\chi(z) - \chi(x)\chi(yz) - \chi(y)\chi(zx) - \chi(y)\chi(yz) + \chi(xyz) + \chi(xzy) = 0.$$

I set down the general law for the definition of these equations in *Gr.* § 3.

In particular, the number of different linear prime functions is equal to the number of solutions of equations (2), § 1 in ordinary numbers (DED. (19)). As I will show in § 6, these solutions are all linearly independent.

If $\Theta(x)$ is non-zero for a certain value x then, from (1), § 4, one has:

$$\Phi(x^{-1}yz + ue) = \Phi(x^{-1})\,\Phi(y + ue)\,\Phi(x) = \Phi(y + ue)$$

and thus $\chi(x^{-1}yz) = \chi(y)$, or, when one replaces y with xy (MOL., theorem 14: *Gr.*, 3 (13)):
(9) $$\chi(xy) = \chi(yx).$$

The matrix of this symmetric bilinear form $F(\chi, x, y)$ is:

(10) $$R_\kappa = R'_\kappa .$$

One then calls ξ a *symmetric parameter* when it satisfies the equations $r_{\alpha\beta}(\xi) = r_{\beta\alpha}(\xi)$ or $R_\xi = R'_\xi$ then χ is then such a parameter.

A quantity x is called an *invariant quantity* of the group (ε) when it commutes with any quantity y of that group $xy = yx$. Therefore:

$$\sum_\gamma a_{\alpha\beta\gamma}x_\gamma = \sum_\gamma a_{\alpha\gamma\beta}x_\gamma \qquad \text{or} \qquad t_{\alpha\beta}(x) = s_{\alpha\beta}(x).$$

Conversely, if the variability of x is restricted by the equation $S(x) = T(x)$ then x is an invariant quantity of (ε).

From (14), § 1, the general equation $S(x) = T(y)$ or:

$$x_1 E_1 + \ldots + x_n E_n = y_1 F_1 + \ldots + y_n F_n$$

can exist only when $x = y$ is an invariant quantity. The group that is defined by the invariant quantities is then the greatest common divisor of the two antistrophic groups (3) and (4), § 2.

If y varies without restriction, but x is an invariant variable then $S(x) = T(y)$ commutes with $S(y)$ and thus the characteristic roots $u_1, u_2, \ldots$ of the matrix $S(x)$ can be associated with the characteristic roots $v_1, v_2, \ldots$ of the matrix $S(y)$ in such a way that $u_1 v_1, u_2 v_2, \ldots$ are the characteristic roots of $S(x)\,S(y) = S(xy)$. Thus, one also has:

$$\Phi(ue - xy) = (u - u_1 y_1)\,(u - u_2 y_2) \ldots (u - u_r y_r),$$

where $u_1, u_2, \ldots, u_r$ depend upon only x and $v_1, v_2, \ldots, v_r$, upon only y. If one sets $y = e$ then one sees that $u_1, u_2, \ldots, u_r$ are the characteristics roots of $\Phi(x)$, and if one sets $x = e$ then $v_1, v_2, \ldots, v_r$ are those of $\Phi(y)$.

For the determination of the way that these roots are associated with each other in the aforementioned decomposition, I restrict myself to the case where $\Phi(y)$ is a prime function. $\Phi(uv - y)$ is then irreducible as a function of u; i.e., it cannot be represented as a product of entire functions of u whose coefficients are rational functions of the u unrestricted variable quantities $y_1, y_2, \ldots, y_r$. Since u_1 is independent of them, the function $\Phi(ue - u_1 y)$ of the variables u is then irreducible in the same sense. The function $\Phi(ue - xy)$ has the factor $u - u_1 e_1$ in common with it, and as a result, one has $\Phi(ue - xy) = \Phi(ue - u_1 y)$. With this, one has $\chi(xy) = u_1 \chi(y)$, so for $y = e$, one has $c(x) = r u_1$. If one sets $u = 0$, $y = e$ then one will have $\Phi(x) = u_1^r$. Then, from the theorem of (*Gr.*, § 6):

If $\chi(x)$ is the trace of the prime factor of r^{th} degree $\Phi(x)$ of the group determinant, and y is an unrestricted variable quantity, but x is an invariant quantity, then:

$$\chi(x)\chi(y) = v\,\chi(xy) = \chi(v)\chi(xy), \tag{11}$$

and:

$$\Phi(x) = \left(\frac{1}{r}\chi(x)\right)^r, \qquad \Phi(ue - x) = \left(u - \frac{1}{r}\chi(x)\right)^r \tag{12}$$

is the r^{th} power of a linear function and the r characteristic roots of $\Phi(x)$ are all equal to $\frac{1}{r}\chi(x)$.

§ 5.

If $t = xy$ then $t + ux = x(y + ue)$, and as a result, if $\Phi(x)$ is a factor of a power of $\Theta(x)$ then $\Phi(t + ux) = \Phi(x)\,\Phi(y + ue)$. If one compares the coefficients of u^{r-1} in this then one obtains:

$$\sum_\alpha \frac{\partial \Phi(x)}{\partial x_\alpha} t_\alpha = \Phi(x)\chi(y),$$

so because one has $t_\alpha = \sum_\alpha s_{\alpha\beta}(x) y_\beta$:

$$\sum_\alpha \frac{\partial \Phi(x)}{\partial x_\alpha} s_{\alpha\beta} = \Phi(x)\chi_\beta .$$

In this way, and through the decomposition $yx + ux = (y + ue)\, x$, one obtains (*Gr.*, § 5, (3)):

(1) $$\sum_\kappa \frac{\partial \Phi(x)}{\partial x_\kappa} s_{\kappa\alpha}(x) = \sum_\kappa \frac{\partial \Phi(x)}{\partial x_\kappa} t_{\kappa\alpha}(x) = \Phi(x)\chi_\alpha .$$

If one then sets $\xi_\chi = \partial\Phi / \partial x_\kappa$ in formula (5′), § 2 then one will have $\zeta_\mu = \Phi\chi_\mu$, and from (5) and (6), § 2, one therefore (*Gr.*, §, (1)) has:

(2) $$S'(x)\ R\left(\frac{\partial \Phi}{\partial x}\right) = R\left(\frac{\partial \Phi}{\partial x}\right) T(x) = \Phi(x)\ R(\chi),$$

where the matrix is:

$$R\left(\frac{\partial \Phi}{\partial x}\right) = \left(\sum_\kappa a_{\kappa\mu\beta} \frac{\partial \Phi}{\partial x_\kappa}\right).$$

If this goes to:

$$R_1\, u^{r-1} + R_2\, u^{r-2} + \ldots + R$$

when one replaces x with $ue - x$ then, from (2), § 4:

$$(R_1\, u^{r-1} + R_2\, u^{r-2} + \ldots + R)\ (Eu - T) = R_\chi(u^r - \Phi_1\, u^{r-1} + \Phi_2\, u^{r-2} - \ldots \pm \Phi_x).$$

If one compares the coefficients of $u^r, u^{r-1}, \ldots, u^0$ then one obtains $r + 1$ equations. If one multiplies them on the right by $T^r, T^{r-1}, \ldots, T^0$ and adds them then this gives (*Gr.*, § 4, (5)):

(3) $$R_\chi(u^r - \Phi_1\, u^{r-1} + \Phi_2\, u^{r-2} - \ldots \pm \Phi_x) = 0$$

or, more simply:

$$R_\chi\, \Phi(xE - eT) = 0.$$

In the same way, one obtains $\Phi(xE - eS')\ R_\chi = 0$, so when one takes the conjugate matrices, from (10), § 4:

(3) $$R_\chi\, \Phi(xE - eS) = 0, \qquad R_\chi\, \Phi(xE - eT) = 0.$$

From (3), § 3, one can also write $S\big(\Phi(x - ((x)\ e)\big)$ for $\Phi(xE - eS)$, i.e., $S(y)$, where $y = g\big((x)\big)$ and $g(u) = \Phi(x - ue)$.

One also arrives at formula (2), upon which the following development essentially rests, when one sets $t = xyz$ (or yzx), and compares the coefficients of $y_\alpha\, z_\beta$ in the equation

$$\sum_\kappa \frac{\partial \Phi(x)}{\partial x_\kappa} t_\kappa = \Phi(x)\ \chi(yz),$$

from which:

$$t_\kappa = \sum a_{\kappa\lambda\mu}\, x_\kappa\, a_{\nu\alpha\beta}\, y_\alpha\, z_\beta\,, \qquad \chi(yz) = \sum r_{\alpha\beta}(\chi)\, y_\alpha\, z_\beta\ .$$

§ 6.

I will call the smallest number of independent linear couplings between the variables by which a function or a system of functions can be expressed its *linear rank*, or also just its *rank*, as long as this will not be confused with the concept of the rank of a determinant or matrix. The linear rank m of a quadratic function:

$$(1) \qquad \sum_{\alpha,\beta} r_{\alpha\beta}(\chi)\, x_\alpha x_\beta = F(\chi, x, x) = \chi(x^2) = \Phi_1^2 - 2\Phi_2$$

is equal to the rank of the symmetric matrix R_χ that is defined by its coefficients, and equal to the linear rank of the system of n linear functions:

$$(2) \qquad \sum_\beta r_{\alpha\beta}(\chi)\, x_\beta$$

which are one-half the derivatives of the quadratic function. $\chi(x^2)$ can be represented by any m of these linear functions, which are independent of each other. Likewise, its covariant, viz., the symmetric bilinear form:

$$(3) \qquad \sum_{\alpha,\beta} r_{\alpha\beta}(\chi)\, x_\alpha y_\beta = F(\chi, x, y) = \chi(xy) = \chi(yx),$$

can be expressed in terms of m and the m of the n variables $\sum_\beta r_{\alpha\beta}\, y_\beta$ that are independent of each other.

The trilinear function $\chi(xyz) = \chi((xy)\, z)$ is a bilinear function of the coordinates of xy and z, and may thus be expressed in terms of the m variables among the n variables $\sum_\beta r_{\alpha\beta}\, z_\beta$ that are independent of each other. It is equal to $\chi(x(yz)) = \chi(z(xy)) = \chi((zx)\, y)$, so it is also independent of the variables $\sum_\beta r_{\alpha\beta}\, x_\beta$ and the variables $\sum_\beta r_{\alpha\beta}\, y_\beta$. The same thing is true of $\chi(xyzt) = \chi(yztx) = \chi(ztxy) = (txyz)$. Therefore:

$$\chi(x^\kappa) = u_1^\kappa + u_2^\kappa + \cdots + u_r^\kappa$$

also depends upon only the variables (2), so the product $\Phi(x) = u_1\, u_2\, \ldots\, u_r$, is as well, and it is an entire function of any power series. Conversely, the functions $\Phi_1(x)$, $\Phi_2(x)$,

… that appear in the development of $\Phi(ue - x)$ may be expressed in terms of linear combinations of the variables $x_1, x_2, \dots x_n$ upon which $\Phi(x)$ depends, as well as $\chi(x^2)$.

1. *If $\Phi(x)$ is a product of prime factors of the group determinant and $\chi(x)$ is the trace of $\Phi(x)$ then the linear rank of $\Phi(x)$ is equal to that of the quadratic function $\chi(x^2)$, so it is equal to the rank of the matrix R_χ, and the function $\Phi(x)$ may be expressed in terms of the derivatives of $\chi(x^2)$.*

Now, let $\Phi, \Phi', \Phi'', \dots$ be the k different prime factors of Θ and let $\chi, \chi', \chi'', \dots$ be their traces. If $C, C', C'', \dots$ are any matrices of n^{th} degree then I will show: A relation:

$$CR_\chi + C' R_{\chi'} + C'' R_{\chi''} + \dots = 0 \tag{4}$$

can exist only when one has:

$$CR_\chi = C' R_{\chi'} = C'' R_{\chi''} = \dots = 0$$

individually. From (2), § 5, one then has:

$$R_\chi = R\left(\frac{\partial l(\Phi)}{\partial x}\right)T(x),$$

and then, since $|\,T(x)\,|$ is non-zero:

$$CR\left(\frac{\partial l(\Phi)}{\partial x}\right)+C'R\left(\frac{\partial l(\Phi')}{\partial x}\right)+C''R\left(\frac{\partial l(\Phi'')}{\partial x}\right)+\cdots = 0.$$

Now, let $\Phi = \Phi' \Phi'' \dots$ be the product of the $k - 1$ prime functions that are different from Φ. If one multiplies this by $\Phi\Psi$ then one sees that the elements of the matrix $\Psi(x)\, CR\left(\frac{\partial \Phi}{\partial x}\right)$ are all divisible by the function of n^{th} degree $\Phi(x)$, so since Φ and Ψ are relatively prime, the same is true for the matrix $CR\left(\frac{\partial \Phi}{\partial x}\right)$. Since this is, however, only of $(r - 1)^{\text{th}}$ degree, it must be zero. With that:

$$0 = CR\left(\frac{\partial l(\Phi)}{\partial x}\right) = CR_\chi .$$

One can also arrive at this result by means of formula (3), § 5. From this, one has $R_{\chi'}\,\Psi(xE - eS) = 0$, and since any two entire functions of the matrix S commute with each other, one also has:

$$R_{\chi'}\Psi(xE - eS) = R_{\chi''}\,\Phi''(xE - eS)\,\Phi'(xE - eS) \dots = 0.$$

As a result, one also has:

$$CR_\chi\big(\Psi(xE - eS) + \Phi(xE - eS)\big) = 0.$$

The determinant of the matrix in the brackets, which is a function of S, is non-zero, because the function $\Psi(xE - eS) + \Phi(xE - eS)$ of the variables u does not vanish for any root of the characteristic equation. As a result, one must have $CR_\chi = 0$.

If $m, m', m'', \ldots$ are the ranks of the matrices $R_\chi, R_{\chi'}, R_{\chi''}, \ldots$ then among the n variables $\sum_\beta r_{\alpha\beta}(\chi')x_\beta$, precisely m' of them are independent, among the n variables $\sum_\beta r_{\alpha\beta}(\chi'')x_\beta$, precisely m'' of them are independent, etc. The $m + m' + m'' + \ldots$ linear combinations of the n variables $x_1, x_2, \ldots, x_n$ that are thus obtained are, however, all linearly independent. An identity relation in the $x_1, x_2, \ldots, x_n$:

$$\sum_\alpha c_\alpha\left(\sum_\beta r_{\alpha\beta}(\chi)x_\beta\right) + \sum_\alpha c'_\alpha\left(\sum_\beta r_{\alpha\beta}(\chi')x_\beta\right) + \sum_\alpha c''_\alpha\left(\sum_\beta r_{\alpha\beta}(\chi'')x_\beta\right) + \ldots = 0$$

can then exist only when the k partial sums vanish individually. The n equations:

$$\sum_\alpha c_\alpha r_{\alpha\beta}(\chi) + \sum_\alpha c'_\alpha r_{\alpha\beta}(\chi') + \sum_\alpha c''_\alpha r_{\alpha\beta}(\chi'') + \ldots = 0$$

have, in fact, the form (4) when C is a matrix in which one row consists of the elements $c_1, c_2, \ldots, c_n$, while the elements of the other row vanish.

Since $\sum_\alpha \chi_\alpha x_\alpha = \sum_\alpha e_\alpha\left(\sum_\beta r_{\alpha\beta}(\chi)x_\beta\right)$ is a linear combination of the n variables (2), the k functions:

(5) $$\chi(x),\ \chi'(x),\ \chi''(x),\ \ldots$$

are also linearly independent.

Furthermore, $\chi(x^2)$ can be expressed as a quadratic function of m independent variables with non-vanishing determinants, $\chi'(x^2)$, as a function of m' variables of rank m', etc., and these $m + m' + m'' + \ldots$ variables are all mutually independent. If $c, c', c'', \ldots$ are constants then the rank of the quadratic function:

$$c\,\chi(x^2) + c'\chi'(x^2) + c''\chi''(x^2) + \ldots$$

is equal to the sum of the ranks of the individual summands. The rank of $c\chi(x^2)$ is 0 or m, according to whether $c = 0$ or not, resp.

II. *The rank of the matrix* $cR_\chi + c'R_{\chi'} + c''R_{\chi''} + \ldots$ *is equal to the sum of the ranks of the matrices* cR_χ, $c'R_{\chi'}$, $c''R_{\chi''}$, …

The linear rank of a product of prime factors of group determinants is equal to the sum of the ranks of their various prime factors.

§ 7.

I denote the traces of the determinants $|S(x)|$ and $|T(x)|$ by:

(1) $$\sigma(x) = \sum_\alpha \sigma_\alpha x_\alpha = \sum_\kappa s_{\kappa\kappa}(x), \qquad \tau(x) = \sum_\alpha \tau_\alpha x_\alpha = \sum_\kappa t_{\kappa\kappa}(x).$$

Their coefficients are:

(2) $$\sigma_\alpha = \sum_\kappa a_{\kappa\alpha\kappa}, \qquad \tau_\alpha = \sum_\kappa a_{\kappa\kappa\alpha},$$

and from (2), § 3, one has:

(3) $$\sigma = s\chi + s'\chi' + s''\chi'' + \cdots = \sum s\chi,$$
$$\tau = t\chi + t'\chi' + t''\chi'' + \cdots = \sum t\chi.$$

Therefore, from theorem II, § 6 the matrices R_σ and R_τ both have rank $m + m' + m'' + \ldots$

I will call a group (ε), for which the determinant of the symmetric matrices:

(4) $$R_\sigma = P = (p_{\alpha\beta})$$

is non-zero, a *DEDEKIND group* (DED. (27)), since DEDEKIND recognized the meaning of this condition, at least for commutative groups. For such a group, from the theorem of § 2, $\Theta(x) = |S(x)| = |T(x)|$, so $\sigma = \tau$, and the rank of R_σ:

$$n = m + m' + m'' + \ldots = \sum m.$$

Thus, a DEDEKIND group can also be defined to be a group whose order is equal to the linear rank of its determinant. If:

$$\psi = \chi + \chi' + \chi'' + \ldots = \Sigma\,\chi$$

is the trace of the product of the k different prime factors of Θ:

$$\Psi(x) = \Phi\,\Phi'\,\Phi'' \ldots = \Pi\,\Phi(x)$$

then, from theorem II, § 6, R_ψ also has rank $\sum m = n$. Thus, $|R_\psi|$ is non-zero. From (3), § 5 is, however:

$$R_\psi\,\Psi(Se - Ex) = 0,$$

and as a result:

(5) $$\Psi(Se - Ex) = 0.$$

Since the function:

(6) $$g(u) = \Psi(ue - x) = \Pi\,\Phi(ue - x)$$

has no a multiple factor, $g(S) = 0$ is the reduced equation for the matrix S (cf., MOL. theorem 24).

One can also see this as follows: From (2), § 5, one has:

$$S'R\left(\frac{\partial\Theta}{\partial x}\right) = \Theta\, R_\chi\,.$$

Now, $\Theta\, S'^{-1} = \bar{S}'$ is the adjoint matrix to S', and therefore one has (DED. (63)):

$$\bar{S}' = R\left(\frac{\partial\Theta}{\partial x}\right)R_\sigma^{-1} \tag{7}$$

The elements of the matrix $\bar{S}'$ are the sub-determinants of $(u - 1)^{\text{th}}$ degree of Θ. They are then linear combinations of the derivatives of Θ in the n variables $x_1, x_2, \ldots, x_n$. Conversely, the derivatives of a determinant are linear combinations of their sub-determinants. Therefore, the greatest common denominator of the sub-determinants is equal to the derivatives, so when x is now replaced by $ue - x$, it equals:

$$\Pi(\Phi(ue - x))^{-1}.$$

From the rules that were mentioned in § 3, however, one obtains the reduced equation $g(S) = 0$ when one divides each expression by:

$$\Theta(ue - x) = \Pi(\Phi(ue - x)),$$

and consequently $g(u)$ is the function (6). The roots $u_1, u_2, \ldots, u_p$ of the equation $g(u) = 0$ are the $p = r + r' + r'' + \ldots$ distinct roots of the n roots $u_1, u_2, \ldots, u_n$ of the equation $\Theta(ue - x) = 0$.

§ 8.

If A and B are two commuting matrices of n^{th} degree then their characteristic roots $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ can be associated in such a way that $f(a_1, b_1), f(a_2, b_2), \ldots, f(a_n, b_n)$ are the characteristic roots of the matrix $f(A, B)$ and this association is independent of the choice of the entire function $f(u, v)$. In § 3 of the present paper, I have further proved:

I. *If the characteristic determinants of two (or more) mutually commuting matrices A and B are decomposable into nothing but linear elementary parts then any matrix f(A, B) has the same property.*

II. *If, moreover, one always has $b_\kappa = b_\lambda$ whenever $a_\kappa = a_\lambda$ then B is an entire function of A.*

Now:

$$y(u, x) = \Pi\, \Phi(ue - x) \tag{1}$$

is an entire function of p^{th} degree of u whose coefficients are entire functions of the n variables $x_1, x_2, \ldots x_n$. The p roots $u_1, u_2, \ldots u_n$ of the equation $g(u, x) = 0$ are all mutually distinct. If $X = S(x)$ then the elementary parts of $|\, uE - X \,|$ are all linear and X satisfies the equation $g(X, x) = 0$.

This identity equation in $x_1, x_2, \ldots, x_n$ will thus be also satisfied by any special system of values for the variables. For such an equation, $u_1, u_2, \ldots, u_n$ do not all need to be different, the elementary parts of $|\, uE - X \,|$ are not all linear, and $g(X, x) = 0$ does not need to be the reduced equation for X. The equation $|\, uE - X \,| = 0$ cannot then have more than p different roots, and if $\psi(X) = 0$ then $\psi(u)$ must vanish for any roots of this characteristic equation. Since the equation $g(u, x) = 0$ has not multiple roots, the elementary parts of $|\, uE - X \,|$ are all linear.

Let x be chosen as above, and let y be a quantity that commutes with x. The matrix $Y = S(y)$ then commutes with $X = S(x)$. Therefore, the determinant:

$$(2) \qquad |\, uX + vY + wE \,| = |\, S(ux + vy + we) \,| = \Pi\big(\Phi(ux + vy + wz)\big)$$

is a product of linear factors $u\, u_\alpha + v\, v_\alpha + w$, in which u_α and v_α are the associated characteristic roots of the two commuting matrices X and Y and therefore:

$$(3) \qquad \Phi(ux + vy + we) = \prod_\alpha (uu_\alpha + vv_\alpha + w)$$

is also a product of linear functions of u, v, w. If one sets $v = 0$ or $u = 0$ then one recognizes that the r characteristic roots $u_1, \ldots, u_r$ of $\Phi(x)$ are associated with the r roots $v_1, \ldots, v_r$ of $\Phi(y)$ in a certain sequence, and consequently the p roots $u_1, \ldots, u_r$ of $g(u, x) = 0$ are associated with the p roots $v_1, \ldots, v_r$ of $g(v, y) = 0$, as well. Since $u_1, \ldots, u_r$ are distinct, moreover, formulas (2) and (3) show that any two equal roots $u_\kappa = u_\lambda$ among the n characteristic roots of X are associated with $v_\kappa = v_\lambda$ equal roots of Y.

If l is a constant then $v_1 - lu_1, \ldots, v_p - lu_p$ are the roots of the equation $g(w, y - lx) = 0$. If l is not equal to one of the ratios $\dfrac{v_\alpha - v_\beta}{u_\alpha - u_\beta}$ then any p roots are all different, and consequently the elementary parts of the characteristic determinants of the matrix $Z = Y - lX$ are all linear. Since X and Z commute and $Y = lX + Z$ is a function of X and Y, from theorem I, the elementary parts of $|\, vE - Y \,|$ are also all linear. From theorem II, Y is therefore an entire function of X (cf., MOL., theorem 27).

The equation of lowest degree that x satisfies has degree p. Therefore, if $x^0, x^1, \ldots, x^{p-1}$ are linearly independent then any entire function of x is, however, independent of these p. The linear equations $xy - yz = 0$ between the n unknowns $y_1, y_2, \ldots, y_n$ thus have precisely p independent solutions. The matrix of their coefficients $S(x) - T(x)$ thus has rank $n - p$.

III. *If the* $r + r' + r'' + \ldots$ *roots of the equation* $\Pi\, \Phi(ue - x) = 0$ *are all different for a certain quantity* x *in a DEDEKIND group then the matrix* $S(x) - T(x)$ *has the rank* $n - (r + r' + r'' + \ldots)$. *Every quantity* y *that commutes with* x *is then an entire function of* x, *and the elementary parts of* $|\, vE - S(y) \,|$ *are all linear for such a quantity* y.

§ 9.

I will take the theory of DEDEKIND groups to its conclusion in §§ 13 to 16. Now, I shall turn back to the general theory. In place of the basis $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ for the group (ε), one can introduce a new basis by a linear substitution:

$$\varepsilon_\beta = \sum_\alpha c_{\alpha\beta}\bar{\varepsilon}_\alpha \tag{1}$$

in which the determinant of the matrix $C = (c_{\alpha\beta})$ is non-zero. Then if:

$$\sum \varepsilon_\alpha x_\alpha = \sum \bar{\varepsilon}_\alpha \bar{x}_\alpha\,, \tag{2}$$

then:

$$\bar{x}_\alpha = \sum_\beta c_{\alpha\beta} x_\beta\,. \tag{3}$$

The quantities $\partial\Phi \,/\, \partial x_\alpha$, ξ_α , χ_α , σ_α are *cogredient* to the basic numbers ε_α , the quantities y_α, z_α are cogredient to the coordinates x_α, and the basic numbers $\bar{\varepsilon}_\alpha$ are *contragredient*. Furthermore:

$$\begin{aligned} S(x) &= C^{-1}\bar{S}(\bar{x})C, \quad T(x) = C^{-1}\bar{T}(\bar{x})C, \\ R(\xi) &= C'\bar{R}(\bar{\xi})C, \quad R'(\xi) = C'\bar{R}(\bar{\xi})C. \end{aligned} \tag{4}$$

Therefore, the exponents and the degrees of the elementary parts into which the determinants of the matrices $R(x)$, $S(x)$, $T(x)$ decompose are *invariants of the group.* The same is true for the determinants of any of the matrices that depend upon $2n$ variables:

$$S(x) + T(x), \qquad\qquad R(\xi) + R'(\eta), \tag{5}$$

in particular, also for the elementary parts of the determinant of the matrix of the bilinear form $uF(\xi, y, z) + vF(\xi, z, y)$ of y and z:

$$uR(\xi) + vR'(\xi). \tag{6}$$

For the explanation for the remark, I consider some examples from the paper of STUDY, "Über Systeme von complexen Zahlen," Göttinger Nachrichten, 1889. If (ST. IX):

$$R(\xi) = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \xi_2 & \xi_1 & \xi_4 & 0 \\ \xi_3 & -\xi_4 & c\xi_1 & 0 \\ \xi_4 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

then its elementary part is:

$$\xi_1^3\xi_4\,(u+v)\,(u+v)\,\big((u-v)^2+c(u+v)^2\big), \tag{8}$$

in which the last factor is a quadratic elementary part for $c = 0$; otherwise it decomposes into two different linear elementary parts. This formula shows that c is a (non-numerical) invariant of the group.

Furthermore, if (ST. XIV):

$$R(\xi) = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \xi_2 & 0 & \xi_4 & 0 \\ \xi_3 & -\xi_4 & 0 & 0 \\ \xi_4 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

then the elementary part of $|\,u\,R(\xi) + c\,R'(\xi)\,|$:

$$\xi_1^3\xi_4\,(u+c)\,(u+v)\,(u-c)\,(u-v), \tag{10}$$

is essentially different from the above (for $c = 0$). Therefore, I would not like to further into the classification of groups here.

It is possible that the quantities (2) also define a group (ϑ) when the coordinates are subject to certain linear relations. (ϑ) will then be called a *subgroup* of (ε). Its basic numbers ϑ_1, ϑ_2, … are linear combinations of the n basic numbers ε_α, and therefore in order for (ϑ) to be a group, each of the products $\vartheta_\alpha\,\vartheta_\beta$ must be a linear combination of the quantities ϑ_1, ϑ_2, …

If each of the products $\varepsilon_\alpha\,\vartheta_\beta$ is a linear combination of the quantities ϑ_1, ϑ_2, …then (ϑ) is an invariant subgroup [1]) of (ε). Formulas (2), § 1 also define a group (η) when the n basic number ε_α are subject to the relations $\vartheta_1 = 0$, $\vartheta_2 = 0$, or when one considers any two quantities of (ε) being equal mod (ϑ) when their difference belongs to the group (ϑ). They will be *complementary* to those of the invariant subgroup (ϑ) that is called the group *homomorphic* to (ε) (*Accompanying number system* in MOLIEN).

In order to see this, one transforms the basis in such a way that ϑ_1, ϑ_2, … become the basic numbers ε_{m+1}, …, ε_n. The subgroup (ϑ) will then be determined by the m linear equations $x_1 = 0$, …, $x_m = 0$ between the coordinates. Therefore, in order for (ϑ) to be a subgroup it is necessary and sufficient that $a_{\alpha\beta\gamma} = 0$ whenever $\alpha < m$, $\beta > m$ and $\gamma > m$; in order for (ϑ) to be an invariant subgroup it is necessary and sufficient that $a_{\alpha\beta\gamma} = 0$ whenever $\alpha < m$ and also only one of the two numbers β or γ is greater than m (MOL,

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[1]) If $\mathfrak{H}$ is a finite group of order n and $\mathfrak{G}$ is a subgroup of order n/m, and if R is an element of $\mathfrak{H}$ and P, an element of $\mathfrak{G}$ then the hypercomplex quantities $\sum R\,x_R$ define a group (ε) of order n, and the quantities $\sum R\,x_R$, a subgroup (ϑ). It does not have to be an invariant subgroup of (ε) when $\mathfrak{G}$ is an invariant subgroup of $\mathfrak{H}$. However, if $\mathfrak{H} = \mathfrak{G} + \mathfrak{G}A + \mathfrak{G}B + \dots$ then one obtains an invariant subgroup of (ε) of order $n - m$ when one restricts the variability of the n coordinates x_R by the n equations $\sum_P x_P = 0$, $\sum_P x_{AP} = 0$, $\sum_P x_{BP} = 0$, … The homomorphic group (χ) that is complementary to it in (ε) has the same relationship with the group $\mathfrak{H}/\mathfrak{G}$ as (ε) does to $\mathfrak{H}$.

theorem 2). Equations (2), § 1 also define a group when one sets $\varepsilon_{m+1} = \dots = \varepsilon_n = 0$. In formulas (4), § 1, $x_1, \dots, x_m$ depend only upon $y_1, \dots, y_m$ and $z_1, \dots, z_m$. Therefore, in the two matrices $S(x)$ and $T(x)$ all elements vanish when they have the first m rows and the last $n - m$ columns in common. Therefore:

$$S = \begin{pmatrix} S_1 & 0 \\ S_{21} & S_2 \end{pmatrix}, \qquad T = \begin{pmatrix} T_1 & 0 \\ T_{21} & T_2 \end{pmatrix},$$

where, e.g., the partial matrix S_1 only includes the first m rows and columns of S. From $S(y)\,S(z) = S(yz)$, it then follows that $S_1(y)\,S_1(z) = S_1(yz)$, and since $S_1(x) = x_1E_1 + \dots + x_m E_m$ depends only upon then the equations:

$$\varepsilon_\beta\,\varepsilon_\gamma = \sum_{\alpha=1}^{m} a_{\alpha\beta\gamma}\varepsilon_\alpha \qquad (\beta, \gamma = 1, 2, \dots, m)$$

define a *group* (η) that emerges from (ε) when one sets $\varepsilon_{m+1} = \dots = \varepsilon_n = 0$. Since $|S| = |S_1|\,|S_2|$ and $|T| = |T_1|\,|T_2|$, (η) also satisfies the conditions (1), § 3. Furthermore, the determinant of the group (ε) is divisible by the determinant of any group (η) that is homomorphic to (ε), and the same is true for the groups that are antistrophic to (ε) and (η).

If $a_{\alpha\beta\gamma} = 0$, in addition, when $\alpha > m$ and one of the two numbers β or γ is less than m then (ε) *decomposes* into two groups, each of which is an invariant subgroup of (ε), as well as a group homomorphic to (ε).

§ 10.

If $m, m', m'', \dots$ are the ranks of the k matrices $R_\chi, R_{\chi'}, R_{\chi''}, \dots$, then Φ depends upon only the m independent ones among the n linear functions $\sum_\beta r_{\alpha\beta}(\chi)x_\beta$, Φ', upon only the m' independent ones among the functions $\sum_\beta r_{\alpha\beta}(\chi')x_\beta$, etc., and these $m + m' + m'' + \dots$ variables are all mutually independent. One can thus choose the basic numbers in such a way that Φ is also independent of $x_1, \dots, x_m$, Φ', depends upon only $x_{m+1}, \dots, x_{m+m'}$, etc. $\chi(x^2) = \sum r_{\alpha\beta}(\chi)\,x_\alpha\,x_\beta$ then also depends upon only $x_1, \dots, x_m$. Thus, $r_{\alpha\beta}(\chi) = 0$ when α or β is greater than m, and since R_χ has rank m, the determinant of m^{th} degree:

$$|\,r_{\nu\lambda}(\chi)\,| \qquad (\nu, \lambda = 1, 2, \dots, m)$$

is then non-zero. Furthermore, from § 6, the trilinear function:

$$\chi(xyz) = \sum r_{\nu\alpha}(\chi)\,a_{\alpha\beta\gamma}\,x_\nu\,y_\beta\,z_\gamma = \sum r_{\nu\lambda}(\chi)\,a_{\lambda\beta\gamma}\,x_\nu\,y_\beta\,z_\gamma,$$

depends upon only $x_1, \ldots, x_m$, $y_1, \ldots, y_m$, $z_1, \ldots, z_m$. Therefore, if β or γ is greater than m then $\sum_{\lambda=1}^{m} r_{\nu\lambda}(\chi)\, a_{\lambda\beta\gamma} = 0$, and thus $a_{\lambda\beta\gamma} = 0$. Therefore, the k different prime factors of Θ correspond to k groups that are homomorphic with (ε).

More simply: All quantities x whose coordinates satisfy the linear equations $\sum r_{\alpha\beta}(\chi)\, x_\beta = 0$ define an invariant subgroup (ϑ) of (ε). The equations then express the idea that $\chi(tx) = 0$ for any quantity t. If y is an arbitrary quantity of (ε) and one replaces y with yt or ty then one obtains $\chi(t(xy)) = 0$ or $\chi(t(yx)) = 0$. If x then belongs to the complex (ϑ) then it also belongs to xy and yx. Consequently, (ϑ) is an invariant subgroup of (ε).

A group (ε) of order r is called *simple* (MOLIEN: *original number system*) when it has no invariant subgroup (except for (ε)), so it has no homomorphic group of order less than n.

From the development above, the determinant Θ of a simple group includes no prime factor whose linear rank is $m < n$. Therefore, from theorem II, § 6, Θ must not include two different prime factors, so $\Theta = \Phi^s$ must be a power of a prime function and therefore $\sigma = s\chi$. Furthermore, the rank m of Φ, and thus, the rank of the matrix $R_\sigma = sR_\chi$, must be equal to $m = n = rs$, so (ε) must be a DEDEKIND group.

Conversely, if $|R_\sigma|$ is non-zero and $\Theta = \Phi^s$ is a power of a prime function then (ε) is a simple group, so it is not homomorphic to a group (η) whose order $n' < n$ (MOL., theorem 23). The determinant of (η) is a divisor of Θ, so it equals $\Theta' = \Phi^{s'}$, and thus has the same rank as Φ, namely, n. The rank n of Θ' cannot, however, be larger than the order n' of (η).

Since $S(x)$ and $T(y)$ are commuting matrices the determinant:

$$| u\, S(x) + v\, T(y) + W\, e |$$

decomposes (*Gr.*, § 10, (1)) into a product of linear functions of u, v, w. If $uu_1 + vv_1 + w$ is one of them then one sees, when one sets $v = 0$ ($u = 0$, resp.) that u_1 (v_1, resp.) is a characteristic root of $\Phi(x) = (\Phi(y))$. If one considers y to be constant, but x to be variable then $\Phi(ux + we)$ is irreducible as a function of w, and thus $\Phi(ux + (vv_1 + w)e)$, as well. The determinant has the factor $uu_1 + vv_1 + w$ in common with this function, and thus, all r factors $uu_\alpha + vv_1 + w$. If one then considers x to be constant and y to be variable then one sees that the determinant of each of the r^2 factors is:

$$uu_\alpha + vv_\beta + w \qquad (\alpha, \beta = 1, 2, \ldots, r),$$

and each of them is included equally often. Therefore:

$$| u\, S(x) + v\, T(y) + w\, E | = (\Pi\, (uu_\alpha + vv_\beta + w))^c$$

and therefore $s = rc$. In particular (*Gr.*, § 10, (8)):

$$(1) \qquad | S(x) - T(x) + w\, E | = w^s \prod_{\beta > \alpha} \left(w^2 - (u_\beta - u_\alpha)^2 \right)^c .$$

The elementary parts of the characteristic determinant of the two commuting matrices $S(x)$ and $T(x)$ are all linear. From theorem I, § 8, the elementary parts of the determinant (1), and especially the s for $w = 0$ also vanish. Consequently, the matrix $S(x) - T(x)$ has rank $n - s$.

From theorem III, § 8, however, they have rank $n - r$. Therefore (MOL., theorem 29):

(2) $$r = s, \quad m = n = r^2,$$

so $c = 1$ and:

(3) $$|\, u\, S(x) + v\, T(y) + w\, E \,| = \prod_{\alpha,\beta=1}^{r} (uu_\alpha + vv_\beta + w).$$

The determinant of a simple group is a power of a prime function whose exponent equals the degree of the prime function. The linear rank of this function, which agrees with the order of the group, is equal to the square of its degree.

§ 11.

If, the r roots of the equation $\Phi(ue - h) = 0$ all vanish for a certain quantity h in the simple group (ε) of order $n = r^2$, and if a is a particular one of these r roots then the determinant $|\, S(ue - h) \,|$ has nothing but linear elementary parts, and consequently the matrix $S(ue - h)$ has rank $n - r$. Therefore, (MOL., § 8), the linear equations $(ae - h)\, t = 0$ in the coordinates $t_1, \ldots, t_n$ of the unknown quantity t have precisely r independent solutions $t = t^{(1)}, t^{(2)}, \ldots, t^{(r)}$.

If x is a variable quantity then $t^{(\kappa)} x$ is also a solution and therefore:

$$t^{(\kappa)} x = \sum_{\lambda} x_{\kappa\lambda} t^{(\lambda)} .$$

Since the r solutions $t^{(\lambda)}$ are independent the quantities $x_{\kappa\lambda} = f_{\kappa\lambda}(x)$ are linear functions of the coordinates $x_1, \ldots, x_n$. If y and z are two other quantities, and one sets $y_{\kappa\lambda} = f_{\kappa\lambda}(y)$, $z_{\kappa\lambda} = f_{\kappa\lambda}(z)$ then one also has:

$$t^{(\kappa)} y = \sum_{\lambda} y_{\kappa\lambda} t^{(\lambda)} , \qquad t^{(\kappa)} z = \sum_{\lambda} z_{\kappa\lambda} t^{(\lambda)}$$

and thus:

$$t^{(\kappa)} yz = \sum_{\lambda} y_{\kappa\lambda} t^{(\lambda)} z = \sum_{\lambda,\mu} y_{\chi\mu} z_{\mu\lambda} t^{(\lambda)} .$$

Thus, if $x = yz$ then:

(1) $$x_{\kappa\lambda} = \sum_{\mu} y_{\kappa\mu} z_{\mu\lambda} .$$

Among the $n = r^2$ linear functions $x_{\mu\lambda} = f_{\mu\lambda}(x)$, let m of them be independent. Therefore, from (1), if $x = yz$ then these m combinations of $x_1, \ldots, x_n$ depend on m combination of $y_1, \ldots, y_n$ and those of $z_1, \ldots, z_n$. Consequently, (ε) has a homomorphic group of order m.

However, since (ε) is simple one has $m = n = r^2$, and one can then introduce r^2 new basic numbers, such that:

$$\sum \varepsilon_\alpha x_\alpha = \sum \varepsilon_{\mu\lambda} x_{\mu\lambda} .$$

For this to be true, formulas (4), § 1 must go over to (1) and the relations (2), § 1 to:

$$(2) \qquad \varepsilon_{\alpha\beta}\,\varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}, \qquad \varepsilon_{\alpha\delta}\,\varepsilon_{\beta\gamma} = 0 \qquad (\beta \neq \delta).$$

As was shown at the outset in § 10, the k different prime factors of Θ correspond to k groups that are homomorphic to (ε). The first one has order m, and its determinant is a part of Θ and depends only upon $x_1, \ldots, x_m$. Since $\Phi, \Phi', \ldots$ are independent of $x_1, \ldots, x_m$ then its determinant is a power of Φ, and thus, like Φ, it has linear rank m. Because its order is likewise m, it is, from § 10, it is a simple group. Consequently, its determinant is equal to Φ^r and the rank of Φ is $m = r^2$.

I. *The linear rank of each prime factor of the group determinant is equal to the square of its degree.*

I will now denote the k different functions Φ by $\Phi_1, \Phi_2, \ldots, \Phi_k$. All elements $s_{\alpha\beta}$ vanish in the group determinant S that have the first m_1 rows in common with the last $n - m_1$ columns, and likewise all elements that have the following m_2 rows in common with the m_1 and the last $n - m_1 - m_2$ columns, and furthermore, the ones that have the following m_3 rows in common with the first $m_1 + m_2$ columns and the last $n - m_1 - m_2 - m_3$ columns, etc., while nothing is known about the last $n - m_1 - \ldots - m_k$ columns. The same is true for the antistrophic matrix T, and also for $u\,S(x) + v\,T(y) + w\,E$. If the first m_1 rows and columns of S (T, resp.), define the matrix S_1 (T_1, resp.) and the following m_2 rows and columns of the matrix S_2 (T_2 , resp.) then let:

$$U = \begin{pmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & S_k \end{pmatrix}, \qquad V = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ . & . & \cdots & . \\ 0 & 0 & \cdots & T_k \end{pmatrix}.$$

Therefore:

$$S = \begin{pmatrix} U & 0 \\ U_0 & S_0 \end{pmatrix}, \qquad T = \begin{pmatrix} V & 0 \\ V_0 & T_0 \end{pmatrix}.$$

This gives us the theorem of MOLIEN:

II. *Any group is homomorphic to a DEDEKIND group whose determinant includes every prime factor of its determinant* Θ *and whose order is equal to the linear rank of* Θ.

This rank m is equal to the rank of the quadratic form $\sigma(x^2) = \sum p_{\alpha\beta}\, x_\alpha\, x_\beta$, and in order to obtain a group one needs only to introduce the m independent ones among the n

functions $\sum_\beta p_{\alpha\beta} x_\beta$ as new variables. Therefore, in formula (4), § 9 the substitution C is chosen in such a way that it transforms the form $\sigma(x^2)$ into a function of m variables, perhaps into a sum of m squares. I have not, however, succeeded in proving this directly; i.e., without any use of the prime function Φ_m .

Furthermore, one has:

$$| S | = | S_1 | \ldots | S_k | | S_0 |, \qquad | T | = | T_1 | \ldots | T_k | | T_0 |,$$

and:

$$| uS(x) + vT(y) + wE | = | uS_1 + vT_1 + wE_1 | \ldots | uS_k + vT_k + wE_k | | uS_0 + vT_0 + wE_0 |.$$

Since S_κ is the matrix of a simple group, from (3), § 10, one has:

$$\Psi_\kappa = | uS_\kappa(x) + vT_\kappa(y) + wE_\kappa | = \Pi\,(uu_\alpha + vv_\beta + w),$$

where $u_1, \ldots, u_r$ are the characteristic roots of $\Phi_\kappa(x)$ and $v_1, \ldots, v_r$ are the characteristic roots of $\Phi_\kappa(y)$. These results remain unchanged when the basic numbers of (ε) are chosen arbitrarily, rather than in the manner described.

$\Psi_0 = | uS_0(x) + vT_0(y) + wE_0 |$ also decomposes into linear factors $uu_0 + vv_0 + w$. Here, u_0 is a characteristic root of a prime factor $\Phi_\kappa(x)$ and v_κ is a characteristic root of a prime factor $\Phi_\lambda(y)$. From the irreducibility of $\Phi_\kappa(ue - x)$ and $\Phi_\lambda(ve - y)$ it then follows that Ψ_0 also includes all $r_\kappa\, r_\lambda$ factors of the products $\Pi\,(uu_\alpha + vv_\beta + w)$ in which u_α runs through the r_κ characteristics roots of $\Phi_\kappa(x)$ and v_β runs through the r_λ roots of $\Phi_\lambda(y)$. If $\kappa = \lambda$ then the factor Φ_κ appears repeatedly. As the example in § 3 shows, however, κ can be different from λ.

If one chooses an invariant quantity for x then, from (12), § 4, the r_κ quantities u_α are all equal to $\frac{1}{r_\kappa}\chi^{(\kappa)}(x)$. However, formulas (2) and (3), § 8 show that the root v_β is then associated with $\frac{1}{r_\lambda}\chi^{(\lambda)}(x)$. Therefore, κ can be different from λ only for an invariant quantity x:

(1) $$\frac{1}{r_\kappa}\chi^{(\kappa)}(x) = \frac{1}{r_\lambda}\chi^{(\lambda)}(x) \qquad (S_x = T_x)$$

Now, let $w = 0$, $u = v = 1$, and:

(2) $$\Psi_{\kappa\lambda}(x, y) = \Pi(u_\alpha + v_\beta),$$

where u_α runs through the r_κ characteristic roots of $\Phi_\kappa(x)$ and v_β runs through the r_λ roots of $\Phi_\lambda(y)$. $\Psi_{\kappa\lambda}$ is then an irreducible function of $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ relative to any of degree r_κ, r_λ , resp., and one has:

(3) $$\Psi(x, y) = | S(x) + T(y) | = \prod \left(\Psi_{\kappa\lambda}(x, y)\right)^{c_{\kappa\lambda}} ,$$

where the whole number $c_{\kappa\lambda} > 0$, but $c_{\kappa\lambda} \geq 0$, and only under the condition (1) can one have $c_{\kappa\lambda} > 0$. Since:

$$\Psi_{\kappa\lambda}(x, 0) = \Phi_{\kappa}(x)^{r_\lambda}, \qquad \Psi_{\kappa\lambda}(0, y) = \Phi_{\lambda}(x)^{r_\kappa},$$

this gives the decompositions:

$$(4) \qquad |\, S(x)\,| = \prod \Phi_{\lambda}(x)^{s_\lambda}, \qquad |\, T(x)\,| = \prod \Phi_{\lambda}(x)^{t_\lambda}$$

where the exponents that appear have the expressions:

$$(5) \qquad \begin{aligned} s_\lambda &= c_{\lambda 1} r_1 + c_{\lambda 2} r_2 + \cdots + c_{\lambda k} r_k, \\ t_\lambda &= c_{1\lambda} r_1 + c_{2\lambda} r_2 + \cdots + c_{k\lambda} r_k, \end{aligned}$$

such that one always has:

$$(6) \qquad s_\lambda \geq r_\lambda, \qquad t_\lambda \geq r_\lambda .$$

The two functions $|\, S(x)\,|$ and $|\, T(x)$ have the same linear rank:

$$(7) \qquad m = r_1^2 + r_2^2 + \cdots + r_k^2,$$

which is equal to the tank of the matrix R_σ and that of R_τ, and is equal to the sum of the ranks of the k matrices $R(\chi^{(n)})$ or the k prime functions $\Phi_\kappa(x)$. By contrast, the order of the group (ε):

$$(8) \qquad n = r_1\, s_1 + \ldots + r_k\, s_k = r_1\, t_1 + \ldots + r_k\, t_k = \sum c_{\kappa\lambda}\, r_\kappa\, r_\lambda,$$

in which $r_\kappa\, r_\lambda$ has the coefficients $c_{\kappa\lambda} + c_{\lambda\kappa}$ in the event that κ is different from λ.

§ 13.

I now recall the theory that of DEDEKIND groups that was begun in §§ 7 and 8. For such a group, $m = n$, so, from (6), (7), and (8), § 12, $s_\kappa = t_\kappa = r_\kappa$.

I. *The exponent of the power of a prime function that enters in the determinant of a DEDEKIND group is equal to the degree of this function.*

The group (ε) decomposes into k simple groups of order r^2, r'^2, … in each of which $S(x)$ decomposes into r identity matrices of r^{th} degree, and the r^2 elements $x_{\kappa\lambda}$ of such a matrix are mutually independent of the variables of the other simple groups. In my paper "Über die Darstellungen der endlichen Gruppen durch lineare Substitutionen II," Sitzungsberichte, 1899, § 5, I derived the theorem:

II. *If a and b are two well-defined quantities of a DEDEKIND group and if the elementary invariants of the two matrices $S(ue - a)$ and $S(ue - b)$ agree then there is a* *quantity c in the group for which* $| S(c) |$ *is non-zero, and which satisfies the condition* $c^{-1}ac = b$.

The determinant of a DEDEKIND group is:

(1) $$\Theta(x) = | S(x) | = | T(x) | = \Pi\, \Phi^r,$$

its trace is:

(2) $$\sigma = \tau = r\,\chi + r'\,c' + r''\,c'' + \ldots = \sum r\,\chi,$$

so one also has:

(3) $$P = R_\sigma = r\,R_\chi + r'\,R_{\chi'} + r''\,R_{\chi''} + \ldots = \sum r\,R_\chi\,.$$

The matrix R_χ has rank r^2, while from theorem II, § 6, the matrix $r\,R_\chi + r'\,R_{\chi'} + \ldots = P - r\,R_\chi$ has rank $r'^2 + r''^2 + \ldots = n - r^2$. The determinant $| uP - rR_\chi |$ thus vanishes for $u = 0$, at least for order $n - r^2$, for $u = 1$, at least of order r^2, and since it does not vanish for more than r values, so one not only has:

(4) $$| uP - rR_\chi | = | P |\, u^{n-r^2} (u-1)^{r^2},$$

but also that the elementary parts of this determinant are all linear (because they would vanish to higher order for either $u = 0$ or $u = 1$). Consequently, the matrix $H = r\,P^{-1}R_\chi$ satisfies the equation $H^2 = H$, so one has:

(5) $$R_\chi\, P^{-1} R_\chi = \frac{1}{r}\, R_\chi\,.$$

If one then adds the factor $R_\chi\, P^{-1}$ to the left-hand side of (3) then one obtains:

$$0 = r'\,(R_\chi\, P^{-1})\, R_{\chi'} + r''\,(R_\chi\, P^{-1})\, R_{\chi''} + \ldots$$

and thus, from (4), § 6:

(6) $$R_\chi\, P^{-1} R_{\chi'} = 0.$$

If the equations:

(7) $$\xi_\alpha = \sum_\beta p_{\alpha\beta}\, x_\beta\,,$$

exist between $\xi_1, \ldots, \xi_n$ and $x_1, \ldots, x_n$, then I call the parameter ξ and the quantity r *conjugate.* One then has:

(8) $$R_\xi = S'_x P = PT_x, \qquad R'_\xi = PS_x = T'_x P\,.$$

Thus, if $R_\xi = R'_\xi$ then $S_x = T_x$, and a symmetric parameter is conjugate to an invariant quantity, and conversely.

I will call the invariant quantity $x = h$ that is conjugate to the *characteristic parameter* $\xi = \chi$ a *characteristic quantity* for the prime function Φ. One then has:

(9) $$\chi_\alpha = \sum_\beta p_{\alpha\beta} h_\beta$$

and

(10) $$R_\chi = PS_h = PT_h\,.$$

From (5) and (10), it follows that $\frac{1}{r} S(h) = S(h)^2 = S(h^2)$ and $0 = S(h)\,S(h') = S(hh')$, hence (*Gr.*, § 5, (9)):

(11) $$h^2 = \frac{1}{r}\,h, \qquad hh' = 0$$

and from (3):

(12) $$e = rh + r'\,h' + r''\,h'' + \ldots = \sum rh\,,$$

such that parameter σ and the principal unit e are conjugate. Furthermore, if:

$$r\,R_\chi\,S_h = R_\chi, \qquad r\sum_\lambda r_{\alpha\lambda}(\chi)\,s_{\lambda\beta}(h) = r_{\alpha\beta}(\chi).$$

If one multiplies this by $e_\alpha\,e_\beta$ and sums over α and β the one obtains (cf., DED. (13)):

(13) $$\sum_\alpha \chi_\alpha h_\alpha = 1, \qquad \sum_\alpha \chi'_\alpha h_\alpha = 0,$$

or when one sets:

$$h(x) = h_1\,x_1 + \ldots + h_n\,x_n$$

one gets:

(14) $$\chi(h) = h(\chi) = 1, \qquad \chi(h') = h(\chi') = 0,$$

or finally, if $P^{-1} = Q = (q_{\alpha\beta})$:

(15) $$\sum p_{\alpha\beta}\,h_\alpha\,h_\beta = \sum q_{\alpha\beta}\,\chi_\alpha\,\chi_\beta = 1, \qquad \sum p_{\alpha\beta}\,h_\alpha\,h'_\beta = \sum q_{\alpha\beta}\,\chi_\alpha\,\chi'_\beta = 0.$$

§ 14.

Let $u_1, \ldots, u_r$ be the characteristic roots of $\Phi(x)$, and let $v_1, \ldots, v_r$ those of $\Phi(y)$. Then, if $g(u, v)$ is an entire function of u and v then, from § 12, the u characteristic roots of the matrix $g(S_x, T_y)$ are the r^2 quantities $g(u_\alpha, v_\beta)$ and the $r'^2, r''^2, \ldots$ are quantities that are obtained in the stated way.

If x is an invariant quantity then, from the theorem of § 4, one has $u_1 = \ldots = u_r = \frac{1}{r}\,\chi(x) = c$. Therefore, from (13), § 13, the characteristic determinant of $S_h = T_h$ equals $\left(u - \frac{1}{r}\right)^{r^2} u^{n-r^2}$ and the characteristic roots of $(S(x) - cE)\;T(h) = S((x - ce)h)$ are all zero. Some power of this matrix then vanishes. However, since it commutes with any group matrix $S(y)$, from theorem III, § 8 the elementary parts of its characteristic determinant are all linear. Thus, the first power vanishes, and therefore:

$$xh = ch = h\,\frac{1}{r}\chi(x). \tag{1}$$

If one then multiplies (12), § 13 by x then one obtains:

$$x = rch + r'c'h' + r''c''h'' + \ldots = h\,\chi(x) + h'\,\chi'(x) + h''\,\chi''(x) + \ldots, \tag{2}$$

where the coefficients $rc = \chi(x)$, $r'c' = \chi'(x)$, … are ordinary quantities.

The k quantities h, h', h'', …, which are linearly independent, from (5), § 6 or (11), § 13, thus define a complete system of solutions to the linear equations $S(x) = T(x)$. Among the n^2 equations $s_{\alpha\beta}(x) = t_{\alpha\beta}(x)$ then $n - k$ of them are independent. Now, when the quantity x satisfies the equations $\chi(x) = \chi'(x) = \ldots = 0$, as well, from (2), one has $x = 0$.

One can also see this as follows: If $xy = yx$ then, from (11), § 4, $r\chi(xy) = \chi(x)\chi(y)$. Thus, if $\chi(x) = 0$ then one also has $\chi(xy) = 0$ or $\sum_\beta r_{\alpha\beta}(\chi)x_\beta = 0$. If this is also true for χ', χ'', … then this gives $r^2 + r'^2 + \ldots = n$ independent linear equations, and therefore $x_\xi = 0$.

I. *Among the n^2 linear functions $s_{\alpha\beta}(x) - t_{\alpha\beta}(x)$ of the n variables $x_1, \ldots, x_n$, $n - k$ of them are independent. Together with the k functions $\chi(x)$, $\chi'(x)$, ..., they define n independent functions. The k characteristic quantities h, h', … define a complete system of independent solutions of the equations $s_{\alpha\beta}(x) = t_{\alpha\beta}(x)$.*

II. *Among the linear functions $r_{\alpha\beta}(\xi) - r_{\beta\alpha}(\xi)$ of the n variables $\xi_1, \ldots, \xi_n$, $n - k$ of them are independent. Together with the k functions $h(\xi)$, $h'(\xi)$, …, they define n independent functions. The k characteristic parameters χ, χ', … define a complete system of independent solutions of the equations $r_{\alpha\beta}(\xi) = r_{\beta\alpha}(\xi)$.*

A special case of this theorem is the elegant criterion that MOLIEN gave in theorems 9 and 10, § 3 for the simplicity of a group. Any simple group is, from § 10, a DEDEKIND group, and in order for it to be simple it is necessary and sufficient that $k = 1$.

III. *In order for a group to be simple, it is necessary and sufficient that the determinant of n^{th} degree $\left|\sum_\kappa \sigma_\kappa a_{\kappa\alpha\beta}\right|$ be non-zero, and that $\xi_\kappa = \sigma_\kappa$ is the only solution to the linear equations:*

$$\sum_\kappa (a_{\kappa\alpha\beta} - a_{\kappa\beta\alpha})\,\xi_\kappa = 0.$$

Furthermore:

IV. *In order for a group to be simple, it is necessary and sufficient that the determinant of nth degree* $\left|\sum_\kappa \sigma_\kappa a_{\kappa\alpha\beta}\right|$ *be non-zero and that* $x_\lambda = e_\lambda$ *is the only solution of the linear equations:*

$$\sum_\lambda (a_{\alpha\lambda\beta} - a_{\alpha\beta\lambda}) x_\lambda = 0.$$

The fact that any theorem appears in two forms has its basis in the fact that the group itself can be presented in two *conjugate* forms. If the parameters ξ, η, ζ are conjugate to the quantities x, y, z then let:

$$F(\xi, y, z) = G(x, y, \zeta) = \sum b_{\alpha\beta\gamma} x_\alpha \eta_\beta \zeta_\gamma .$$

One then has:

$$a_{\alpha\beta\gamma} = \sum_\lambda b_{\beta\lambda\alpha} p_{\lambda\gamma} = \sum_\lambda b_{\gamma\alpha\lambda} p_{\lambda\beta} . \tag{3}$$

The group (ε) goes to an equivalent group $(\bar{\varepsilon})$ under this transformation, namely, the *conjugate* group. If one appeals to the relations (4), § 9 for this then, as long as one sets $C = P$ in each formula, this yields:

$$R_x = \bar{T}_x P, \qquad R'_x = \bar{S}_x P, \qquad S_x = \bar{R}'_x P, \qquad T_x = \bar{R}_x P . \tag{4}$$

Furthermore, $\bar{\sigma}_\alpha = e_\alpha$. Just as one sets $\bar{a}_{\alpha\beta\gamma} = b_{\alpha\beta\gamma}$, one sets $\bar{p}_{\alpha\beta} = q_{\alpha\beta}$. One then has $q_{\alpha\beta} = \sum e_\kappa b_{\kappa\alpha\beta}$, and therefore:

$$\sum_\lambda p_{\alpha\lambda} q_{\lambda\beta} = \sum_{\kappa,\lambda} p_{\alpha\lambda} e_\kappa b_{\kappa\lambda\beta} = \sum_\kappa e_\kappa a_{\beta\kappa\alpha} = e_{\alpha\beta} ,$$

so $Q = P^{-1}$. Consequently, the conjugate group of $(\bar{\varepsilon})$ is again the original group:

$$b_{\alpha\beta\gamma} = \sum_\lambda a_{\beta\lambda\alpha} q_{\lambda\gamma} = \sum b_{\gamma\alpha\lambda} q_{\lambda\beta} . \tag{5}$$

§ 15.

In § 6, (5) it was shown that the k functions $\chi(x)$, $\chi'(x)$, … are linearly independent. From theorem I, § 14, this is also still true when x does not vary without restrictions, but only runs through the invariant quantities of (ε). This defines a subgroup (η) of (ε), whose order is k (in general, not an invariant subgroup). One choose the basic numbers $\eta_1, \eta_2, \ldots, \eta_k$ of (η) to be the k quantities rh, $r'h'$, … One then has:

$$\eta_\kappa^2 = \eta_\kappa , \qquad \eta_\kappa \eta_\lambda = 0. \tag{1}$$

Consequently, (η) is a DEDEKIND commutative group whose determinant equals $\prod \frac{1}{r}\chi(x)$, as long as the quantities $\eta_1, \ldots, \eta_k$ are expressed in terms of the $\varepsilon_1, \ldots, \varepsilon_k$. In fact, these k linear factors, in which the variability of the coordinates is restricted by the conditions $s_{\alpha\beta}(x) = t_{\alpha\beta}(x)$, are linearly independent.

I. *If k is the number of different prime factors of the determinant of a DEDEKIND group then its invariant quantities define a commutative group of order k, which is likewise a DEDEKIND group. Its determinant is* $\prod \frac{1}{r}\chi(x)$, *while the determinant of the given group is equal to* $\prod \left(\frac{1}{r}\chi(x)\right)^{r^2}$ *for an invariant variable x.*

If the degree $r = r' = \ldots = 1$ then $k = n$, so, from theorem I, § 12, one has $s_{\alpha\beta}(x) = t_{\alpha\beta}(x)$, $a_{\alpha\beta\gamma} = a_{\alpha\gamma\beta}$, identically.

II. *The determinant of a DEDEKIND group always decomposes when the group is commutative, but only into nothing but linear factors.*

In order for the determinant of an arbitrary group to decompose into nothing but linear factors, from theorem II, § 12, it is necessary and sufficient that the DEDEKIND group of order $m = \sum r^2$ that it is homomorphic to possess this property, and is therefore a commutative group. One will obtain the same thing when one introduces the m independent variables among the n functions $\sum_\beta p_{\alpha\beta} t_\beta$ as coordinates of a quantity t. If one then sets $t = yz$ then these bilinear functions of $y_1, \ldots, y_n, z_1, \ldots, z_n$ must remain unchanged under exchanges of y and z. If x is a third variable then one has the function:

$$\sum_{\alpha,\beta} p_{\alpha\beta} x_\alpha t_\beta = \sigma(xt) = \sigma(xyz)$$

have the same property. From § 6, this always remains true under a cyclic permutation of x, y, z, and thus, under any permutation, in the case considered. It is the trace of the matrix $S(xyz) = S(x)\, S(y)\, S(z)$, so it equals:

$$\sigma(xyz) = \sum_{\kappa,\lambda,\mu} s_{\lambda\mu}(x) s_{\mu\kappa}(y) s_{\kappa\lambda}(z) = \sum a_{\lambda\alpha\mu}\, a_{\mu\beta\kappa}\, a_{\kappa\gamma\lambda}\, x_\alpha\, y_\beta\, z_\gamma . \tag{2}$$

If one employs the antistrophic group then $\tau(xyz)$ appears in place of $\sigma(xyz)$. Thus, yields the theorem (cf., CARTAN, *I. Thèse, Sur la structure des groupes de transformations finis et continus,* Paris, 1894, pp. 48, (5)):

III. *In order for the determinant of a group to be decomposable into nothing but linear factors, it is necessary and sufficient that the trilinear functions* $s(xyz)$ (*or* $t(xyz)$)

remains unchanged under permutation of y and z, so it is a symmetric function of the three sequences of variables, such that the expression:

(3) $$\sum_{\kappa,\lambda,\mu} a_{\lambda\alpha\mu} a_{\mu\beta\kappa} a_{\kappa\gamma\lambda} \qquad or \qquad \sum_{\kappa,\lambda,\mu} a_{\lambda\mu\alpha} a_{\mu\kappa\beta} a_{\kappa\lambda\gamma}$$

also remains unchanged under a transposition of α, β, γ.

Since $s(xyz)$ is also the trace of $S(x)\ S(yz)$, in place of the sums (3), one can also take the expressions:

(4) $$\sum_{\kappa,\lambda,\mu} a_{\kappa\lambda\mu} a_{\mu\alpha\kappa} a_{\lambda\beta\gamma} \qquad or \qquad \sum_{\kappa,\lambda,\mu} a_{\kappa\lambda\mu} a_{\lambda\kappa\alpha} a_{\mu\beta\gamma}\ .$$

By means of the formulas (1) and (2), § 7, one finally obtains the expressions:

(5) $$\begin{aligned}&\sum_{\kappa,\lambda,\mu} a_{\kappa\lambda\kappa} a_{\lambda\alpha\mu} a_{\mu\beta\gamma} \ \ or \ \ \sum_{\kappa,\lambda,\mu} a_{\kappa\kappa\lambda} a_{\lambda\alpha\mu} a_{\mu\beta\gamma},\\ &\sum_{\kappa,\lambda,\mu} a_{\kappa\lambda\kappa} a_{\lambda\mu\alpha} a_{\mu\beta\gamma} \ \ or \ \ \sum_{\kappa,\lambda,\mu} a_{\kappa\kappa\lambda} a_{\lambda\mu\alpha} a_{\mu\beta\gamma}.\end{aligned}$$

The decomposition of the determinant of a DEDEKIND group into its prime factors may be performed with the help of the theorem above. The linear equations $s_{\alpha\beta}(x) - t_{\alpha\beta}(x) = 0$ have k independent solutions. One know poses the quadratic equations between the unknowns $\chi_1, \chi_2, \ldots, \chi_n$, which are obtained from:

(6) $$\chi(x)\ \chi(y) = \chi(e)\ \chi(xy),$$

when one replaces x with a sequence of any k solutions and y, with the n basic numbers. In combination with the linear equations $r_{\alpha\beta}(\chi) = r_{\beta\alpha}(\chi)$ the deliver k different systems of values for the ratios $\chi_1, \chi_2, \ldots, \chi_n$. One chooses the constant factor in such a way that $\sum q_{\alpha\beta}\, \chi_\alpha\, \chi_\beta = 1$ and $\sum \chi_\alpha\, e_\alpha$ is positive. It will then be equal to a positive whole number r, which is the degree of the prime function Φ that is determined from the characteristic parameter χ.

One denotes k independent solutions of the equations $s_{\alpha\beta}(x) = t_{\alpha\beta}(x)$ by $g, g', \ldots$ or also by $\eta_1, \eta_2, \ldots, \eta_k$. They define the basis for a commutative DEDEKIND group of order k, so one has $\eta_\beta\, \eta_\gamma = \sum_\alpha c_{\alpha\beta\gamma} \eta_\alpha$, where $c_{\alpha\beta\gamma} = c_{\alpha\gamma\beta}$. Its determinant is a product of k independent linear factors, and when the basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is again introduced it goes to $\Pi\ \psi(x)$, where $\psi(x) = \sum \psi_\alpha\, x_\alpha$. Since $\psi_1, \psi_2, \ldots, \psi_n$ include an arbitrary constant factor, one then determines the ratios of the unknowns $\chi_1, \chi_2, \ldots, \chi_n$ from the linear equations:

$$r_{\alpha\beta}(\chi) - r_{\beta\alpha}(\chi) = 0, \quad g(\chi) = g(\psi), \quad g'(\chi) = g'(\psi), \ \ldots$$

and from them, the arbitrary values, as before.

§ 16.

If $E_1, E_2, \dots, E_n$ are n matrices of m^{th} degree that satisfy the conditions:

$$E_\beta E_\gamma = \sum_\alpha a_{\alpha\beta\gamma} E_\alpha \tag{1}$$

then they define a *representation* of the group (ε). If these n matrices are not linearly independent (by comparison, cf., MOL., pp. 126, condition 1) then they represent a group that is homomorphic to (ε).

Thus, if:

$$(x_{\kappa\lambda}) = \sum x_\alpha E_\beta \tag{2}$$

then, when $x = yz$, one has:

$$(x_{\kappa\lambda}) = (y_{\kappa\lambda})(z_{\kappa\lambda}).$$

Therefore, I call the matrix (2) an *associated matrix* to the group (ε). If its determinant is non-zero then from (I), § 4:

$$|\, x_{\kappa\lambda} \,| = \Pi\, \Phi(x)^s \tag{3}$$

is a product of prime factors of the group determinant Θ.

If C is a constant matrix of m^{th} degree whose determinant is non-zero then the n matrices $C^{-1}E_1C, C^{-1}E_2\, C, \dots, C^{-1}E_nC$ also define a representation of (ε) that is said to be *equivalent* to the first one. If one cannot choose C in such a way that these n matrices assume the form:

$$\begin{pmatrix} E_1' & 0 \\ E_1^{(0)} & E_1'' \end{pmatrix}, \quad \begin{pmatrix} E_2' & 0 \\ E_2^{(0)} & E_2'' \end{pmatrix}, \dots, \begin{pmatrix} E_n' & 0 \\ E_n^{(0)} & E_n'' \end{pmatrix}$$

then I will call the representation *primitive* or irreducible, when this is possible, and otherwise, I call it *imprimitive* or reducible, and when $E_1^{(0)}, \dots, E_n^{(0)} = 0$, as well, *decomposed* or *decomposable.*

The methods that I developed in my paper "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, II," Sitzungsberichte, 1899, may be immediately carried over to the arbitrary DEDEKIND groups (with the modification that was suggested at the end of § 3).

The r^2 linear functions of $x_1, \dots, x_n$ that I denoted by $x_{\kappa\lambda}$ in § 11, defined a matrix of r^{th} degree that is associated with the group (ε) and whose determinant $|\, x_{\kappa\lambda} \,| = \Phi(x)$. I denote the corresponding representation of (ε) by $[\Phi]$. The k different representations thus obtained that correspond to then prime factors of the group determinant Θ constitute all of the primitive representations of the group (ε).

A representation that is equivalent to the representation (2) then decomposes into s primitive representations $|\, \Phi \,|$, s' representations $[\Phi']$, etc. The coefficient of u^{m-1} in the determinant $|\, x_{\kappa\lambda} + v\, e_{\kappa\lambda} \,|$, namely:

$$\varphi(x) = \sum s\, \chi(x),$$

is called the *trace* of the representation (2). In order for two representations of a DEDEKIND group to be equivalent, it is necessary and sufficient that their traces agree.