INTRODUCTION

TO

DIFFERENTIAL GEOMETRY

FOLLOWING THE METHOD OF H. GRASSMANN

BY

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PREFACE

The book that we are publishing today contains a brief exposition of the *geometric calculus,* and some of its applications to *elementary differential geometry*.

The geometric calculus was first conjectured by Leibniz (1679) ([1]) whose first recognized the opportunity – or rather, the necessity – of operating directly on the geometric elements, whereas analytic geometry operates numbers that have an indirect relationship to the elements that they represent. However, the geometric operation that was introduced by Leibniz does not have the usual properties of algebraic operations; the author did push the geometric research quite further.

Nevertheless, Leibniz's idea was destined propagate and to produce great results. In 1797, Caspar Vessel ([2]) gave an analytic representation of direction that contained Argand's (1806) geometric representation of complex numbers and several operations that had been introduced by Hamilton (1843-1853) for quaternions. Möbius, with his *barycentric calculus* (1827-1842) and Bellavitus, with the method of *equipollences* (1830-1854) gave two methods for a geometric calculus, independently of each other, that the authors applied to several questions of pure geometry and mechanics. In 1843, Hamilton published his first essay on quaternions, and that theory, which was developed completely by 1854, gave a complete geometric calculus that was soon know, appreciated, and even applied by Hamilton's contemporaries; today, one especially applies it to physics.

The papers of Hamilton are preceded by the *Ausdehnungslehre* of H. GRASSMANN (1844), who, by the power and simplicity of the operations, surpassed all of the other geometric calculi. The excessively abstract form of exposition that was adopted by Grassmann has retarded the diffusion of the *Ausdehnungslehre*, in such a way that today one employs the barycentric calculus, the theory of equipollences, or quaternions, and even more frequently, Cartesian geometry, in order to solve geometric questions that have a very simple solution using the Grassmann method. The applications that Grassmann made to the generation of lines and surfaces soon foreshadowed the power of the method; however, in order to make it known and applied by the whole world, he further constructed a concrete link to Euclidian geometry.

Professor Peano was the first one to give a concrete geometric interpretation of the forms and operations of the *Ausdehnungslehre*. Taking the common idea of a tetrahedron for his point of departure, he defined the product of two and three points. He then defined the products of these elements by numbers, and finally, he defined the sums of these products. The theory of forms of first order gave the barycentric calculus, along with that of vectors (or directions). The forms of second order represented lines, orientations and systems of forces that were applied to a rigid body. The forms of the

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([1]) LEIBNITZENS, Math. Schriften, v. II and V, Berlin, 1849.

([2]) "Essai sur la représentation analytique de la direction" (Om Directionens analytiske Betegning). Published by the Royal Academy of Sciences and Letters of Denmark, on the occasion of the centenary of its presentation to the Academy on 10 March 1797. Copenhagen, 1897.

third order represent planes and the plane at infinity. Among the operations, the progressive and regressive products give the geometric operations of *projection and intersection*. The internal product gives the orthogonal projections and the quantities that one refers to in mechanics as *work, moment,* …

In this book, we give the elements of the geometric calculus according to Grassmann's method in a very simple and concrete form. The goal that we shall propose is that of giving young students the means to easily comprehend that powerful instrument for calculation, and, at the same time, to give them the means to apply it to the questions of higher differential geometry.

We believe that the latter objective of our book is quite important. Indeed, in ordinary differential geometry, one obtains some very simple properties with very complicated developments. In general, that complication is due to the use of coordinates, because with coordinate we make algebraic transformations on numbers in order to obtain, after calculations that are frequently very complicated, a small formula – viz., an *invariant* – that is susceptible to a geometric interpretation. The geometric calculus makes no use at all of coordinates. It operates directly on the geometric elements, and each formula – which is, in itself, an invariant – has a very simple geometric significance that leads very easily to the graphic representation of the element considered. One can thus predict a simplification when compared to the ordinary methods. Our book proves that the simplification is possible in regard to ordinary differential geometry, and leaves a vast field of transformations and research in higher geometry to the younger students.

The importance The of the role that is played by the *Ausdehnungslehre* in geometry, mechanics, and physics is explained quite well by V. Schlegel in his important historical paper: "*Die Grassmann'sche Ausdehnungslehre*…([3]), to which we will refer the reader. Today, Grassmann's method has no need for recommendation; it only needs to be known and applied by the whole world. It is by *constant* application to all parts of mathematics that one can comprehend the power and simplicity of Grassmann's method.

Turin, April 1897.

([3]) Zeitschrift für Mathematik und Physik, Leipzig, 1896.

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CHAPTER I.

GEOMETRIC FORMS

§ 1. – DEFINITIONS AND RULES OF CALCULATION.

1. Tetrahedra. – We express the idea that the points A, B, C, D are situated on the same plane by writing $ABCD = 0$ or by saying that the points A, B, C, D are the summits of a *null tetrahedron*. One always has:

$$AABC = ABAC = \ldots = AAAB = \ldots = 0.$$

If A, B, C, D are points that are not situated on the same plane ($ABCD \neq 0$) then the notation $ABCD$ shall denote a real number. The absolute value of $ABCD$ is the number that measures – and with an arbitrary unit, moreover – the volume of the tetrahedron whose summits are precisely the points A, B, C, D. The sign of this number is + or – according to whether an observer that is placed on the line AB with their head at A and their feet at B and regards the line CD sees the point D to his *right* or *left*, resp., or else he sees the point C to his *left* or *right*, resp. ([4]).

No matter what the points A, B, C, D are, the real number $ABCD$ will be well-defined once the unit of measure for volumes is fixed. Moreover, one obviously has:

$$ABCD = -BACD = -ACBD = -ABDC.$$

In an expression such as $ABCD$, one can therefore choose the order in which one desires the letters to appear, on the condition that one nonetheless recall that every exchange of two consecutive letters implies a change of sign.

Furthermore, the definition that we just stated gives meaning to the expressions:

$$ABCD + A_1B_1C_1D_1 + \ldots + A_n B_n C_n D_n , \qquad ABCD - EFGH, \qquad mABCD,$$

where m is a number. One can always – and in an infinitude of ways – determine the points P, Q, R, S in such a way that the number $PQRS$ is equal to a given number, and in turn, to any of the cited expressions.

If $ABCD \neq 0$ then we will naturally say that the tetrahedron whose summits are arranged in the order A, B, C, D has the *direct sense* or the *inverse sense* according to whether the number $ABCD$ is *positive* or *negative*. In order to abbreviate the language,

([4]) The consideration of the sense of a sequence of points A, B, C, D as the sense or sign of a tetrahedron is due to Möbius. This idea is nowhere to be found in the books of Euclid.

we shall indeed say that *ABCD* is a tetrahedron, but that word does not possess its ordinary meaning here, because the equality $ABCD = EFGH$ expresses the idea that the tetrahedra with summits A, B, C, D and E, F, G, H have not only the same volume, but also the same sense.

2. Geometric forms. Equality of forms. – We call entities such as:

(1) $$x_1 A_1 + x_2 A_2 + \ldots + x_n A_n,$$

(2) $$x_1 A_1 B_1 + x_2 A_2 B_2 + \ldots + x_n A_n B_n,$$

(3) $$x_1 A_1 B_1 C_1 + x_2 A_2 B_2 C_2 + \ldots + x_n A_n B_n C_n,$$

where the $x_1, \ldots$ are real numbers and $A_1, \ldots, B_1, \ldots, C_1, \ldots$ represent points, *forms of the first, second, and third order,* respectively.

Under these conditions, the symbolic equalities:

(1)′ $$x_1 A_1 + \ldots + x_n A_n = x'_1 A'_1 + \ldots + x'_m A'_m,$$

(2)′ $$x_1 A_1 B_1 + \ldots + x_n A_n B_n = x'_1 A'_1 B'_1 + \ldots + x'_m A'_m B'_m,$$

(3)′ $$x_1 A_1 B_1 C_1 + \ldots + x_n A_n B_n C_n = x'_1 A'_1 B'_1 C'_1 + \ldots + x'_m A'_m B'_m C'_m,$$

express the idea that one has for any points P, Q, R:

(1)″ $$x_1 A_1 PQR + \ldots + x_n A_n PQR = x'_1 A'_1 PQR + \ldots + x'_m A'_m PQR,$$

(2)″ $$x_1 A_1 B_1 PQ + \ldots + x_n A_n B_n PQ = x'_1 A'_1 B'_1 PQ + \ldots + x'_m A'_m B'_m PQ,$$

(3)″ $$x_1 A_1 B_1 C_1 P + \ldots + x_n A_n B_n C_n P = x'_1 A'_1 B'_1 C'_1 P + \ldots + x'_m A'_m B'_m C'_m P.$$

The terminology of *first-order form*, for example, is not defined by the expression (1). We consider the entity (1) to be an *abstract geometric element* that is common to all forms $x'_1 A'_1 + \ldots + x'_m A'_m$ satisfy the condition (1)′, a condition that takes on a precise significance by virtue of the equality (1)″. The same remarks apply to the expressions (2) and (3) ([5]).

We likewise say that one of these forms – (1), for example – is zero, and we write:

$$x_1 A_1 + x_2 A_2 + \ldots + x_n A_n = 0$$

([5]) The definitions of the entities (1), (2), (3) in a form that is analogous to the one that we just stated is due to Peano (*Calcolo geometrico*; Bocca, Turin, 1888). It then results that a simple relationship is established between the geometric forms and the elements that one considers in Euclidian geometry, and Grassmann's abstract calculus acquires a concrete value that is susceptible to geometric applications.

when for any points P, Q, R, one has:

$$x_1 A_1 PQR + x_2 A_2 PQR + \ldots + x_n A_n PQR = 0.$$

A tetrahedron that is a sum of tetrahedra or an expression like $x_1 A_1 B_1 C_1 D_1 + \ldots + x_n A_n B_n C_n D_n$ is called a *fourth-order form*, by analogy.

If A, B, C are points then we will also write $1\,A$, $1\,AB$, $1\,ABC$, instead of A, AB, ABC. This simply amounts to assuming that A is a first-order form (i.e., that each point is a first-order form), AB is a second-order form, and ABC is a third-order form.

3. Points. – We remark, first of all, that if A corresponds to a point then one will necessarily have that $A \neq 0$.

One will express the idea that *the point A coincides with the point B* by writing $A = B$. Indeed, the relation $A = B$ is equivalent to $APQR = BPQR$ for any points P, Q, R. Each plane that contains A ($APQR = 0$) will then also contain B ($BPQR = 0$), and consequently, A will be identical to B.

4. Line segments. – The definition of the equality of second-order forms shows that the equation $AB = 0$ *corresponds to the equality* $A = B$ *in a necessary and sufficient manner*. Indeed, $AB = 0$ is equivalent to $ABPQ = 0$ for any P and Q. Therefore, the four points A, B, P, Q must be in the same plane, which demands precisely the coincidence of the points A and B.

Similarly, for any P and Q the two tetrahedra $ABPQ$ and $BAPQ$, which has the opposite sense, will have the same volume; i.e., $ABPQ = -BAPQ$, and *one will always have the relation* $AB = -BA$ *between two points A and B*.

If one denotes the modulus of AB by "mod AB," which is the positive or zero number that measures the distance between the two points A and B, then one will always have that mod AB = mod BA, as well as mod $AB = 0$ only when the two points A and B coincide, or if $A = B$.

THEOREM I. – *If x is a non-zero real number and one has that* $AB = xCD$ *then the four points A, B, C, D are situated on the same line and* mod AB *is equal to* mod CD, *multiplied by the absolute value of the number x*.

Proof. – If $AB = 0$ then one will have $CD = 0$, and the theorem is proved. If $AB \neq 0$ then one must also have that $CD \neq 0$, and conversely. If P, Q are two arbitrary points then one will have $ABPQ = xCDPQ$, and if the point P is situated on the line AB ([6]), then it will be likewise situated on the line CD, since the two sides of the equation will be zero, and this will be true for any point P on the line AB, which amounts to saying that the four points A, B, C, D will be situated on the same line. But then if $ABPQ \neq 0$ then the tetrahedra $ABPQ$, $xCDPQ$ can be considered to have the same heights (viz., the

([6]) To abbreviate, we will say "line AB," instead of "line that joins the points A and B," "plane ABC," instead of "plane that passes through the points A, B, C." In § 3 of this chapter, we will give a somewhat difference significance to these expression.

distance from the point Q to the plane ABP) and two equivalent triangles for bases that have the summit P in common and their bases along the line AB. Consequently, the distance between the two points A and B will be equal to the distance of the two points C and D, multiplied by the absolute value of the number x.

THEOREM II. – *If A, B, C, D are points on the same line and $CD \neq 0$ then one can determine only one real number x such that $AB = xCD$.*

Proof. – If $AB = 0$ then one will have $x = 0$. If $AB \neq 0$ then let P and Q be two points such that $ABPQ \neq 0$. There then indeed exists a real number x that is defined by the equality $ABPQ = xCDPQ$. Moreover, it immediately results, from some notions of elementary geometry, that the relation $ABPQ = xCDPQ$ persists for any points P, Q, and in turn (no. 2), the number x that will answer the question – i.e., that will make $AB = xCD$ – is therefore determined in a unique fashion.

Remark. – Under these conditions, we let the symbol $\frac{AB}{CD}$ (i.e., the ratio of AB to CD) denote the number x such that $AB = xCD$. If $\frac{AB}{CD} \neq 0$ then we will say that the form AB has the *direct sense* or the *opposite sense* relative to the form CD according to whether the number $\frac{AB}{CD}$ is positive or negative, respectively.

If $AB = CD$ with $CD \neq 0$ then:

1. The points A, B, C, D will be situated on the same line.
2. The distance between two points A and B will be equal to the distance between the two points C and D.
3. The forms AB, CD have the same sense.

Therefore, the form AB is an abstract geometric element that is a function of the unbounded line that joins the points A and B, the distance between these two points, and the sense of the form AB. We will say that the form AB is a *line segment*, an expression that will not have its usual significance of *a bounded line* here.

If $AB \neq 0$ and $CD \neq 0$ then if the line AB is parallel to the line CD and the line AC is parallel to CD we will say that the forms AB and xCD are *parallel* and *do or do not have the same sense* according to whether x, which is assumed to be real and non-zero, is positive or negative, respectively.

5. Triangles. – Let A, B, C, D, E, F be points. The equality of two third-order forms easily shows that the equality $ABC = 0$ demands that the points A, B, C be on a straight line. Similarly, one has $ABC = -BAC = -ACB$. One denotes the modulus of ABC by "mod ABC," which is the positive or zero number that measures the area of the triangle whose summits are A, B, C, in such a way that mod $ABC = 0$ if $ABC = 0$ and mod ABC = mod BAC = mod ACB.

THEOREM I. – *If one has $ABC = xDEF$, where x is a non-zero real number, then the points A, B, C, D, E, F are situated in the same plane and* mod ABC *is equal to* mod DEF, *multiplied by the absolute value of the number x.*

THEOREM II. – *If A, B, C, D, E, F are points on the same plane and $DEF \neq 0$ then one can determine just one real number x such that $ABC = xBEF$.*

These two theorems are proved just as theorems I and II of no. 4 were.

Remarks. – Under the hypotheses of the preceding theorem, we further let the symbol $\frac{ABC}{DEF}$ (i.e., the ratio of ABC to DEF) denote the number x such that $ABC = xDEF$. If $\frac{ABC}{DEF} \neq 0$ then we can say that the form ABC has *the direct sense* or *the opposite sense* relative to DEF according to whether the number $\frac{ABC}{DEF}$ is positive or negative, respectively.

If $ABC = DEF$ and $ABC \neq 0$ then:

1. The points A, B, C, D, E, F are situated on the same plane.
2. The area of the triangle whose summits are A, B, C is equal to the area of the triangle whose summits are D, E, F.
3. The forms ABC, DEF have the same sense.

As before, the form ABC is an abstract geometric element that is a function of the plane of the points A, B, C, the area of the triangle whose summits are A, B, C, and the sense of the form ABC. We will say that the form ABC is a *triangle* and attribute a special significance to that word.

If $ABC \neq 0$, $DEF \neq 0$, the plane ABC is parallel to the plane DEF, and the lines AD, BE, CF are mutually parallel then the forms ABC, $xDEF$ will be called *parallel and with the same sense or not* according to whether x, which is real and non-zero, is positive or negative, respectively.

Upon supposing that $ABC \neq 0$, an observer that is standing on the plane ABC is either placed in the region of all points P such that $PABC$ is a positive number, or else in one whose points P are such that $PABC$ is a negative number. For example, if the observer is in the first of these regions, and he traverses the perimeter of the triangle ABC from A to B, B to C, and C to A then he will have the area of the triangle ABC on his right. If he is always situated in the same region then if he traverses the perimeter of a triangle DEF in the same plane in the sense D, E, F then he will have the area to his right or left according to whether $\frac{ABC}{DEF}$ is positive or negative. In this manner, one can quite easily recognize whether two triangles in the same plane have the same or opposite senses.

6. Sum and product with a number. – Let:

$$x_1 A_1 + \ldots + x_n A_n , \qquad y_1 B_1 + \ldots + y_m B_m$$

be first-order forms, and let h be a real number. We set:

$$(x_1 A_1 + \ldots + x_n A_n) + (y_1 B_1 + \ldots + y_m B_m) = x_1 A_1 + \ldots + x_n A_n + y_1 B_1 + \ldots + y_m B_m ,$$
$$h(x_1 A_1 + \ldots + x_n A_n) = hx_1 A_1 + \ldots + hx_n A_n .$$

These equalities, which define the addition of two first-order forms or the product of a form by a number will further allow us to define the same operations for forms of second or third order.

All of the rules of calculation for algebraic polynomials apply to the sum of forms of the same order that are finite in number and to the product of a form by a real number. One introduces the – sign from algebra by agreeing that $-A = (-1) A$ and that $A - B = A + (-B)$, where A, B are forms of the same order. If x is a non-zero number then we can write A / x, instead of $1/x\, A$.

A geometric form is the (algebraic) sum of a finite number of forms that are themselves separately the products of a point, a line segment, or a triangle with a number.

7. Progressive product. – For example, set:

$$(x_1A_1 + x_2A_2)\,(y_1B_1C_1 + y_2\,B_2\,C_2 + y_3\,B_3\,C_3)$$
$$= x_1y_1\,A_1B_1C_1 + x_1y_2\,A_1B_2C_2 + x_1y_3\,A_1B_3C_3 + x_2y_1\,A_2B_1C_1 + x_2y_2\,A_2B_2C_2 + x_2y_3\,A_2B_3C_3 .$$

In a word, we have operated as if:

$$x_1A_1 + x_2A_2 , \quad y_1B_1C_1 + y_2\,B_2\,C_2 + y_3\,B_3\,C_3 ,$$

were polynomials, and performed the multiplication while respecting the order of the large letters.

It is easy to generalize this rule in order to take the product of two or more forms, with the single restriction that the sum of their orders must not exceed 4. The product thus defined is called the *progressive product*, or simply the *product*, when there is no possible confusion. The line segment AB is therefore the product of the point A with the point B, the triangle ABC is the product of the point A with the line segment BC, or of the line segment AB with the point C, or finally the double product of the point A with the point B and the point C. Of course, the same thing will be true for the tetrahedron $ABCD$.

It results easily from these definitions that the rules of algebraic calculation apply to the products of forms [if $A = B$ then one will have $AB = BC$, $(A + B)C = AC + BC$, $mAB = (mA)\,B$], except for the ones that depend upon a commutative property. In that case, one must have recourse to the following rule: If A and B are two forms of orders r and s, respectively, with the condition that $r + s \leq 4$, then one will have:

$$AB = (-1)^{rs}\,BA.$$

That is, in a product of forms, one can permute two consecutive factors of orders r and s at will, with the caveat that one must multiply the product by $(-1)^{rs}$. The formulas $AB = - BA$, $ABC = - BAC$ are only particular cases of this rule.

§ 2. – VECTORS AND THEIR PRODUCTS.

8. Vectors. – One calls *the difference of two points* a *vector*. If A and B are points then $B - A$ will be a *vector*. One sees immediately that $B - A = 0$ when $A = B$, and conversely.

THEOREM. – *In order that the (non-zero) vectors $B - A$, $D - C$ be equal, it is necessary and sufficient that the line segments AB, CD be parallel and have the same sense and modulus.*

Proof. – If P, Q, R are three arbitrary points then:

$$(B - A)\ PQR = BPQR - APQR$$

will represent the tetrahedron that has the triangle PQR for its base and the distance from the orthogonal projections of A and B onto the perpendicular to the plane PQR for its height. Consequently, we assert that no matter what the points P, Q, R are, the equality:

$$(B - A)\ PQR = (D - C)\ PQR$$

will amount to an assertion of the necessary and sufficient conditions for the vectors $B - A$, $D - C$ to be equal.

Remarks. – Say that the non-zero vectors $B - A$, $D - C$ are *parallel* when the line segments AB, CD are parallel. What we will call the *direction* of a vector $\mathbf{I}$ is an abstract geometric function of $\mathbf{I}$ that $\mathbf{I}$ has in common with all of the vectors that are parallel to it. One deduces from the preceding theorem that *equal vectors have the same direction.*

If the vectors $B - A$, $D - C$ are parallel then we will say that they have *the same sense* or *the opposite sense* according to whether the line segments AB, CD do or do not have the same sense, respectively. The *sense* of a vector $\mathbf{I}$ is therefore an abstract geometric element that is a function of $\mathbf{I}$ that $\mathbf{I}$ has in common with the other vectors that are parallel to $\mathbf{I}$. One deduces from the preceding theorem that *equal vectors have the same sense.*

Furthermore, set $\mathrm{mod}(B - A) = \mathrm{mod}\ AB$, and agree that $\mathbf{I}$ is a *unit vector* when $\mathrm{mod}\ \mathbf{I} = 1$. It likewise results that *equal vectors have the same modulus*.

It also results from the preceding conventions that: *In order for two vectors to be equal, it is necessary and sufficient that they have the same direction, sense, and modulus*. Therefore, a vector is an abstract geometric element that is a function of its direction, sense, and magnitude; i.e., a vector is given when one knows its direction, sense, and magnitude.

Graphically, one will represent the vector $B - A$ with the points A, B linked by an arrow whose head is at B. One thus comprehends that in mechanics one can represent a velocity by means of a vector, because a velocity can be defined as an element that is known when one has its direction, sense, and magnitude.

If A, B, C are three points then the preceding theorem will immediately give us the construction of the point D such that $B - A = D - C$. A is called the *origin* and B, the *extremity* of the vector $B - A$. It also results from this that one can take an arbitrary point to be the origin of a vector $\mathbf{I}$, but once the origin is chosen, the extremity will be a perfectly well-defined point.

9. *a*. The sum of a point and a vector $\mathbf{I}$ is a new point that one deduces from A by a translation whose magnitude, direction, and sense is determined by the vector $\mathbf{I}$. Indeed, if A is the origin of the vector $\mathbf{I}$ then its extremity B will be determined by the condition $\mathbf{I} = B - A$; it will follow from this that $A + \mathbf{I} = B$ is a point, etc.

b. The product of a point O and a vector $\mathbf{I}$ is a line segment, because $O\mathbf{I} = OO + O\mathbf{I} = O(O + \mathbf{I})$. Conversely, a line segment is the product of a point by a vector. Indeed, if A, B are points then one will have the equalities:

$$AB = AB - AA = A(B - A).$$

Likewise, mod $O\mathbf{I}$ = mod $\mathbf{I}$, since, by definition:

$$\text{mod } AB = \text{mod}(B - A).$$

c. The sum of two vectors is a vector. Indeed, if $\mathbf{I}$, $\mathbf{J}$ are vectors and O is a point then $A = O + \mathbf{I} + \mathbf{J}$ will be a well-defined point and $A - O = \mathbf{I} + \mathbf{J}$ will indeed be a vector. The construction of the expression $\mathbf{I} + \mathbf{J}$ is the same as the one that will give the resultant of two velocities that are represented by the vectors $\mathbf{I}$, $\mathbf{J}$, respectively. One will also easily find the construction of the sum of a finite number of vectors, and one will confirm that the result is independent of the order that was adopted in the operation.

d. If $\mathbf{I}$, $\mathbf{J}$ are vectors then one will have:

$$\text{mod}(\mathbf{I} + \mathbf{J}) \leq \text{mod } \mathbf{I} + \text{mod } \mathbf{J},$$

because that relation is nothing but the one that relates the distance between the three points O, $O + \mathbf{I}$, $O + \mathbf{I} + \mathbf{J}$.

10. Let $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, $\mathbf{U}$ be non-zero vectors.

a′. If x is a non-zero real number then $x\mathbf{I}$ will be a vector that is parallel to $\mathbf{I}$, and will have the same or opposite sense to $\mathbf{I}$ according to whether x is positive or negative, respectively. The modulus of $x\mathbf{I}$ is equal to the modulus of $\mathbf{I}$, multiplied by the absolute value of x. Indeed, if $\mathbf{I} = B - A$ then the point C such that $xAB = AC$ is completely determined. All that one needs to do now is to imitate the proof of the theorem in no. 8 in order to see that $x(B - A) = C - A$, which proves the theorem.

a''. If **I** is parallel to **J** then there will exist exactly one real number x such that $\mathbf{J} = x\mathbf{I}$. Indeed, if O is an arbitrary point then $O\mathbf{I}$, $O\mathbf{J}$ will be two line segments on the same line, and the number x will be such that $O\mathbf{J} = xO\mathbf{I}$ or such that $O\mathbf{J} = O(x\mathbf{I})$ is well-defined. It then indeed results that $\mathbf{J} = x\mathbf{I}$. Let x' be another number such that $\mathbf{J} = x'\mathbf{I}$, so one will have $0 = (x - x')\,\mathbf{I}$, and thus $x = x'$, which proves that the number x will indeed be independent of the chosen point O.

a. The condition for the parallelism of **I** and **J** is thus that **J** *be a multiple of* **I**.

One can express the same thing with the relation $\mathbf{IJ} = 0$. Indeed, if **I** is parallel to **J**, and if O is an arbitrary point then the points O, $O + \mathbf{I}$, $O + \mathbf{J}$ will be on the same line, namely:

$$O(O + \mathbf{I})(O + \mathbf{J}) = O\mathbf{IJ} = 0;$$

i.e., $\mathbf{IJ} = 0$. Conversely, if $\mathbf{IJ} = 0$ then:

$$O(O + \mathbf{I})(O + \mathbf{J}) = 0;$$

i.e., the points O, $O + \mathbf{I}$, $O + \mathbf{J}$ will be on the same line, or furthermore, the vectors **I**, **J** will be parallel.

If **I**, **J** are parallel vectors then the sign $\mathbf{I} / \mathbf{J}$ will always denote the number x such that $\mathbf{J} = x\mathbf{I}$, and, in addition, we agree that the symbol $0 / \mathbf{I} = 0$.

b'. We say that the vector **I** is parallel to the plane α or that the plane α is parallel to the vector **I** when there exist two points A, B on α such that **I** is parallel to $B - A$, and that the vectors **I**, **J**, **K** are *coplanar* when **I**, **J**, **K**, … are parallel to the same plane.

If **I**, **J**, **K** are coplanar and $\mathbf{IJ} \neq 0$ then the numbers x, y such that $\mathbf{K} = x\mathbf{I} + y\mathbf{J}$ will be well-defined. Indeed, if O is a point then the points $O + \mathbf{I}$, $O + \mathbf{J}$, $O + \mathbf{K}$ will be on the same plane that is parallel to the vectors **I**, **J**, **K**. If a line is drawn parallel to the line $O\mathbf{I}$ through the point $O + \mathbf{K}$ and it meets the line $O\mathbf{J}$ at the point A then one will have $\mathbf{K} = (A - O) + [(O + \mathbf{K}) - A]$. However, $A - O$ and $(O + \mathbf{K}) - A$ are vectors that are parallel to the vectors **I**, **J**, and the proposition shows that one can determine numbers x, y such that $\mathbf{K} = x\mathbf{I} + y\mathbf{J}$. The numbers x, y will not be functions of O; indeed, if x', y' are other numbers such that $\mathbf{K} = x'\mathbf{I} + y'\mathbf{J}$ then one must have:

$$(x - x')\,\mathbf{I} + (y - y')\,\mathbf{J} = 0 \qquad \text{and} \qquad (x - x')\,\mathbf{IJ} = (y' - y)\,\mathbf{IJ} = 0,$$

and it results immediately from the fact that $\mathbf{IJ} \neq 0$ that one must have:

$$x = x' \qquad \text{and} \qquad y = y'.$$

b''. If x, y are numbers, and if $\mathbf{K} = x\mathbf{I} + y\mathbf{J}$ then the vectors **I**, **J**, **K** will be coplanar. Let O be a point, so one has:

$$O(O + \mathbf{I})(O + \mathbf{J})(O + \mathbf{K}) = O\mathbf{IJK},$$

and upon replacing **K** with $x\mathbf{I} + y\mathbf{J}$, one will have:

$$O(O + \mathbf{I})(O + \mathbf{J})(O + \mathbf{K}) = 0,$$

which proves the theorem.

b. The condition of coplanarity of three vectors **I**, **J**, **K** is therefore: **K** *is the sum of a multiple of* **I** *and a multiple of* **J**, or **K** *is a linear function of* **I** *and* **J**, which one can, moreover, express by the condition that $\mathbf{IJK} = 0$.

c. If $\mathbf{IJK} \neq 0$ then the real numbers x, y, z such that:

$$\mathbf{U} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

will be well-defined. It suffices to imitate the proof of proposition (b').

The vectors $x\mathbf{I}$, $y\mathbf{J}$, $z\mathbf{K}$ are called the *components* of **U** relative to the vectors **I**, **J**, **K**. The numbers x, y, z are called the *coordinates* of **U** relative to the vectors **I**, **J**, **K**.

d. One always has $\mathbf{IJKU} = 0$; i.e., that the product of four vectors is always zero. Indeed, if $\mathbf{IJK} = 0$ then one will indeed have $\mathbf{IJKU} = 0$. If $\mathbf{IJK} \neq 0$ then $\mathbf{U} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, and in turn, $\mathbf{IJKU} = 0$.

11. Bivectors. – One calls the product of two vectors a *bivector*; i.e., if **I**, **J** are vectors then **IJ** will be a bivector. One has $\mathbf{IJ} = 0$ when one of the vectors **I** or **J** is zero, or furthermore if **I** is parallel to **J**, and conversely.

THEOREM I. – *In order for the bivectors* **IJ**, **KU** *to be equal, it is necessary and sufficient that for any point O the triangles* $O\mathbf{IJ}$, $O\mathbf{KU}$ *be equal*.

Proof. – By virtue of the definition of the equality of two second-order forms, the condition $\mathbf{IJ} = \mathbf{KU}$ is equivalent to $PO\mathbf{IJ} = PO\mathbf{KU}$ for any points P and O. From the definition of the equality of two third-order forms, that equality is equivalent to $O\mathbf{IJ} = O\mathbf{KU}$ for any O. If one observes that $O\mathbf{IJ} = O(O + \mathbf{I})(O + \mathbf{J})$ then it will result that $O\mathbf{IJ}$ is a triangle.

THEOREM II. – *If* **I**, **J** *are vectors then for any points P, Q the triangles* $P\mathbf{IJ}$, $Q\mathbf{IJ}$ *will be parallel and will have the same sense and modulus*.

Proof. – One proves this theorem by observing that:

$$Q - P = (Q + \mathbf{I}) - (P + \mathbf{I}) = (Q + \mathbf{J}) - (P + \mathbf{J}).$$

Remarks. – We say that the non-zero bivectors **IJ**, **KU** are *parallel* or *coplanar* when for any point O the triangles $O\mathbf{IJ}$, $O\mathbf{KU}$ are in the same plane. We shall call the *orientation* of the bivector **IJ** an abstract geometric element that is a function of **IJ** that **IJ**

has in common will all of the bivectors that are parallel to **IJ**. It then results that *equal bivectors have the same orientations*.

If the non-zero bivectors **IJ**, **KU** are parallel then we will say that they do or do not have the same *sense* according to whether the triangles *O***IJ**, *O***KU** do or do not have the same sense, respectively, for any point *O*. The *sense* of a non-zero bivector **IJ** is therefore an abstract geometric element that **IJ** has in common with the other bivectors that are parallel to **IJ**. *Equal bivectors have the same sense*.

If **IJ** is a bivector then for any point *O* set mod **IJ** = 2 mod *O***IJ**. The modulus of a bivector is then the positive or zero number that measures the area of the parallelogram whose three summits are the points *O*, *O* + **I**, *O* + **J**. It further results that *equal bivectors have the same modulus*.

Theorem I and the definitions that we just stated likewise imply that *in order for two bivectors to be equal, it is necessary and sufficient that they have the same orientation, sense, and modulus.* Thus, a bivector is an abstract geometric element that is a function of its orientation, sense, and magnitude.

Graphically, if one excludes its sense, then one can represent the bivector $(B - A)(C - A)$ by the parallelogram whose three summits are the points *A*, *B*, *C*, and whose edges will be the two vectors $B - A$, $C - A$. If *A*, *B*, *C* are points in a plane α then one will obtain a point *E* on the plane α such that:

$$(B - A)(C - A) = (D - A)(E - A)$$

by constructing the representative parallelogram of:

$$(B - A)(C - A)$$

in such a way that the triangles *ABC*, *ADE* have the same sense. The transformation of a bivector into another one that is equal to it is therefore reduced to the problem of elementary geometry that consists of transforming a parallelogram into an equivalent one. Consequently, the equality:

$$\mathbf{I}(\mathbf{J} + \mathbf{K}) = \mathbf{IJ} + \mathbf{IK}$$

expresses Varignon's theorem.

12. *a*. The product of a point with a bivector is a triangle (*see* proof of Theorem I, no. 11). Conversely, every triangle is the product of a point with a bivector, since $ABC = A(B - C)(C - A)$.

b. We say that the non-zero bivector **IJ** is parallel to the non-zero vector **K** (or **K** is parallel to **IJ**) when the three vectors **I**, **J**, **K** are coplanar; i.e., when:

$$\mathbf{IJK} = 0.$$

The sum of the non-zero line segment *AB* with the bivector *u* that is parallel to $B - A$ is a line segment that one deduces from *AB* by a translation. Indeed, let **K** be a vector such that:

$$\mathbf{K}(B - A) = u.$$

By virtue of:

$$AB = A(B - A),$$

one will then have:

$$AB + u = (A + \mathbf{K})(B - A),$$

which proves the theorem. The translation **K** is not well-defined; on the other hand, the line upon which the segment $AB + u$ is situated is defined completely.

If X is a point on the line that joins A and B (i.e., it is a point such that $ABX = 0$), and if Y is a point of the line $AB + u$ then the translation $\mathbf{K} = Y - X$ will be such that:

$$AB + u = (A + \mathbf{K})(B + \mathbf{K}).$$

c. The sum of two bivectors is a bivector. Indeed, if u, v are bivectors then there will always exist a vector **I** that is parallel to the two bivectors u, v. Consequently, one can determine two vectors **J**, **K** such that $u = \mathbf{IJ}$, $v = \mathbf{IK}$. However, one will then have $u + v = \mathbf{I}\,(\mathbf{J} + \mathbf{K})$, which proves the theorem.

d. One proves very easily that the *condition of parallelism between two non-zero bivectors u, v is that u be a multiple of v*.

If u, v are non-zero, parallel bivectors then we will further let the symbol v / u denote the number x such that:

$$v = xu.$$

We also agree that $0 / u = 0$.

e. A bivector is always reducible to the sum of three line segments that are the edges of a triangle, because if u is a bivector then one will indeed have:

$$u = (B - A)(C - A) = BC - CA + AB,$$

which will prove the stated property.

By analogy, if a line segment represents a force that is applied to a rigid body then a bivector can represent a couple.

13. Trivectors. – One calls the product of three vectors a *trivector*.

If **I**, **J**, **K** are vectors and P is a point then by virtue of the equality $P\mathbf{IJK} = P(P + \mathbf{I})(P + \mathbf{J})(P + \mathbf{K})$, one sees that $P\mathbf{IJK}$ will be defined to be the tetrahedron whose summits are the points P, $P + \mathbf{I}$, $P + \mathbf{J}$, $P + \mathbf{K}$. If Q is a point then one will always have $P\mathbf{IJK} = Q\mathbf{IJK}$; i.e., the number $P\mathbf{IJK}$ is not a function of P, but only of the trivector **IJK**, so we will call the *sense* of the trivector **IJK** the sense of the tetrahedron $P\mathbf{IJK}$.

Let α be a trivector, and let **I** and **J** be two non-zero vectors such that $\mathbf{IJ} \neq 0$. The vectors **K** such that $\alpha = \mathbf{IJK}$ are always well-defined. The determination of **K** depends upon the following problem of elementary geometry: *Transform a tetrahedron into another equivalent tetrahedron.*

The sum of the non-zero triangle ABC with the triangle α is a triangle that one deduces from ABC by a translation. Indeed, let $\mathbf{K}$ be a vector such that:

$$\mathbf{K}(B - A)(C - A) = \alpha.$$

One will then have:

$$ABC + \alpha = A(B - A)(C - A) + \mathbf{K}(B - A)(C - A) = (A + \mathbf{K})(B - A)(C - A),$$

which proves the theorem. The translation $\mathbf{K}$ is not defined; on the other hand, the plane on which the triangle $ABC + \alpha$ is situated is determined completely.

If α, β are two trivectors such that $\alpha \neq 0$ then the number $P\beta / P\alpha$ will be independent of the point P. We therefore set $\beta / \alpha = P\beta / P\alpha$ for any point P, and it is clear that β / α is precisely the number x such that $\beta = x\alpha$. Therefore, if α is a non-zero trivector then the trivector β – whatever it is – will be a multiple of α. It then further results that the sum of two trivectors is a trivector.

One can always represent the triangle $\mathbf{IJK}$ – except for its sense – by three vectors with the same origin O and the parallelepiped whose four summits are precisely the points O, $O + \mathbf{I}$, $O + \mathbf{J}$, $O + \mathbf{K}$. The volume of that parallelepiped, which is affected with a sign (viz., the sign of the number $O\mathbf{IJK}$), moreover, is called the *magnitude* of $\mathbf{IJK}$ (mag $\mathbf{IJK}$); this amounts to setting mag $\mathbf{IJK} = 6\, O\mathbf{IJK}$ for any point O. We can then suppress the sign mag, with no possible ambiguity, and give the symbol $\mathbf{IJK}$ the double significance of a trivector and a number.

We call the trivector – which we shall always denote by ω – such that for any point O the number $O\omega = 1$ the *unit trivector*. If ω is considered to be a number then $\omega = 6$.

14. Rotation. – We shall now consider a plane whose three fixed points A, B, C are such that $ABC \neq 0$.

Let O, O' be two arbitrary points of the plane ABC. One can always determine three points P, Q, R of the plane ABC such that the points O, O' are interior to the triangle whose summits are P, Q, R, and in addition, PQR / ABC is a positive number.

Let a then be a line of the given plane that passes through the point O, and let a' be a line in the same plane that passes through the point O'. The lines a, a' each meet the perimeter of the triangle PQR at two points M, N and M', N', respectively. If we make the point M traverse the perimeter of the triangle PQR – for example, in the sense P, Q, R (*see* no. 5, pp. 5) – then the point N will simultaneously traverse the perimeter of the triangle in the same sense, and the line a, as well as each of its points will *turn* around the point O in the plane. We say that the lines a, a' (or each point of these lines) *turn around the points O and O'*, respectively, *in the same sense* when the points M, M' (or N, N') traverse the perimeter of the triangle PQR in the same sense.

Let $P_1Q_1R_1$ be another triangle in the plane ABC that enjoys the same properties as the triangle PQR. If M_1, N_1, for example, are the points of intersection of the line a with the perimeter of the triangle $P_1Q_1R_1$, and if the point M traverses the perimeter of the triangle PQR in the sense P, Q, R then the point M_1 will traverse the perimeter of the triangle $P_1Q_1R_1$ in the sense P_1, Q_1, R_1 . Therefore, the *sense of rotation* of a line (or a point) around a fixed point of the plane *ABC* is an abstract geometric element that is a function of the fixed triangle *ABC* in the plane.

We can fix the *positive sense* or the *negative sense* of the rotation in the plane *ABC* by saying that *the line has turned around the point O in the* POSITIVE (NEGATIVE, resp.) SENSE when *the point M traverses the perimeter of the triangle PQR in the sense P*, *Q*, *R* (*P*, *R*, *Q*¸ resp.). Now, suppose that an observer is placed in the region of the points *S* such that *SABC* corresponds to a negative number. If the line *a* turns in the positive sense then the observer will turn in the inverse sense to the hands of a clock with respect to the face.

14. *cont*. Let **I** be a non-zero vector in the plane *ABC*, and let φ be a non-zero number. We say that **J** *is equal to the vector* **I**, *but turned through the angle* φ, when:

1. **J** is a vector.
2. mod **J** = mod **I**.
3. For any point *O* in the plane *ABC*, the line *O***I** can coincide with the line *O***J** by turning in the positive or negative sense according to whether φ is positive or negative, respectively, where the path that is traversed by the point $O + \mathbf{I}$ has a measure that is equal to the absolute value of the number φ mod **I**.

Furthermore, one has that **I** *is equal to the vector* **I** *turned through a zero angle*, and *a zero vector is equal to itself turned through an arbitrary angle* φ*.*

Let **I**, **J** be two non-zero vectors in the plane *AB* such that mod **I** = mod **J**. There exists an infinitude of real numbers φ such that **J** is equal to the vector **I**, turned through the angle φ. Among these numbers, the ones that are positive have a minimum φ_1 and the ones that are negative have a maximum φ_2 . One always has that $\varphi_1 - \varphi_2 = 2\pi$, and every number φ will be of the form $\varphi_1 + 2n\pi$, $\varphi_2 + 2n\pi$, where *n* is an arbitrary integer number that is positive, negative, or zero.

We say *the angle between* **I** *and* **J**, and denote it by (**I**, **J**), when we mean the smallest of the positive or zero numbers φ such that **J** is equal to the vector **I**, turned through the angle φ. One should observe that by the notation (**I**, **J**), the positive sense of rotation in the plane *ABC* is intended; i.e., the number (**I**, **J**) is not just a function of the vectors **I** and **J**, but also of the positive sense that is chosen for the rotation in the plane.

If (**I**, **J**) = 0 then we shall call the angle (**I**, **J**) *radiant*. If the angle (**I**, **J**) = $\pi / 2$ then the angle (**I**, **J**) will be called a *right angle*, and we will further say that the vector **I** is perpendicular to the vector **J**, or conversely, in the cases of (**I**, **J**) = $\pi / 2$ and (**I**, **J**) = $3\pi / 2$.

If **U**, **V** are non-zero vectors in the plane *ABC* then we shall say the *angle* between **U** and **V**, and denote it by (**U**, **V**), to mean the angle between the vector **U** / mod **U** and the vector **V** / mod **V**, which amounts to setting:

$$(\mathbf{U}, \mathbf{V}) = \left(\frac{\mathbf{U}}{\operatorname{mod}\mathbf{U}}, \frac{\mathbf{V}}{\operatorname{mod}\mathbf{V}}\right).$$

One has:

$$(\mathbf{U}, -\mathbf{U}) = \pi, \quad (\mathbf{U}, \mathbf{V}) = (-\mathbf{U}, -\mathbf{V}),$$

and if:

$$\frac{\mathbf{U}}{\operatorname{mod}\mathbf{U}} \neq \frac{\mathbf{V}}{\operatorname{mod}\mathbf{V}}$$

then one will have:

$$(\mathbf{U},\, \mathbf{V}) + (\mathbf{V},\, \mathbf{U}) = 2\pi.$$

15. If **I** is a non-zero vector in the plane then we will let $i\mathbf{I}$ denote the vector **I**, when turned through the positive right angle π / 2; we also set $i0 = 0$. We then let **I** and **J** be two arbitrary vectors on the plane and let x be a real number. Instead of the notations:

$$i(x\mathbf{I}), \quad x(i\mathbf{I}), \quad (i\mathbf{I}) + \mathbf{J}, \quad \mathbf{I}(i\mathbf{J}),$$

which presently have a precise significance, we will employ simply:

$$ix\mathbf{I}, \quad xi\mathbf{I}, \quad i\mathbf{I} + \mathbf{J}, \quad \mathbf{I}i\mathbf{J}.$$

a. If $\mathbf{I} = \mathbf{J}$ then $i\mathbf{I} = i\mathbf{J}$.

b. If n is a positive or zero whole number then we will set $i^0\mathbf{I} = \mathbf{I}$, $i^n\mathbf{I} = i(i^{n-1}\mathbf{I})$; in other words, we will let $i^n\mathbf{I}$ denote the vector that one deduces from **I** by applying the operation n times. One thus has:

$$i^2\mathbf{I} = -\,\mathbf{I}, \quad i^3\mathbf{I} = -\,i\mathbf{I}, \quad i^4\mathbf{I} = \mathbf{I}, \quad i^5\mathbf{I} = i\mathbf{I}, \quad \ldots,$$

which shows that the sign i^n has the same properties as the symbol $\left(\sqrt{-1}\right)^n$.

c. One can easily prove the formulas:

$$\begin{aligned} ix\mathbf{I} &= xi\mathbf{I}, & i(\mathbf{I} + \mathbf{J}) &= i\mathbf{I} + i\mathbf{J}, \\ (i\mathbf{I})(i\mathbf{J}) &= \mathbf{I}\mathbf{J}, & \mathbf{I}i\mathbf{J} &= \mathbf{J}i\mathbf{I}, \\ \mathbf{I}i(\mathbf{J} + \mathbf{K}) &= \mathbf{I}i\mathbf{J} + \mathbf{I}i\mathbf{K}, & (\mathbf{I},\, \mathbf{J}) &= (i\mathbf{I},\, i\mathbf{J}). \end{aligned}$$

The first one expresses the idea that one can change the order of the following two operations: *multiplication by a number* and *turning through a right angle*. The second one shows that the operation i has the distributive property with respect to the sum.

d. The orthogonality condition for two non-zero vectors **I**, **J** is $\mathbf{I}i\mathbf{J} = 0$.

16. Let **I** be a unit vector in the plane. The modulus of the bivector $\mathbf{I}i\mathbf{I}$ is 1, and for any unit vector **J**, one will have $\mathbf{J}i\mathbf{J} = \mathbf{I}i\mathbf{I}$, which leads us to call $\mathbf{I}i\mathbf{I}$ the *unit bivector* in the plane.

a. If u is a bivector in the plane [*see* no. 12 (*d*)] then u / $\mathbf{I}i\mathbf{I}$ will represent the number x such that $u = x\mathbf{I}i\mathbf{I}$. Of course, this number x is positive or negative according to whether the bivectors u and $\mathbf{I}i\mathbf{I}$ do or do not have the same sense, resp., and its absolute value is mod u. We agree to denote the number u / $\mathbf{I}i\mathbf{I}$ by the symbol u, as we have already done for trivectors; i.e., we give the symbol u the double meaning of a bivector and a number. If the number u is positive and the angle Ou is equal to the triangle OAB then the observer who traverses the triangle from O to A to B will see the area of the triangle on his left.

b. If $\mathbf{U}$, $\mathbf{V}$ are non-zero vectors and one supposes that the theory of circular functions is known then one will prove the following formulas quite easily:

$$\begin{aligned}\mathbf{UV} &= \text{mod}\,\mathbf{U}\,\text{mod}\,\mathbf{V}\,\sin(\mathbf{U},\mathbf{V}),\\ \mathbf{U}i\mathbf{V} &= \text{mod}\,\mathbf{U}\,\text{mod}\,\mathbf{V}\,\cos(\mathbf{U},\mathbf{V}),\end{aligned}$$

from which, one will deduce that:

$$\sin(\mathbf{U},\mathbf{V}) = \frac{\mathbf{UV}}{\text{mod}\,\mathbf{U}\,\text{mod}\,\mathbf{V}}, \qquad \cos(\mathbf{U},\mathbf{V}) = \frac{\mathbf{U}i\mathbf{V}}{\text{mod}\,\mathbf{U}\,\text{mod}\,\mathbf{V}}, \qquad \tan(\mathbf{U},\mathbf{V}) = \frac{\mathbf{UV}}{\mathbf{U}i\mathbf{V}}.$$

c. For any vector $\mathbf{U}$, one thus has:

$$\mathbf{U}i\mathbf{U} = (\text{mod}\,\mathbf{U})^2.$$

Upon agreeing to write $\mathbf{U}^2$, instead of $\mathbf{U}i\mathbf{U}$, one will have:

$$\mathbf{U}^2 = (\text{mod}\,\mathbf{U})^2, \qquad (\mathbf{U}+\mathbf{V})^2 = \mathbf{U}^2 + 2\mathbf{U}i\mathbf{V} + \mathbf{V}^2$$

and

$$(\mathbf{U}+\mathbf{V})\,i\,(\mathbf{U}-\mathbf{V}) = \mathbf{U}^2 - \mathbf{V}^2.$$

d. If $\mathbf{U}$, $\mathbf{V}$ are vectors then the number $\mathbf{U}i\mathbf{V}$ will be called the *inner product of* $\mathbf{U}$ *with* $\mathbf{V}$ ([7]). This inner product enjoys the commutative property and the distributive property with respect to the sum. If mod $\mathbf{U}$ = mod $\mathbf{V}$ = 1 then $\mathbf{U}i\mathbf{V}$ will be the cosine of the angle $(\mathbf{U}, \mathbf{V})$. If mod $\mathbf{U}$ = 1 then $\mathbf{U}i\mathbf{V}$ will give the magnitude and sense of the vector that is the orthogonal projection of the vector $\mathbf{V}$ onto $\mathbf{U}$; i.e., the vector $(\mathbf{U}i\mathbf{V})\,\mathbf{U}$ will be the orthogonal projection of the vector $\mathbf{V}$ onto the vector $\mathbf{U}$.

Examples. – 1. If $A_1, A_2, \ldots, A_n$ are points and $\mathbf{I}$ is a unit vector in the plane then the identity:

$$\mathbf{I}\,i\,(A_2 - A_1) + \mathbf{I}\,i\,(A_3 - A_2) + \ldots + \mathbf{I}\,i\,(A_n - A_{n-1}) = \mathbf{I}\,i\,(A_n - A_1)$$

will show that *the (algebraic) sum of the projections of the edges of a broken line onto a line is equal to the projection onto that same line of the bounded line that joins the extremities of the broken line* (*d*).

([7]) The reader must appreciate the importance of the inner product. The inner product, which was introduced by Grassmann as an abstract operation, is reduced here to the progressive product of two vectors by means of the operation i.

2. If $\mathbf{I}$, $\mathbf{J}$ are non-zero vectors and have the same modulus then the identity $(\mathbf{I} + \mathbf{J})\, i\, (\mathbf{I} - \mathbf{J}) = \mathbf{I}^2 - \mathbf{J}^2 = 0$ will express the idea that *the bisectors of two adjacent angles are rectangular*.

3. Let A, B, C be the summits of a triangle in the plane; set:

$$\mathbf{I} = C - B, \qquad \mathbf{J} = A - C, \qquad \mathbf{K} = B - A.$$

One has:

$$\mathbf{I} + \mathbf{J} + \mathbf{K} = 0. \tag{1}$$

One deduces from formula (1) that:

$$(-\mathbf{J})\,\mathbf{K} = (-\mathbf{K})\,\mathbf{I} = (-\mathbf{I})\,\mathbf{J}, \tag{2}$$

$$\mathbf{I}^2 = \mathbf{J}^2 + \mathbf{K}^2 - 2(-\mathbf{J})\, i\, \mathbf{K}, \tag{3}$$

$$\mathbf{I}^2 = -\mathbf{J}\, i\, \mathbf{I} - \mathbf{K}\, i\, \mathbf{I} = (-\mathbf{I})\, i\, \mathbf{J} + (-\mathbf{K})\, i\, \mathbf{I}. \tag{4}$$

If one divides equations (2) and (4) by mod $\mathbf{I}$. mod $\mathbf{J}$. mod $\mathbf{K}$ and mod $\mathbf{I}$, respectively, then one will get:

$$\frac{\sin(-\mathbf{J},\mathbf{K})}{\operatorname{mod}\mathbf{I}} = \frac{\sin(-\mathbf{K},\mathbf{I})}{\operatorname{mod}\mathbf{J}} = \frac{\sin(-\mathbf{I},\mathbf{J})}{\operatorname{mod}\mathbf{K}},$$

$$(\operatorname{mod}\mathbf{I})^2 = (\operatorname{mod}\mathbf{J})^2 + (\operatorname{mod}\mathbf{K})^2 - 2 \operatorname{mod}\mathbf{J} \operatorname{mod}\mathbf{K} \cos(-\mathbf{J}, \mathbf{K}),$$

$$\operatorname{mod}\mathbf{I} = \operatorname{mod}\mathbf{J} \cos(-\mathbf{I}, \mathbf{J}) + \operatorname{mod}\mathbf{K} \cos(-\mathbf{K}, \mathbf{I}),$$

which are the same as the formulas in plane trigonometry that are known in the form:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

$$a^2 = b^2 + c^2 - 2\, bc \cos A, \qquad a = b \cos C + c \cos B.$$

17. If x, y are real numbers and $\mathbf{I}$ is a vector in the plane then when one sets:

$$(x + iy)\,\mathbf{I} = x\mathbf{I} + y\, i\mathbf{I},$$

one will have:

$$[(x + iy)\,\mathbf{I}]^2 = (x^2 + y^2)\,(\operatorname{mod}\mathbf{I})^2,$$

and, in turn:

$$\operatorname{mod}[(x + iy)\,\mathbf{I}] = \sqrt{x^2 + y^2}\,.$$

If $\mathbf{I} \neq 0$ and the numbers x, y are not both zero then if one is given that:

$$\mathbf{I}\,[(x + iy)\,\mathbf{I}] = y(\operatorname{mod}\mathbf{I})^2, \qquad \mathbf{I}\, i[(x + iy)\,\mathbf{I}] = x\,(\operatorname{mod}\mathbf{I})^2$$

then one will have:

$$\tan [\mathbf{I}, (x + iy) \mathbf{I}] = \frac{y}{x}.$$

Therefore, *multiplying the vector* **I** *by the complex number* $x + iy$ *means multiplying* **I** *by the modulus of the complex number and turning the vector thus obtained through an angle that is equal to the argument of the complex vector.*

If we write the complex number $x + iy$ in the form:

$$x + iy = \rho (\cos \varphi + i \sin \varphi)$$

or in the form:

$$x + iy = \rho\, e^{i\varphi}$$

then we will see that $\cos \varphi + i \sin \varphi$ – or $e^{i\varphi}$ – is the symbol for the operation that will make a vector turn through the angle φ when it is applied to that vector; i.e., for any number φ, $(\cos \varphi + i \sin \varphi)\, \mathbf{I}$ – or, more simply, $e^{i\varphi}\, \mathbf{I}$ – represents the vector **I**, but turned through the angle φ.

Examples. – Let O be a point, and let **I** be a unit vector in the plane.

1. If x, y are numbers, and $P = O + (x + iy)\, \mathbf{I}$ then x, y will be the rectangular Cartesian coordinates of the point P, if one takes the point O to be the origin and the lines $O\mathbf{I}$, $O(i\, \mathbf{I})$ to be the axes.

2. If ρ, φ are numbers, and $P = O + \rho\, e^{i\varphi}\, \mathbf{I}$ then ρ, φ will be the polar coordinates of the point P, where O is the pole and the $O\mathbf{I}$ is the chosen polar axis.

3. When φ varies from 0 to 2π, the point $P = O + r\, e^{i\varphi}\, \mathbf{I}$ will describe the circle whose center is O and whose radius is r.

4. The point $P = O + r\, \varphi\, e^{i\varphi}\, \mathbf{I}$ describes an Archimedean spiral.

5. Let O be the common center of two circles, one of which has radius a, while the other one has radius b ($a > b$). A radius of the first one that makes the angle φ with **I** will meet the first one at M and the second one at N. The parallels to the vectors **I**, i**I** that are drawn through the points N, M, respectively, will meet at a point P. When φ varies from 0 to 2π, the point P will describe an ellipse whose center is at O and whose semi-axes are a and b, respectively.

One easily sees that:

$$P = O + a \cos \varphi\, \mathbf{I} + ib \sin \varphi\, \mathbf{I}.$$

If we recall that:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi, \qquad e^{-i\varphi} = \cos \varphi - i \sin \varphi$$

then we will have:

$$P = O + \frac{a+b}{2} e^{i\varphi}\mathbf{I} + \frac{a-b}{2} e^{-i\varphi}\mathbf{I}.$$

In general, when φ varies, the point $P = O + h\, e^{i\varphi}\, \mathbf{I} + k\, e^{-i\varphi}\, \mathbf{I}$ will describe an ellipse whose semi-axes are $h + k$, $h - k$.

6. If a circle of radius r rolls without slipping on the line m then the locus of successive points that a point of the circle must occupy will be a cycloid.

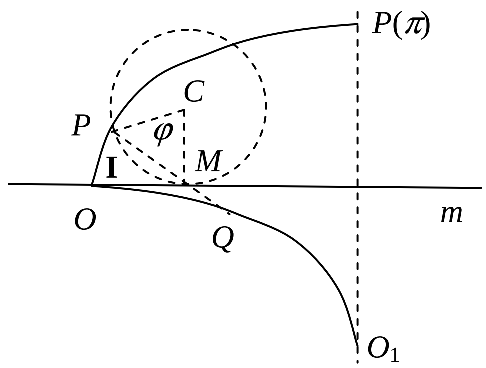

Figure 1.

Let O, M be two points of m such that mod $OM < 2\pi r$ and, in addition, let C (*fig.* 1) be the center of the circle of radius r that is tangent to m at M. If O is a position of the point that describes the cycloid then a point P on the circle of center C that is such that MP = mod OM will be a point of the cycloid.

Therefore, let $\varphi = (P - C, M - C)$; one will then have mod $OM = r\varphi$. Consider the unit vector $\mathbf{I}$ that is parallel and in the same sense as the vector $M - O$. We will have:

$$P - O = (M - O) + (C - M) + (P - C),$$

$$M - O = r\varphi\,\mathbf{I}, \quad C - M = ri\,\mathbf{I}, \quad P - C = -\,r\,e^{-i\varphi}\,i\,\mathbf{I};$$

therefore:

$$P = O + r\varphi\,\mathbf{I} + ri\,\mathbf{I} - r\,e^{-i\varphi}\,i\,\mathbf{I},$$

and the point P will describe the cycloid when φ varies from $-\infty$ to $+\infty$.

7. The point:

$$P = O + a\,e^{im\varphi}\,\mathbf{I} + b\,e^{in\varphi}\,\mathbf{I},$$

when φ varies from $-\infty$ to $+\infty$ describes an epicycloid. The radius of the fixed circle is $a\frac{n-m}{n}$ $\left(\text{or } b\frac{m-n}{m}\right)$. The radius of the moving circle is $a\frac{m}{n}$ $\left(\text{or } b\frac{m}{n}\right)$, and the distance from the point P to the center of the moving circle is b (or a).

18. Index operation. – Let **U** be a non-zero vector, u, a non-zero bivector, and let O be a point. If the line O**U** is perpendicular to the plane Ou then the line P**U** will be perpendicular to the plane Pu, and this will be true for any point P. We express this property by saying that *the vector* **U** *is perpendicular to the bivector* u, or u *is perpendicular to* **U**. We give the same significance to the phrases: *the bivector u is perpendicular to the bivector v, and the vector* **U** *is perpendicular to the vector* **V**.

If u is a non-zero bivector then we will let the symbol $\mid u$ denote the vector such that:

1. $\mid u$ is perpendicular to the bivector u.
2. $\operatorname{mod}(\mid u) = \operatorname{mod} u$.
3. The trivector $u(\mid u)$ has the direct sense.

The vector $\mid u$ is therefore well-defined, and one calls it the *index of u*.

We agree to give to the relation:

$$\mathbf{U} = \mid u,$$

the reciprocal form:

$$u = \mid \mathbf{U},$$

in order to be able to call *u the index of* **U**. Consequently, when the operation whose symbol is |, which one calls the *index operation*, is applied to a non-zero bivector, it will produce a vector, and when it is applied to a non-zero vector, it will produce a bivector. Upon also setting:

$$0 = \mid 0,$$

this convention will succeed in defining the index operation, which then persists for any vectors and bivectors.

If, for example, u, v are two bivectors, **U** is a vector, and x is a number then instead of the symbols:

$$\mid (\mid u),\quad (\mid u) + \mathbf{U},\qquad x(\mid u),\quad \mid (xu),\quad u(\mid v),$$

which currently mean something, we shall write, more simply:

$$\parallel u,\quad \mid u + \mathbf{U},\quad x \mid u,\quad \mid xu,\quad u \mid v.$$

a. If u, v, w are bivectors or vectors and x is a real number then we will have the formulas:

(1) $$\parallel u = u,$$
(2) $$\mid xu = x \mid u,$$
(3) $$u \mid v = v \mid u,$$
(4) $$\mid (u + v) = \mid u + \mid v,$$
(5) $$(u + v) \mid w = u \mid w + v \mid w.$$

If $\mathbf{U} = \mid u$ then $u = \mid \mathbf{U}$, and consequently, $u = \parallel u$, which proves formula (1). Formulas (2), (3) are also proved easily.

Here is the proof of formula (4), which gives the distributive property of the index operation with respect to the sum: Let u, v be bivectors. If one of them is zero then formula (4) will be obvious; therefore, suppose that u and v are non-zero bivectors. One can then determine vectors **I**, **J**, **K** such that **I** is perpendicular to **J** and **K**, and:

$$\text{mod}\,\mathbf{I} = 1, \qquad u = \mathbf{IJ}, \qquad v = \mathbf{IK}.$$

If one observes that $u + v = \mathbf{I}\,(\mathbf{J} + \mathbf{K})$ then one will see that $\mid u$, $\mid v$, $\mid (u + v)$ are the vectors **J**, **K**, **J** + **K**, which have received a rotation through a right angle in the same sense around an arbitrary point O in the plane O**IJ**. Formula (4) is therefore proved when u, v are bivectors. In the case where u and v are vectors, one will deduce formula (4) for the vectors in formulas (1) and (4).

Formula (5) is only a consequence of formula (4) and the distributive property of the product with respect to the sum.

b. One has:

$$u \mid u = (\text{mod}\,u)^2.$$

Upon writing u^2 instead of $u \mid u$, this will become:

$$u^2 = (\text{mod}\,u)^2, \qquad (u + v)^2 = u^2 + 2u \mid v + v^2, \qquad (u + v) \mid (u - v) = u^2 - v^2.$$

c. The perpendicularity condition of u with respect to v is:

$$u \mid v = 0.$$

d. Let **U**, **V** be non-zero vectors. If we fix the sense of positive rotation on a plane that is parallel to **U** and **V**, and we let φ represent the angle between **U** and **V** then we will easily see that the number **U** | **V** is equal to the product of mod **U** mod **V** with either cos φ or – cos φ. We cannot call φ or $\pi - \varphi$ the angle between the two vectors **U**, **V**, because φ is a function of not only **U**, **V**, but also the sense of positive rotation in the plane, and we cannot establish a relationship between the directions of positive rotation on two arbitrary planes.

Thus, in order to introduce the angle between two arbitrary vectors **U**, **V** in space – that angle being regarded as a function of only **U** and **V** – we set:

$$\mathbf{U} \mid \mathbf{V} = \text{mod}\,\mathbf{U}\;\text{mod}\,\mathbf{V}\,\cos(\mathbf{U} \mid \mathbf{V}),$$

or better yet:

(1) $$\cos(\mathbf{U} \mid \mathbf{V}) = \frac{\mathbf{U}}{\text{mod}\,\mathbf{U}} \,\Bigg|\, \frac{\mathbf{V}}{\text{mod}\,\mathbf{V}},$$

and we define (**U**, **V**) by saying that it is the smallest positive number or zero that verifies equation (1). One deduces from this, notably, that (**U**, **V**) can vary from 0 to π, and in turn, sin(**U**, **V**) will be a number that is always positive, and one will have:

$$\text{mod}(\mathbf{UV}) = \text{mod}\,\mathbf{U}\;\text{mod}\,\mathbf{V}\,\sin(\mathbf{U}, \mathbf{V}).$$

There is no contradiction between the results that we just obtained and the ones that we obtained in no. 16, because in the latter case (**U**, **V**) was not just a function of **U** and **V**.

c. One lets **U** | **V** denote the inner product of **U** by **V**, and that operation has some properties that are analogous to the operation that we already referred to as the *inner product on a plane*.

Examples. – 1. If *A*, *B*, *C*, *D* are arbitrary points then one will always have:

$$(A - B) \mid (C - D) + (B - C) \mid (A - D) + (C - A) \mid (B - D) = 0.$$

In order to prove this formula, it suffices to reduce the vectors $A - B$, $B - C$, $C - A$ to the difference of two vectors that have the same origin *D* [for example, $A - B = (A - D) - (B - D)$]. If one then develops the inner products then one will find that the first number will be equal to zero. The identity that we just stated can be interpreted geometrically in the following manner: *If, in the tetrahedron whose summits are the points A, B, C, D, the opposite edges AB and CD, BC and AD, respectively, are pair-wise perpendicular then the last two edges AC and BD will also be rectangular.* Even better: *The three altitudes of a triangle will have a common point.*

2. If *A*, *B*, *C* are the summits of a triangle and **I** is a vector then one will have:

$$(B - B) \mid \mathbf{I} + (C - A) \mid \mathbf{I} + (A - B) \mid \mathbf{I} = 0.$$

Therefore, if two of the vectors $B - C$, $C - A$, $A - B$ are perpendicular to the vector **I** then the third one will also be perpendicular to **I**, which expresses a theorem that is already well-known from elementary geometry.

3. Let **I**, **J**, **K** be unit vectors. Set:

$$a = (\mathbf{J}, \mathbf{K}), \qquad b = (\mathbf{K}, \mathbf{I}), \qquad c = (\mathbf{I}, \mathbf{J})$$

and decompose **J** into two vectors **J′** and **J″**, and **K** into two vectors **K′** and **K″**, where the first two (viz., **J′** and **K′**) are perpendicular to **I** and the second two (viz., **J″** and **K″**) are parallel to **I**. With $\alpha = (\mathbf{J}', \mathbf{K}')$, one easily finds:

$$\mathbf{J}' \mid \mathbf{K}' = \sin b \sin c \cos a, \qquad \mathbf{J}'' \mid \mathbf{K}'' = \cos b \cos c,$$

but:

$$\cos a = \mathbf{J} \mid \mathbf{K} = (\mathbf{J}' + \mathbf{J}'') \mid (\mathbf{K}' + \mathbf{K}'') = \mathbf{J}' \mid \mathbf{K}' + \mathbf{J}'' \mid \mathbf{K}''.$$

Therefore:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha,$$

which is nothing but the fundamental formula of spherical trigonometry ([8]).

([8]) E. CARVALLO, "Sur une généralisation du théorème des projections," Nouvelles Annales de Mathématiques **9** (1891).

§ 3. – REDUCTION OF FORMS.

19. First-order forms. – One calls the number $x_1 + x_2 + \dots + x_n$ *the mass of the first-order form:*

$$S = x_1 A_1 + x_2 A_2 + \dots + x_n A_n .$$

a. The mass of the first-order form S is the number $S\omega$. Indeed:

$$A_1\omega = A_2\omega = \dots = A_n\omega = 1 \quad \text{and} \qquad S\omega = x_1 + x_2 + \dots + x_n .$$

b. If a first-order form has zero mass then that form will be reducible to a vector, and conversely.

If O is an arbitrary point then we will have:

$$S = (x_1 + x_2 + \dots + x_n)\, O + x_1(A_1 - O) + x_2(A_2 - O) + \dots + x_n(A_n - O),$$

and if we set:

$$\mathbf{I} = x_1(A_1 - O) + x_2(A_2 - O) + \dots + x_n(A_n - O)$$

then we will get:

$$S(S\omega)O + \mathbf{I}.$$

If $S\omega = 0$ then $S = \mathbf{I}$. However, $\mathbf{I}$ is a vector, so S is indeed reducible to a vector. Conversely, if $\mathbf{S}$ is a vector such that $\mathbf{S} = B - A$ then one will have:

$$\mathbf{S}\omega = B\omega - A\omega = 1 - 1 = 0.$$

c. If a first-order form does not have mass zero then that form will be reducible to the product of its mass by a point. Indeed, if $S\omega \neq 0$ then one will have:

$$S = (S\omega)\left(O + \frac{1}{S\omega}\mathbf{I}\right).$$

If $S\omega \neq 0$ then the point $S / S\omega$ will be called the *barycenter of the form S*. In mechanics, $S / S\omega$ is the center of gravity of the massive points $A_1, A_1, \dots, A_n$ which have the masses:

$$x_1, x_2, \dots, x_n ,$$

respectively.

d. If A, B are two points and x, y are non-zero numbers, and furthermore, $x + y \neq 0$ then the point $C = \dfrac{xA + yB}{x + y}$ will be situated on the line AB and will decompose the line segment AB into two parts that are inversely proportional to the numbers x, y. Indeed, if one multiples the two sides of the equality:

$$xA + yB = (x + y)\, C$$

by AB then one will have $ABC = 0$; i.e., the points A, B, C will be situated on the same line. If one multiples the same equality by C then one will have $xAC + yBC = 0$; i.e., that $\frac{AC}{CB} = \frac{y}{x}$, which will prove the last part of the theorem.

One then easily deduces the graphic construction of the point $\frac{xA+yB}{x+y}$, and consequently, the construction of the barycenter of an arbitrary first-order form.

Examples. – Let A, B, C be points.

1. The point $\frac{A+B}{2}$ is the middle of the line segment AB.

2. If $ABC \neq 0$ then $\frac{A+B+C}{3}$ will be the point through which pass the three medians of the triangle ABC. One proves this property by observing that $\frac{A+B+C}{3}$ is the barycenter of the points $\frac{A+B}{2}$, C, which are affected with the masses 2, 1, respectively.

3. The identity:

$$\frac{A+B+C+D}{4} = \frac{\frac{A+B}{2}+\frac{C+D}{2}}{2} = \frac{\frac{B+C}{2}+\frac{D+A}{2}}{2} = \frac{\frac{A+C}{2}+\frac{B+D}{2}}{2}$$

shows that the lines that join the middles of the opposite edges and the diagonals of a quadrilateral have a common point that is precisely the barycenter of the quadrilateral.

4. If the lines AD, BC intersect at the point E, and the lines AB, CD, at the point F then if one observes that:

$$EAD = EBC = FBA = FCD = 0$$

then one can write:

$$(A + C)(B + C)(E + F) = ABE + ADF + CBF + CDE = (CDE - BAE) - (DAF - CBF),$$

and the two forms in parentheses will give us the area of the (plane) quadrilateral whose summits are A, B, C, D. Consequently:

$$(A + C)(B + D)(E + F) = 0;$$

i.e., *the middles of the diagonals of a complete quadrilateral are on the same line.*

5. The point O (*fig.* 2) is the common center to two circles. Through a point P, for example, of the inner circle, one draws the rectangular chords PA and PBC, which one, in turn, makes turn around the fixed point P.

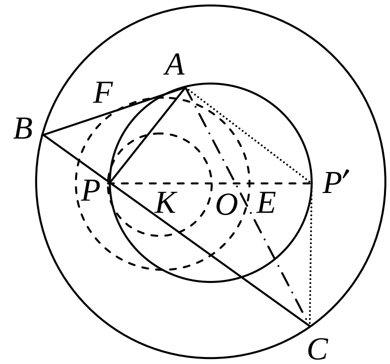

Figure 2.

1. Prove that the barycenter of the triangle ABC is fixed.
2. Prove that the sum of the squares of the distances from the point P to the points A, B, C is constant.
3. Find the locus that is described by the middles of the edges of the triangle ([9]).

If $P' = O + (O - P)$ then the trapezoid whose summits are A, B, C, P' will be isosceles, and one will easily see that:

$$(A - P) + (B - P) + (C - P) = 2(O - P). \tag{1}$$

This relation (1) is, moreover, identical to the relation:

$$\frac{A+B+C}{3} = \frac{2O+P}{3},$$

which proves the first part.

The squares (i.e., inner products) of the two sides of the equality (1) give:

$$(A - P)^2 + (B - C)^2 + (C - P)^2 + 2(B - P)(C - P) = 4(O - P)^2,$$

because:

$$(A - P)\ i\ (B - P) = (A - P)\ i\ (C - P) = 0.$$

However, if r, R are the radii of the two circles then one will have:

$$4(O - P)^2 = 4r^2,$$
$$2(B - P)\ i\ (C - P) = - 2(R + r)(R - r),$$

and, in turn:

$$(A - P)^2 + (B - C)^2 + (C - P)^2 = 2(R^2 + r^2),$$

([9]) *Composition de mathématique à l'École spéciale militaire de Saint-Cyr*. Course given in 1895.

which proves the second part.

Finally, set:

$$D = \frac{B+C}{2}, \quad E = \frac{C+A}{2}, \quad F = \frac{A+B}{2}, \quad K = \frac{P+O}{2}.$$

If we put the relation (1) into the form:

$$A + (B - P) + C = 2O,$$

or even in the form:

$$(A - O) + (B - P) + (C - O) = 2O,$$

then it will result that:

$$D - K = \frac{B-P}{2} + \frac{C-O}{2} = -\frac{A-O}{2}.$$

Therefore, mod(D – K) is constant, and D describes a circle of center K and radius r / 2. One will likewise find that the points E, F are always on the circle of center K and radius R / 2.

20. Second-order forms. – Every line segment is reducible to the form $A\mathbf{I}$, where A is a point and $\mathbf{I}$ is a vector, because $AB = A(B - A)$, and one can do likewise for the product of a line segment with a number, because $x(A\mathbf{I}) = A(x\mathbf{I})$. A second-order form s (which is the sum of a finite number of line segments) is therefore reducible to the general form:

$$s = A_1\,\mathbf{I}_1 + A_2\,\mathbf{I}_2 + \ldots + A_n\,\mathbf{I}_n, \tag{1}$$

where $A_1, A_2, \ldots, A_n$ are points, and $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_n$ are vectors.

a. Call the vector $\mathbf{I}$ the *vector of the line segment* $A\mathbf{I}$. Call the sum of the vectors of the line segments that comprise a second-order form the *vector of the second-order form*. The vector of s will then be $\mathbf{I}_1 + \mathbf{I}_2 + \ldots + \mathbf{I}_n$.

b. A form s is reducible – in an infinitude of ways – to the sum of a bivector and a line segment whose vector is the same vector as that of s. Indeed, let O be an arbitrary point. Since:

$$A_1\,\mathbf{I}_1 = (A_1 - O)\,\mathbf{I}_1 + O\mathbf{I}_1,$$

the relation (1) will take the form:

$$s = O(\mathbf{I}_1 + \mathbf{I}_2 + \ldots + \mathbf{I}_n) + [(A_1 - O)\,\mathbf{I}_1 + (A_2 - O)\,\mathbf{I}_2 + \ldots + (A_n - O)\,\mathbf{I}_n],$$

which proves the theorem.

In general, if O is an arbitrary point then one will have $s = O\mathbf{I} + u$, where $\mathbf{I}$ is the vector of s, and u is a bivector that depends upon s and O.

c. If s is a second-order form then one will call the tetrahedron ss the *invariant of* s.

In order for the second-order form s to be reducible to a line segment or a bivector (i.e., to the product of two second-order forms), it is necessary and sufficient that the invariant of s be zero (viz., $ss = 0$). Indeed, if $s = O\mathbf{I} + u$ then $ss = 2O\mathbf{I}u$. One indeed has $ss = 0$ when $\mathbf{I}u = 0$. If $\mathbf{I}$ or u are zero then s will be a line segment or a bivector, but if $\mathbf{I}$ and u are not zero then u will be parallel to $\mathbf{I}$, and $O\mathbf{I} + u$ will be a line segment that is parallel to $O\mathbf{I}$. The condition is therefore indeed also sufficient.

d. Let A, B be two points, and let $\mathbf{I}$, $\mathbf{J}$ be two vectors in a plane. Consider the second-order form $s = A\,\mathbf{I} + B\mathbf{J}$. If the segments $A\mathbf{I}$, $B\mathbf{J}$ are not parallel and O is the point that is common to the lines that carry the line segments $A\mathbf{I}$, $B\mathbf{J}$ then the identity $s = O(\mathbf{I} + \mathbf{J}) + (A - O)\,\mathbf{I} + (B - O)\mathbf{J}$ will give $s = O(\mathbf{I} + \mathbf{J})$, because the vectors $A - O$, $B - O$ are parallel to the vectors $\mathbf{I}$, $\mathbf{J}$, respectively, and the formula $s = O(\mathbf{I} + \mathbf{J})$ will immediately provide the reduction of s to a line segment. If the line segments $A\mathbf{I}$, $B\mathbf{J}$ are parallel then $\mathbf{J} = x\mathbf{I}$ and $s = (A + xB)\mathbf{I}$. If $x \neq -1$ then one will have $s = \dfrac{A+xB}{1+x}(\mathbf{I} + \mathbf{J})$, and one will reduce s to a line segment by using the construction of the barycenter of the form $A + xB$. If $x = -1$ then $s = (A - B)\mathbf{I}$, and s will be reducible to a bivector. In general, a second-order form that is the sum of line segments in the same plane is always reducible to a line segment or a bivector.

In mechanics, one can represent a force that is applied to a rigid body by a line segment and the resultant of a system of identical forces by a second-order form. A couple is represented by a bivector. If A, B, C, D are four points then the number $ABCD$ will be proportional to the moment of the force AB with respect to the axis CD. If u is a bivector then $|\,u$ will be the moment axis of the couple.

Examples. –

1. Let $A_1, A_2, \ldots, A_n$ be points in the same plane. The form:

$$s = A_1\,A_2 + A_2\,A_3 + \ldots\, A_{n-1}\,A_n + A_n\,A_1$$

has a zero vector. It is therefore reducible to a bivector. For any point P in the plane, the triangle Ps has the same value, and one can call the area that is bounded by the closed polygonal line $A_1, A_2, \ldots, A_n$ the area of the triangle Ps. If the line is convex then the area thus defined will be the one that one considers in elementary geometry.

2. Let A, B, C be the summits of a triangle. Set:

$$a = \frac{BC}{\operatorname{mod} BC}, \qquad b = \frac{CA}{\operatorname{mod} CA}, \qquad c = \frac{AB}{\operatorname{mod} AB}.$$

The line segments $b + c$, $c + a$, $a + b$ are on the external bisectors, and the line segments $b - c$, $c - a$, $a - b$ are on the internal bisectors of the triangles. The identities:

$$a + b + c = (b + c) + a = (c + a) + b = (a + b) + c$$

prove that the points of intersection of the external bisectors with the opposite edges are placed on the same line. One will likewise find the geometrical significance of the identities $a + b - c = (a + b) - c = (a - c) + b = (b - c) + a, \ldots, b - c = (b - a) - (c - a), \ldots$

21. Third-order forms. – Every triangle is reducible to the form Au, where A is a point and u is a bivector, because $ABC = A(B - C)(C - A)$, and one can, as before, likewise make it the product of a triangle by a number because $m(Au) = A(mn)$. As a result, a third-order form σ that is the sum of a finite number of triangles is always reducible to the general form:

(1) $$\sigma = A_1 u_1 + A_2 u_2 + \ldots + A_n u_n ,$$

where $A_1, A_2, \ldots, A_n$ are points, and $u_1, u_2, \ldots, u_n$ are bivectors.

a. Call the bivector *u the bivector of the triangle Au*, and similarly, *the bivector of a third-order form* will be the sum of the bivectors of the triangles that give that form by addition. The bivector of σ is then:

$$u_1 + u_2 + \ldots + u_n .$$

b. A third-order form is reducible to a trivector or to a triangle according to whether its bivector is zero or not, respectively. For example, consider an arbitrary point O. One has, among other things:

$$A_1 u_1 = (A_1 - O) u_1 + Ou,$$

and consequently:

$$\sigma = O(u_1 + u_2 + \ldots + u_n) + (A_1 - O) u_1 + \ldots + (A_n - O) u_n ;$$

i.e., σ is precisely reducible to the form:

$$\sigma = Ou + \alpha,$$

where O is an arbitrary point, u is the bivector of σ, and α is a trivector that depends upon O and α. If $u \neq 0$ then one can determine the vector $\mathbf{I}$ such that $\alpha + \mathbf{I}u$, and in turn:

$$\sigma = (O + \mathbf{I})\, u,$$

and σ will indeed be reducible to a triangle. On the contrary, if $u = 0$ then one will have $\sigma = \alpha$, and σ will be reducible to a trivector.

22. Projective elements. – For any order of non-zero form A, we will write posit A, instead of the *position of A*, to abbreviate.

a. If S is a first-order form such that $S\omega \neq 0$ then set:

$$\text{posit } S = \frac{S}{S\omega}.$$

In other words, we will denote the barycenter of the form S by the symbol posit S. If S' is a multiple of S then one will have posit S' = posit S, and conversely.

If **I** is a non-zero vector then we will always agree that the symbol posit **I** is equivalent to the *direction* of **I**. One deduces from this that for two non-zero parallel vectors **I**, **J** (which amounts, as we have seen, to saying that **I** is a multiple of **J**), one will have posit **I** = posit **J**, and conversely. In the language of ordinary projective geometry, one would say that posit **I** is a *point at infinity*.

We likewise write *projective point*, instead of *position of a non-zero, first-order form*. A projective point can then be a point (à la Euclid) or a point at infinity. If S is a non-zero, first-order form then we will also write point S, instead of posit S.

b. Let a be a non-zero, second-order form with zero invariant (viz., $aa = 0$). We let the symbol posit a denote the locus of projective points that are positions of non-zero, first-order forms A such that $Aa = 0$. If A, B are two points such that $AB \neq 0$ then posit AB will contain all of the points of the (unbounded) line that joins A to B, as well as the point at infinity that is the position of the vector $B - A$. Similarly, let **I**, **J** be two vectors, with the condition that $\mathbf{IJ} \neq 0$. From the preceding conventions, posit **IJ** will represent a set of points at infinity that can be identified with the orientation of the bivector **IJ**. If a, b are two non-zero, second-order forms whose invariants we suppose to be identically zero and are such that a is a multiple of b then we will have:

$$\text{posit } a = \text{posit } b.$$

We further agree to write *projective line*, instead of *position of a non-zero, second-order form whose invariant is zero*, and *line at infinity*, instead of *position of a non-zero bivector*, and line a, instead of posit a.

c. Let α be a non-zero, third-order form. We let posit α denote the locus of projective points that are the positions of first-order forms A such that $A\alpha = 0$. Under these conditions, if A, B, C are three points that are subject to the relation $ABC \neq 0$ then posit ABC will contain all of the points of the plane that passes through A, B, C, as well as all of the points of the line at infinity that is the position of the bivector $(B - A)(C - A)$. If α is a non-zero trivector then posit α = posit ω, and posit ω is the locus of all points at infinity. If α and β are non-zero, third-order forms such that α is a multiple of β then one will have posit α = posit β, and conversely.

As before, we shall write *projective plane*, instead of *position of a non-zero, third-order form*, *plane at infinity*, instead of posit ω, and plane α, instead of posit α ([10]).

([10]) All of the ordinary properties persist for the projective elements that are defined as we just did. By following the method that we just presented and applying the theory of linear transformations, one can very easily obtain all of the properties of ordinary projective geometry. [*See*, C. BURALI-FORTI, "Il metodo del Grassmann nella Geometria proiettiva," Rend. Circ. Matem. di Palermo, Note I, 1896; Note II, 1897. The power of Grassmann's method is likewise confirmed in the field of synthetic geometry.

d. If $AB \neq 0$ then line AB will be the projective line that joins the points A and B. With $Aa \neq 0$, plane Aa will be the projective plane that passes through the point A and the line a. The conditions $\alpha \neq 0$, $AB \neq 0$, $A\alpha = B\alpha = 0$ show that all of the points of the line AB are situated on the plane α.

23. The first-order forms A, B, …, are called *collinear* when the ones that are not zero have their positions on the same projective line. Likewise, the first-order forms A, B, …, and the second-order forms a, b, … with zero invariants are called *coplanar* when the ones that are not zero have their positions on the same projective plane.

Example. – Parallel vectors in the same plane have collinear forms. Vectors and bivectors are coplanar forms.

If $AB \neq 0$, and the first-order forms A, B, C, D are collinear then the notation CD / AB will have a unique significance (no. 4 and no. 12, *d*), since AB and CD will be bivectors or line segments according to whether line AB is or is not a line at infinity, respectively. We then agree to identify the symbol CD with the number CD / AB (the form AB being fixed) when we consider only first-order forms that are collinear with A and B.

b. Likewise, if $ABC \neq 0$, and the first-order forms A, B, C, D, E, F are coplanar then the notation DEF / ABC will take on a unique significance because ABC, DEF will be trivectors or triangles according to whether plane ABC is or is not the plane at infinity, respectively. When the form ABC is fixed, we further agree to identify the symbol DEF with the number DEF / ABC, on the condition that we consider only first-order forms that are coplanar with A, B, and C.

24. Identity between first-order forms. – The theorems that we shall state give relations that exist between five arbitrary first-order forms, or even between four coplanar, first-order forms and three collinear forms of the same order.

THEOREM I. – *If A, B, C, D, E are first-order forms then one will have:*

$$BCDE.A + CDEA.B + DEAB.C + EABC.D + ABCD.E = 0. \tag{1}$$

Proof. – If A, B, C, D are points that satisfy $ABCD \neq 0$ then one can determine numbers x, y, z such that:

$$E - A = x(B - A) + y(C - A) + z(D - A),$$

or furthermore:

$$E = (1 - x - y - z)\, A + xB + yC + zD. \tag{1$'$}$$

Upon multiplying the two sides of the equality (1)′ by BCD, one will have:

$$1 - x - y - z = -\frac{BCDE}{ABCD},$$

and other analogous formulas for x, y, z. If we then substitute these values for $1 - x - y - z$, x, y, z in the equality (1)′ then we will find that the formula (1) is proved when A, B, C, D, E are points and A, B, C, D are not coplanar. However, equation (1) is symmetric with respect to all symbols, which proves that it is likewise true when A, B, C, D, E are non-coplanar points. Moreover, if A, B, C, D, E are coplanar points then every term of that equation will be zero. Therefore, equation (1) is indeed established for the case where A, B, C, D, E are arbitrary points.

If A, B, C, D, E are first-order forms then one will have:

$$A = m_1 A_1 + m_2 A_2\,, \quad B = n_1 B_1 + n_2 B_2\,, \quad C = p_1 C_1 + p_2 C_2\,,$$
$$D = q_1 D_1 + q_2 D_2\,, \quad E = r_1 E_1 + r_2 E_2\,,$$

where A_1, … represent points and m_1, … represent numbers. The analogous equalities to equation (1) that one can define with the points A_1, …, B_1, … are then verified. On the other hand, the ones that are multiplied by the products $mnpqr$ and summed will give formula (1).

THEOREM II. – *If A, B, C, D are coplanar, first-order forms then one will have:*

$$BCD.A - CDA.B + DAB.C - ABC.D = 0.$$

Proof. – Let E be a first-order form that is not coplanar with A, B, C, D, so theorem I will say that one will have:

$$BCDE.A - CDAE.B + DABE.C - ABCE.D = 0,$$

because $ABCD = 0$. However, the numbers (i.e., tetrahedra) $BCDE$, $CDAE$, … are proportional to BCD, CDA, …, and the theorem is this found to be proved.

THEOREM III. – *If A, B, C are collinear, first-order forms then one will have:*

$$BC.A + CA.B + AB.C = 0.$$

Proof. – If D is a first-order form that is coplanar with A, B, C then one will deduce from theorem II that:

$$BCD.A + CAD.B + ABD.C = 0,$$

since $ABC = 0$. Now, the numbers BCD, CAD, ABD are proportional to BC, CA, AB, which succeeds in establishing the stated theorem.

§ 4. – REGRESSIVE PRODUCTS.

25. Second and third-order forms. – Let A, B, P, Q, R be first-order forms. Upon setting:

(1) $$AB.PQR = APQR.B - BPQR.A,$$

we will say that $AB.PQR$ is the *regressive product* – or simply, the *product* – of AB with PQR. Upon comparing the definition (1) with the identity in theorem I of no. 24, one sees that:

(2) $$AB.PQR = APQR.B + ABRP.Q + ABPQ.R.$$

a. If one has $A'B' = AB$ and $P'Q'R' = PQR$ then formulas (1), (2) will prove that:

$$AB.P'Q'R' = AB.PQR, \qquad A'B'.PQB = AB.PQB,$$

and, in turn, the product $AB.PQR$ can be regarded as a function of the forms AB, PQR. If we further set:

$$a = AB, \qquad \alpha = PQR$$

then we can write:

(1)′ $$AB.\alpha = A\alpha.B - B\alpha.A,$$

(2)′ $$a.PQR = aQR.P + aRP.Q + aPQ.R,$$

which defines the product of a second-order form with zero invariant by a third-order form.

As for the product of an arbitrary second-order form a with a third-order form a, we agree that it is found to be defined by formula (2)′. If one agrees that $\alpha a = a\alpha$ then the product of a second or third-order form with a second or third-order form will remain defined in a general manner. Moreover, these products have all of the properties of algebraic products that obey the commutative property. For example, one will have:

$$\alpha a = a\alpha, \qquad (a + b)\ \alpha = \alpha a + b\alpha, \quad a(\alpha + \beta) = a\alpha + a\beta,$$

and if x is a real number then:

$$x(a\alpha) = (xa)\ \alpha = a(x\alpha).$$

b. It further results from equations (1) and (2) that $(a\alpha)\ a = 0$, $(a\alpha)\ \alpha = 0$, which proves that $a\alpha$ or αa is a first-order form that belongs to the forms a, α. If $a \neq 0$, $\alpha \neq 0$, with $aa = 0$, then one will have $a\alpha = 0$ when the line a is contained in the plane α, and conversely.

c. If A is a point and $\mathbf{I}$ is a vector then one will have:

$$A\mathbf{I}.\omega = A\omega\mathbf{I} - \mathbf{I}\omega A = \mathbf{I},$$

which amounts to saying that $A\mathbf{I}.\omega$ is the vector of the line segment $A\mathbf{I}$. In general, if s is a second-order form then $s\omega$ will be the vector of the form s, and one will have, for any point O:

$$s = O(s\omega) + u,$$

and expression in which u will denote a bivector that is a function of O and s.

Examples. – Let A, B, … be points, a, b, …, non-zero line segments, and α, β, …, triangles that are likewise non-zero. One can state the following properties:

1. The parallel to the line a that is drawn through the point A is the position of the form $A.a\omega$.

2. The plane that is perpendicular to the line a and passes through A is the position of the form $A \mid a\omega$.

3. The position of the form $Aa.A \mid a\omega$ is the line that passes through A perpendicularly to the line a, and meets precisely that line at the point that is the position of the form $a.A \mid a\omega$.

4. The conditions for the parallelism of the lines a and b is $a\omega.b\omega = 0$, and the condition for perpendicularity of the same lines is $a\omega \mid b\omega = 0$.

5. If $AB.\alpha \neq 0$ and $AB.\alpha$ is not a vector then $AB.\alpha$ will be the point at which the line AB meets the plane α, where the mass of that point will be, moreover, equal to $A\alpha.B\omega - B\alpha.A\omega$.

26. Third-order forms. – Let A, B, C, P, Q, R be first-order forms. Set:

$$ABC.PQR = APQR.BC + BPQR.CA + CPQR.AB,$$

and call $ABC.PQR$ the *regressive product* – or simply, the *product* – of ABC with PQR.

a. It is obvious that $ABC.PQR$ is a function of the third-order form PQR. However, one can prove that $ABC.PQR$ is also a function of the form ABC. Therefore, if one is given two third-order forms α and β then the product $\alpha\beta$ of these two third-order forms with each other will be found to be well-defined. Moreover, these products will have all of the properties of algebraic products, except for the commutative property. For example, one will have:

$$\alpha\beta = -\beta\alpha, \qquad \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma,$$

and if x is a number then:

$$x(\alpha\beta) = (x\alpha)\beta = \alpha(x\beta).$$

b. From the definition itself of the product $\alpha\beta$, one deduces that $(\alpha\beta)(\alpha\beta) = 0$, $(\alpha\beta)\alpha = 0$, $(\alpha\beta)\beta = 0$; i.e., that $\alpha\beta$ is a second-order with null invariant that belongs to the forms α, β. If $\alpha\beta \neq 0$ then posit $\alpha\beta$ will be the projective line that is the intersection of the planes α and β. One deduces from this that two projective planes will always have at least one projective line in common. If $\alpha \neq 0$, with $\beta \neq 0$ then one will have $\alpha\beta = 0$ if plane α = plane β, and conversely.

c. If we let A be a point and let $\mathbf{I}$, $\mathbf{J}$ be two vectors then:

$$A\mathbf{IJ}.\omega = A\omega.\mathbf{IJ} + \mathbf{I}\omega.\mathbf{J}A + \mathbf{J}\omega.A\mathbf{I} = \mathbf{IJ}.$$

$A\mathbf{IJ}.\omega$ is the bivector of the triangle. In a general manner, if σ is a third-order form then $\sigma\omega$ will be the bivector of σ, and one will have for any point O:

$$\sigma = O(\sigma\omega) + \alpha,$$

where α is a trivector that is a function of O and σ.

Examples. – If A, B, … are points, a, b, … are non-zero line segments, and α, β, … are non-zero triangles then one can say that:

1. The position of the form $A.\alpha\omega$ is the plane that is parallel to α and drawn through the point A.

2. The position of the form $A \mid (\alpha\omega)$ is the line that is perpendicular to the plane α and issues from the point A.

3. The position of the form $a \mid (\alpha\omega)$ is the plane that is perpendicular to the plane α and passes through the line a.

4. The condition for the parallelism of two planes α and β is $(\alpha\omega)(|\beta\omega) = 0$, which amounts to saying that $\alpha\omega$ is a multiple of $\beta\omega$; the condition of perpendicularity is $(\alpha\omega)(\beta\omega) = 0$. The condition for the parallelism of the line a with the plane α is $(a\omega)(\alpha\omega) = 0$, and the perpendicularity of the line and the plane is expressed by $(a\omega) \mid (\alpha\omega) = 0$.

5. If $a\omega.b\omega \neq 0$ then the vector $\mid (a\omega.b\omega)$ will be perpendicular to the lines a and b. Consequently, the line:

(1) $$[a \mid (a\omega.b\omega)]\ [b \mid (a\omega.b\omega)]$$

will be the points that are common to the line (1) and the lines a and b.

If A, B are the positions of the forms (2) then one will have:

$$a = A.a\omega, \qquad b = B.b\omega$$

and

$$ab = A.a\omega.B.b\omega = A(A - B).a\omega.b\omega.$$

However, the vector $A - B$ is perpendicular to the bivector $b\omega.b\omega$. As a result:

$$\text{mod}(ab) = \tfrac{1}{2}\,\text{mod}(AB)\ \text{mod}(a\omega.b\omega),$$

or

(3) $$\text{mod}(AB) = \frac{6\,\text{mod}(ab)}{\text{mod}(a\omega.b\omega)},$$

which gives us the shortest distance between the points of the line a and those of the line b when these two lines are not parallel.

27. General properties of products. – We shall group the general properties of the progressive and regressive products here. Therefore, let r and s be two positive whole numbers that are less than 4, while A_r, A'_r are forms of order r, and A_s, A'_s are forms of order s. One has:

1. If $r + s \leq 4$ then the product of A_r with A_s will be progressive, and $A_r A_s$ will be a form order $r + s$.
2. If $r + s > 4$ then the product of A_r with A_s will be regressive, and $A_r A_s$ will be a form order $r + s - 4$.
3. If $A_r = A'_r$ and $A_s = A'_s$ then $A_r A_s = A'_r A'_s$.
4. $A_r A_s = (-1)^{rs} A_s A_r$.
5. $A_r (A_s + A'_s) = A_r A_s + A_r A'_s$.
6. If x is a number then $x(A_r A_s) = (xA_r) A_s = A_r (xA_s)$.

28. Duality. – Let r, s, t be positive whole numbers that are smaller than 4, and in addition, let A_r, A_s, A_t be forms of order r, s, t, respectively. One obtains the product $A_rA_sA_t$ by two multiplications. If these two multiplications are progressive then $A_r A_s \,.\, A_t$ will be a form of order $r + s + t$. If one of these multiplications is progressive and the other one is regressive then $A_r A_s \,.\, A_t$ will be a form of order $r + s + t - 4$. Finally, if the two multiplications are both regressive then $A_r A_s \,.\, A_t$ will be a form of order $r + s + t - 8$.

If A, B, …, a, b, …, α, β, … are forms of first, second, and third order, respectively, then one can easily prove the following formulas for the products of three factors:

(1)	$AB.C = A.BC,$	(1)′	$\alpha\beta.\gamma = \alpha.\beta\gamma,$
(2)	$AB.a = A.Ba,$	(2)′	$\alpha\beta.a = \alpha.\beta a,$
(3)	$AB.\alpha = A\alpha.B - B\alpha.A,$	(3)′	$\alpha\beta.A = \alpha A.\beta - \beta A.\alpha,$
(4)	$ab.C = aA.b + bA.a,$	(4)′	$ab.\alpha = a\alpha.b - b\alpha.a,$

$$Aa.\alpha = A\alpha.a + a\alpha.A. \tag{5}$$

One deduces formulas (1)′-(4)′ from formulas (1)-(4) by changing the form of the first and third order into forms of third and first order, respectively. When an analogous permutation is performed on formula (5), that will give that same formula, but solved with respect to the second term on the right-hand side.

The formulas that we just wrote down express the *principle of duality* for geometric forms. From these formulas, equations (3), (4), (3)′, (4)′, (5) relate to projective geometric elements of space, since a property that concerns the projections and intersections ([11]) will provide another one by changing points into projective planes and planes into projective points.

Formulas (3), (4), (3)′, (4)′, (5), and the ones that one can deduce from them by solving with respect to a term in the right-hand side permit one to state the following two general rules:

1. If the sum $r + s + t$ is equal to 5 or 7 then:

$$A_r\,A_s\,.\,A_t = (-1)^{r+st}\,A_r\,A_t\,.\,A_s - (-1)^{s+1}\,A_s\,A_t\,.\,A_r\,.$$

2. If $r + s + t = 6$, without one having $r = s = t$, however, then:

$$A_r\,A_s\,.\,A_t = A_r\,A_t\,.\,A_s - (-1)^t\,A_s\,A_t\,.\,A_r\,.$$

29. Regressive products in a projective plane. – We shall now consider forms of the first, second, and third order whose positions are in the same given projective plane. Every third-order form can be identified with a number.

We set:

(1) $$AB.PQ = APQ.B - BPQ.A,$$

and call $AB.PQ$ the *regressive product* – or simply, the *product* – of AB with PQ, upon supposing that A, B, P, and Q are first-order forms. Upon comparing the identity in theorem II (no. 24) with that definition, one can write:

(2) $$AB.PQ = ABQ.P - ABP.Q.$$

a. If one has $A'B' = AB$ and $P'Q' = PQ$ then it will result that:

$$AB.P'Q' = AB.PQ, \qquad A'B'.PQ = AB.PQ.$$

Therefore, the product $AB.PQ$ is a function of the forms AB, PQ, and if one sets $a = AB$, $b = PQ$ then the regressive product of a and b will be well-defined. Moreover, these products have all the properties of algebraic products, except for the commutative property; namely:

$$ab = -\,ba, \qquad a(b + c) = ab + ac$$

and

$$x(ab) = (xa)b = a(xb),$$

if x is a number.

([11]) In the case where the product AB is progressive, posit AB will represent the projective element (viz., line or plane) that is determined by the condition that it must contain posit A and posit B; i.e., that posit AB is the projective element that *projects* posit A, posit B being the center of projection. On the contrary, if the product AB is regressive then posit AB will be the projective element (point or line) that is determined by the condition that it be continuous in posit A and posit B. In a word, the *progressive* product will represent the *projecting* element and the *regressive* product will represent the *projected* element.

b. From formulas (1) and (2), one deduces that $(ab)\,a = 0$, $(ab)\,b = 0$; i.e., that ab is a first-order form that belongs to the forms a and b, while the point ab is the projective point of intersection of the lines a and b in the case where $ab \neq 0$. One further deduces that two coplanar projective lines will always have at least one point in common. When one has $a \neq 0$ with $b \neq 0$, ab will be zero when line a = line b, and conversely.

c. If u¸ v are bivectors then the progressive product of u and v will always be zero. However, if one considers u, v to be second-order forms that have their positions in the plane at infinity then the regressive product of u with v will be a vector, and posit uv will be a point at infinity that is common to the positions of the bivectors u, v ([12]), or a line at infinity in the case where $uv \neq 0$.

30. Let r, s be positive whole numbers that are less than 3, and let A_r, A'_r, A_s, A'_s be forms of order r, s, respectively. The following general properties result for the progressive and regressive products in the plane:

1. If $r + s \leq 3$ then the product of A_r with A_s will be progressive, and $A_r A_s$ will be a form of order $r + s$.

2. If $r + s > 3$ (or $r + s = 4$) then the product of A_r with A_s will be regressive, and $A_r A_s$ will be a form of order $r + s - 3$.

3. If $A_r = A'_r$, with $A_s = A'_s$, then one will have $A_r A_s = A'_r A'_s$.

4. $A_r A_s = (-1)^{r+s-1} A_s A_r$.

5. $A_r (A_s + A'_s) = A_r A_s + A_r A'_s$.

6. If x is a number then $x(A_r A_s) = (xA_r)\,A_s = A_r\,(xA_s)$.

For the products of three factors in the plane, one can also write:

(1)	$AB.C = A.BC,$	(1)′	$ab.c = a.bc,$
(2)	$AB.a = Aa.B - Ba.A,$	(2)′	$ab.A = aA.b - ba,$

which gives the principle of duality for geometric forms and projective elements in the plane.

([12]) The condition of parallelism of two planes α and β that was pointed out in example 4 of no. 26 still remains $\alpha\omega.\beta\omega = 0$ when $\alpha\omega.\beta\omega$ is a regressive product on the plane at infinity (*see b*).

§ 5. – COORDINATES.

31. THEOREM I. – *If one is given first-order forms A_1, A_2, A_3, A_4 such that $A_1A_2 A_3A_4 \neq 0$ and arbitrary forms S, s, σ that are of first, second, and third order, respectively, then the numbers $x_1, \ldots, x_4, y_1, \ldots, y_6, z_1, \ldots, z_4$ such that:*

(1) $S_1 = x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4$,

(2) $s = y_1 A_1 A_2 + y_2 A_1 A_3 + y_3 A_1 A_4 + y_4 A_2 A_3 + y_5 A_3 A_4 + y_6 A_4 A_2$,

(3) $\sigma = z_1 A_2 A_3 A_4 + z_2 A_3 A_4 A_1 + z_3 A_4 A_1 A_2 + z_4 A_1 A_2 A_3$

will be well-defined.

Proof. – The identity in theorem I of no. 24 gives:

$$A_1 A_2 A_3 A_4 \,.\, S + A_2 A_3 A_4 S \,.\, A_1 - \ldots + S A_1 A_2 A_3 \,.\, A_4 = 0.$$

In order to get formula (1), it suffices to divide by $A_1 A_2 A_3 A_4$, with:

$$x_1 = -\frac{A_2 A_3 A_4 S}{A_1 A_2 A_3 A_4}, \qquad x_2 = -\frac{A_2 A_4 S\, A_1}{A_1 A_2 A_3 A_4}, \qquad x_3 = \ldots$$

Now, if x is the product of the two first-order forms S, S' then:

$$s = SS' = (x_1 A_1 + \ldots)(x_1' A_1 + \ldots),$$

which gives a formula that is analogous to formula (2) upon developing the product in the right-hand side. In general, s, when considered to be a sum of two products of first-order forms, is reducible to the form (2). Similarly, σ, when considered to be the product of three first-order forms, is reducible to the form (3). At present, it suffices to prove that the numbers y, z are determined in a unique manner. To that effect, consider:

$$s = y_1' A_1 A_2 + y_2' A_1 A_3 + \ldots$$

Thus:

$$(y_1 - y_1') A_1 A_2 + (y_2 - y')_2 A_1 A_3 + \ldots = 0,$$

and if one multiplies by A_3A_4, A_2A_4, …, successively then one will get:

$$y_1 = y_1', \qquad y_2 = y_2', \qquad \ldots$$

The numbers x, y, z are called the *coordinates of the forms S, s, σ*, respectively, for the reference elements A_1, A_2, A_3, A_4 .

THEOREM II. – *If A_1, A_2, A_3 are first-order forms such that $A_1 A_2 A_3 \neq 0$, and S, s are arbitrary forms of the first and second order, respectively, that are coplanar with A_1, A_2, A_3 then the numbers x_1, x_2, x_3, y_1, y_2, y_3 such that:*

$$S = x_1 A_1 + x_2 A_2 + x_3 A_3 ,$$
$$\sigma = y_1 A_2 A_3 + y_2 A_3 A_1 + y_3 A_1 A_2$$

will be well-defined ([13]).

Similarly:

THEOREM III. – *If A_1, A_2 are first-order forms with $A_1 A_2 \neq 0$ then for any first-order form S that is collinear with A_1 and A_2, the numbers x_1, x_2 such that:*

$$S = x_1 A_1 + x_2 A_2$$

will be well-defined.

a. If:

$$S = x_1 A_1 + \ldots + x_4 A_4 , \qquad S' = x_1' A_1 + \ldots + x_4' A_4$$

then one will have:

$$S + S' = (x_1 + x_1') A_1 + \ldots + (x_4 + x_4') A_4 ,$$

and for a number m:

$$mS = (mx_1) A_1 + \ldots + (mx_4) A_4 ,$$

which are formulas that determine the coordinates of the forms $S + S'$ and mS.

One will get analogous results for the forms $s + s'$, ms', $\sigma + \sigma'$, $m\sigma$. One can, moreover, easily obtain the coordinates of the progressive and regressive product of two forms.

b. If:

$$S_1 = x_{11} A_1 + x_{12} A_2 + x_{13} A_3 + x_{14} A_4 ,$$
$$S_2 = x_{21} A_1 + x_{22} A_2 + x_{23} A_3 + x_{24} A_4 ,$$
$$S_3 = x_{31} A_1 + x_{32} A_2 + x_{33} A_3 + x_{34} A_4 ,$$
$$S_4 = x_{41} A_1 + x_{42} A_2 + x_{43} A_3 + x_{44} A_4 ,$$

([13]) One calls a set of elements a *linear system* when the sum of elements, as well as the product by a number, is defined, and these operations will have the ordinary (i.e., the ones in algebra) properties. We say that the elements $u_1, u_2, \ldots, u_n$ of a linear system are *independent* when it is impossible to determine numbers $m_1, m_2, \ldots, m_n$ that are not all zero and are such that:

$$m_1 u_1 + m_2 u_2 + \ldots + m_n u_n = 0.$$

A linear system is called *n-dimensional* when there exist n elements $u_1, u_2, \ldots, u_n$ of the system that are independent, and $n + 1$ elements of the system would always be dependent. If u is an arbitrary element of the system then the numbers $m_1, m_2, \ldots, m_n$ such that $u = m_1 u_1 + m_2 u_2 + \ldots + m_n u_n$ will be well-defined.

The preceding theorems thus express the ideas that *first (and third) order forms in space are elements of a four-dimensional linear system,* and that *the second-order forms in space are elements of a six-dimensional linear system.* Similarly, *the forms of first (and second) order on a projective plane are elements of a three-dimensional linear system,* etc.

For the other linear systems of geometric forms, *see* C. BURALI-FORTI, "Il metodo del Grassmann nella Geometria proiettiva," *loc. cit.*

where $A_1, \ldots, A_4$ are first-order or third-order forms and the x are numbers, then one will have:

$$(1) \qquad S_1\, S_2\, S_3\, S_4 = \begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{vmatrix} A_1\, A_2\, A_3\, A_4 .$$

Indeed, one term in the product $S_1\, S_2\, S_3\, S_4$ is:

$$x_{11}\, x_{22}\, x_{33}\, x_{44}\, A_1\, A_2\, A_3\, A_4,$$

and one can obtain all of the other terms by permuting the second indices of the $x_{11}\, x_{22}\, x_{33}\, x_{44}$, and giving the term thus obtained a + sign or a – sign according to whether the number of inversions of consecutive indices is or is not an even number, respectively. This law is nothing but the law of formation for the terms in the determinant in formula (1) ([14]).

Here is an example: Let A, B, C be the summits of a triangle, A', B', C', the points of intersection with the opposite edges of the interior bisectors of the angles A, B, C. Set a = mod BC, b = mod CA, c = mod AB. Under these conditions, the point A' will be the position of the regressive product:

$$BC\left(\frac{CA}{b} - \frac{AB}{c}\right) = \frac{ABC}{b}C + \frac{ABC}{c}B.$$

The mass of that form will be:

$$\frac{ABC}{b} + \frac{ABC}{c} = ABC\,\frac{b+c}{bc},$$

and consequently ([15]):

$$A' = \frac{b}{b+c}B + \frac{c}{b+c}C .$$

Similarly, one will have:

$$B' = \frac{c}{c+a}C + \frac{a}{c+a}A,$$

$$C' = \frac{a}{a+b}A + \frac{b}{a+b}B,$$

and consequently:

([14]) For the theory of determinants, as deduced from Grassmann's operations, *see*, E. CARVALLO, "Théorie des détérminants," Nouv. Ann. (1893).

([15]) One deduces from this formula that the interior bisector of the angle A will decompose the opposite side into parts that proportional to the edges AB, AC. Conversely, that property will give the formula:

$$A' = \frac{b}{b+c}B + \frac{c}{b+c}C .$$

$$A'B'C' = \begin{vmatrix} 0 & \dfrac{b}{b+c} & \dfrac{c}{b+c} \\ \dfrac{a}{c+a} & 0 & \dfrac{c}{c+a} \\ \dfrac{a}{a+b} & \dfrac{b}{a+b} & 0 \end{vmatrix} ABC = \frac{2abc}{(b+c)(c+a)(a+b)} ABC,$$

which provides the area of the triangle $A'B'C'$ as a function of the numbers a, b, c, and the area of the triangle ABC.

32. Now, take the reference elements to be a point O and three non-coplanar unit vectors **I**, **J**, **K**. If S is a first-order form then one will have:

$$S = mO + x\mathbf{I} + y\mathbf{J} + z\mathbf{K},$$

and since $S\omega = m$, one will see that m is the mass of the form S. For an arbitrary point P, one will thus have:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K},$$

and the numbers x, y, z will be the Cartesian coordinates of the point P, if one takes the point O to be the origin and the axes to be the lines $O\mathbf{I}$, $O\mathbf{J}$, $O\mathbf{K}$. Likewise, if **U** is a vector then, as we have already seen, one will have:

$$\mathbf{U} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}.$$

In what follows, we will agree that the vectors **I**, **J**, **K** are pair-wise perpendicular and that the trivector **IJK** is positive, which amounts to supposing that the vectors **I**, **J**, **K** satisfy the conditions:

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2,$$
$$\mathbf{J} \mid \mathbf{K} = 0, \qquad \mathbf{K} \mid \mathbf{I} = 0, \qquad \mathbf{I} \mid \mathbf{J} = 0,$$
$$\mathbf{I} = \mid \mathbf{JK}, \qquad \mathbf{J} = \mid \mathbf{KI}, \qquad \mathbf{K} = \mid \mathbf{IJ}.$$

The properties that we shall state prove how Cartesian analytic geometry can be deduced quite easily from the general theory of forms ([16]).

a. If $\mathbf{U} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ then one will have $\mathbf{U}^2 = x^2 + y^2 + z^2$, and the modulus of the vector **U** will be the number $\sqrt{x^2 + y^2 + z^2}$.

b. If $\mathbf{U} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ then one will have $\mathbf{U} \mid \mathbf{I} = x$, and in turn:

([16]) For the other coordinate systems, one can consult: C. BURALI-FORTI, "Il metodo del Grassmann nella Geometria proiettiva," (*loc. cit.*). However, it is good to observe that in the development of Grassmann's method the chosen coordinates have no importance, and that the theorems of no. 31 or no. 34 are all that is necessary.

$$\cos(\mathbf{U}, \mathbf{I}) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \cos(\mathbf{U}, \mathbf{J}) = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad \ldots$$

In a word, the coordinates of **U** will be proportional to the cosines of the angle that **U** makes with **I**, **J**, **K**, and:

$$\cos^2(\mathbf{U}, \mathbf{I}) + \cos^2(\mathbf{U}, \mathbf{J}) + \cos^2(\mathbf{U}, \mathbf{K}) = 1.$$

c. If $\mathbf{U} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $\mathbf{U}' = x'\mathbf{I} + y'\mathbf{J} + z'\mathbf{K}$ then the condition that **U** be parallel to **U′** will be:

$$\mathbf{U}\mathbf{U}' = 0,$$

or:

$$\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} \mathbf{JK} + \begin{vmatrix} z & x \\ z' & x' \end{vmatrix} \mathbf{KI} + \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} \mathbf{IJ} = 0,$$

or even:

$$\begin{vmatrix} y & z \\ y' & z' \end{vmatrix} = 0, \quad \begin{vmatrix} z & x \\ z' & x' \end{vmatrix} = 0, \quad \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = 0,$$

which we agree to write in the form:

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}.$$

The orthogonality condition will be:

$$\mathbf{U} \mid \mathbf{U}' = 0,$$

which will now be written:

$$xx' + yy' + zz' = 0.$$

d. Let u be a bivector. One then has:

$$u = x\mathbf{IJ} + y\mathbf{KI} + z\mathbf{IJ} \quad \text{and} \quad \mid u = x\mathbf{I} + y\mathbf{J} + z\mathbf{K},$$

which reduces the properties of the coordinates of a bivector to those of a vector.

e. If:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K} \quad \text{and} \quad P' = O + x'\mathbf{I} + y'\mathbf{J} + z'\mathbf{K}$$

then one will have:

$$\begin{aligned} \text{mod } PP' &= \text{mod } (P - P'), \\ &= \text{mod } [(x - x')\,\mathbf{I} + (y - y')\,\mathbf{J} + (z - z')\,\mathbf{K}], \\ &= \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \end{aligned}$$

an expression that provides the distance between P and P'.

f. The points:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K} \qquad \text{and} \qquad P' = O + x'\mathbf{I} + y'\mathbf{J} + z'\mathbf{K}$$

are situated on the line t that is parallel to the vector $\mathbf{P} = p\mathbf{I} + q\mathbf{J} + r\mathbf{K}$ when the vectors $\mathbf{U}$, $P - P'$ are parallel; i.e., when one has:

$$\frac{x-x'}{p} = \frac{y-y'}{q} = \frac{z-z'}{r},$$

which is nothing but the equation of the line t.

g. If α is a triangle then one will have:

$$\alpha = a\, O\mathbf{JK} + b\, O\mathbf{KI} + c\, O\mathbf{IJ} - d\, \mathbf{IJK},$$

and the point $P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ **will** belong to the plane α when:

$$ax + by + cz + d = 0,$$

which is precisely the equation of the plane α.

With $\alpha\omega = a\,\mathbf{JK} + b\,\mathbf{KI} + c\,\mathbf{IJ}$ and $|\,(\alpha\omega) = a\,\mathbf{I} + b\,\mathbf{J} + c\,\mathbf{K}$, one sees that the numbers a, b, c will be proportional to the cosines of the angles that the normal to the plane α makes with the axes.

Since:

$$\text{mod}\ \alpha = \frac{1}{2}\,\text{mod}\,(\alpha\omega) = \frac{1}{2}\ \text{mod}\,|\,(\alpha\omega) = \frac{1}{2}\sqrt{a^2+b^2+c^2}\,,$$

the number:

$$\frac{3P\alpha}{\text{mod}\,\alpha} = -\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}$$

will be the distance – with a sign – from the point P to the plane a.

h. If a is a second-order form then one will have:

$$a = p\,O\mathbf{I} + q\,O\mathbf{J} + r\,O\mathbf{K} + p'\mathbf{JK} + q'\,\mathbf{KI} + r'\,\mathbf{IJ},$$

and its invariant will be zero since:

(1) $$pp' + qq' + rr' = 0.$$

The numbers p, q, r, p', q', r', which are coupled by the relation (1), are then the coordinates of the line a, which are coordinates to which one can easily give a geometric significance here, if one observes that:

$$\alpha\omega = p\,\mathbf{I} + q\,\mathbf{J} + r\,\mathbf{K}, \qquad |\,(Oa)\ \omega = p'\mathbf{I} + q'\,\mathbf{J} + r'\,\mathbf{K}.$$

If a, a are two second-order forms such that:

$$aa = 0, \qquad a_1\, a_1 = 0, \qquad a\omega\,.\, a\omega \neq 0$$

then (no. 26, example 5) the shortest distance between points of the line a and those of the line a_1 will be given by the formula:

$$\frac{\operatorname{mod}(pp_1' + qq_1' + rr_1' + p_1 p' + q_1 q' + r_1 r)}{\sqrt{\begin{vmatrix} q & r \\ q_1 & r_1 \end{vmatrix}^2 + \begin{vmatrix} r & p \\ r_1 & p_1 \end{vmatrix}^2 + \begin{vmatrix} p & q \\ p_1 & q_1 \end{vmatrix}^2}}\ .$$

CHAPTER II

VARIABLE FORMS

§ 1. – DERIVATIVES.

33. Definitions. – As in analysis, we will let $f(t)$ denote a geometric form f that is a function of a numerical variable t, and we will always suppose, without having to repeat ourselves in each case, that $f(t)$ is a well-defined function in the interval considered.

Therefore, let $f(t)$ be a first-order form, a second form with zero invariant, or a third-order form. With the restriction that the form $f(t)$ not be annulled for any value of t in the interval of variation for t, posit $f(t)$ will be a projective point, line, or plane, respectively, that will be a function of t.

When there is no possible confusion, we will write simply f, instead of $f(t)$, and posit f, instead of posit $f(t)$.

34. Limit of a form. – Let $f(t)$ be a geometric form, and let t_0 be an arbitrary number that is finite or infinite. Consider a fixed form f_0 of the same order as f. When t tends to the value t_0, we will say that f_0 is the limit of $f(t)$, and write either:

$$\lim_{t=t_0} f(t) = f_0 \qquad \text{or} \qquad \lim_{t=t_0} f = f_0 \qquad \text{or} \qquad \lim f = f_0 .$$

One will always have for any points P, Q, R:

$$\lim_{t=t_0} f(t)PQR = f_0\, PQR,$$

or

$$\lim_{t=t_0} f(t)PQ = f_0\, PQ,$$

or

$$\lim_{t=t_0} f(t)P = f_0\, P,$$

according to whether $f(t)$ is a form of first, second, or third order, respectively. It will be implicit that the variation of t must take place in an interval in which the function f is constantly *defined*, and the knowledge of the limit of a form is thus reduced to that of a variable number, which leads us to suppose that the theory of limits of numerical functions is known.

Example. – In a given plane, consider a fixed point O, a vector $\mathbf{I}$, and the points A_1, A_2, A_3, …, whose sequence is determined by the following law:

$$\begin{aligned} A_1 &= O + \mathbf{I}, \\ A_2 &= A_1 + i(A_1 - O), \\ A_3 &= A_2 + \tfrac{1}{2} i(A_2 - A_1), \\ A_4 &= A_3 + \tfrac{1}{3} i(A_3 - A_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

One easily sees that:

$$\begin{aligned} A_1 &= O + \mathbf{I}, \\ A_2 &= A_1 + i\,\mathbf{I}, \\ A_3 &= A_2 - \frac{1}{2!}\mathbf{I}, \\ A_4 &= A_3 - \frac{1}{3!} i\,\mathbf{I}, \\ A_5 &= A_4 + \frac{1}{4!}\mathbf{I}, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

so:

$$A_{2n} = O + \left(1 - \frac{1}{2!} + \frac{1}{4!} - \cdots \pm \frac{1}{(2n-2)!}\right)\mathbf{I} + \left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots \mp \frac{1}{(2n-1)!}\right) i\,\mathbf{I}.$$

Knowing the series developments of sin x and cos x then permits us to write:

$$\lim_{n=\infty} A_{2n} = O + (\cos 1 + i \sin 1)\,\mathbf{I} = O + e^i\,\mathbf{I},$$

which proves that the variable point A_{2n} or A_n (which is a function of the whole number n) has a certain point A for its limit position when n increases indefinitely, whose distance from the point O is mod $\mathbf{I}$, such that the vector $\mathbf{I}$ makes an angle of one radian with the vector $A - O$. This sequence A_1, A_2, … thus permits one to construct the angle of one radian by approximations.

35. Now, suppose that $A(t)$, $B(t)$ that have well-defined limits for $t = t_0$ ([17]).

([17]) If we accept the modern notion of limit (G. PEANO, Rivista di Matematica, vol. II) then $\lim_{t=t_0} f(t)$ will be a set of numbers [when $f(t)$ is a numerical function]. If that set contains just one element then we will say that $\lim_{t=t_0} f(t)$ is well-defined, which agrees with the usual meaning of limit. [For the limit of a variable set, *see*, C. BURALI-FORTI, "Sul limite di una classe variabile d'ensembles," Atti Acc. Sc. Torino, vol. XXX; "Sur quelques propriétés des ensemble d'ensembles," Math. Ann., B. 47.]

The condition that "$A(t)$, $B(t)$ have well-defined limits" is necessary. For example, proposition d expresses the idea that the limit of a product is the product of the limits. Therefore, if one sets $A = t\,\mathbf{I}$, B

a. If A, B are forms of the same order then we will have:

$$\lim(A + B) = \lim A + \lim B \quad [\text{because } (A + B)\, PQR = APQR + BPQR].$$

b. If x is a number that is itself a function of the numerical variable t, and $\lim\limits_{t=t_0} x$ is well-defined then:

$$\lim(xA) = (\lim x)(\lim A).$$

With the restriction that the number x not be annulled for any value of t that is considered, even at the limit, one will have:

$$\lim \frac{A}{x} = \frac{\lim A}{\lim x}.$$

c. The coordinates of $\lim A$ are the limits of the coordinates of A. Indeed, if, for example, $A = xA_1 + yA_2$, where x and y are numbers that are functions of t and A_1, A_2 are constant forms, then one will have:

$$\lim A = \lim(xA_1) + \lim(yA_2) = (\lim x)\, A_1 + (\lim y)\, A_2 .$$

d. $\lim(AB) = (\lim A)(\lim B)$.

e. If $\mathbf{A}$ is a vector (bivector, trivector, resp.) then $\lim \mathbf{A}$ will be a vector (bivector, trivector, resp.), because $\mathbf{A}\omega = 0$ and $\lim(\mathbf{A}\omega) = (\lim \mathbf{A})\,\omega = 0$, which proves the theorem.

f. If $\mathbf{A}$ is a vector in a given plane then:

$$\lim(i\, \mathbf{A}) = i\, (\lim \mathbf{A}).$$

Similarly, if $\mathbf{A}$ is a vector or bivector then one will have:

$$\lim(|\, \mathbf{A}) = |\, (\lim \mathbf{A}).$$

g. If $\operatorname{mod} \mathbf{A}$ is defined then one will have:

$$\lim(\operatorname{mod} \mathbf{A}) = \operatorname{mod}(\lim \mathbf{A}).$$

36. Limit of a projective element. – Let $f(t)$ be a geometric form such that in the interval of variation for t the projective element posit $f(t)$ is well-defined and for $t = t_0$ the function $f(t)$ has a well-defined, non-zero limit. We set:

$= \left(\frac{1}{t} + \sin\frac{1}{t}\right) \mathbf{J}$, where $\mathbf{I}$, $\mathbf{J}$ are, for example, vectors, then $AB = \left(1 + t\sin\frac{1}{t}\right) \mathbf{IJ}$. One then sees that $\lim\limits_{t=0} AB = \mathbf{IJ}$, while $\lim\limits_{t=0} A = 0$, and $\lim\limits_{t=0} B$ has no meaning.

$$\lim_{t=t_0} [\text{posit}\, f(t)] = \text{posit}\left[\lim_{t=t_0} f(t)\right],$$

which amounts to saying that the limit of the position of f is the position of the limit of f.

Moreover, we assume that the forms $A(t)$, $a(t)$, $\alpha(t)$ of first, second, and third order, respectively, have the same properties as $f(t)$, and set:

$$\lim A = A_0, \qquad \lim a = a_0, \qquad \lim \alpha = \alpha_0 .$$

Under these hypotheses, one will have the following propositions:

a. If A and A_0 are not vectors then the limit of the distance between the point A and the point A_0 will be zero. Indeed, let d be that distance, so one will have:

$$d = \text{mod}\left(\frac{A}{A\omega} - \frac{A_0}{A_0\omega}\right).$$

Now:

$$\lim AA_0 = A_0\, A_0 = 0, \qquad \lim(A\omega \,.\, A\omega) = (A_0\,\omega)^2,$$

and consequently:

$$\lim d = 0.$$

b. If a point A of the line a has a well-defined limit ([18]) then that limit can only be a point on the line a_0, because it results from Aa = that:

$$\lim (Aa) = A_0\, a_0 = 0.$$

c. If the line a_0 is not entirely at infinity and A_1, which is at a finite distance and is situated on the line a, has a well-defined limit then the distance from the point A to the line a_0 will tend to zero, because if d is precisely that distance then one will have:

$$(\text{mod}\, a_0)^{1/2}\, d = \text{mod}(A\, a_0).$$

However:

$$\lim (A\, a_0) = (\lim A)\, a_0 = 0 \qquad \text{(prop. } b\text{)},$$

and, in turn:

$$\lim d = 0.$$

d. If a point or a line in the plane α has a well-defined limit then that limit will be a point or a line in the plane.

([18]) *See* the note on page 46 for the restrictive conditions that we must impose upon propositions *a*, *b*, …

e. If the plane α_0 is not the plane at infinity and A, which is at a finite distance in the plane α, has a well-defined limit then the distance from the point A to the plane α_0 will have zero for its limit.

37. Derivatives. – Let $f(t)$ be a geometric form. If the function:

$$\frac{f(t+h)-f(t)}{h}$$

has a well-defined limit for $h = 0$ then we set:

$$\frac{df(t)}{dt} = \lim_{h=0}\frac{f(t+h)-f(t)}{h},$$

in order to call $df(t) / dt$ (or df / dt) the *derivative* of the form $f(t)$, following the language of analysis.

We further denote the expression $df(t) / dt$ (or df / dt) by $\frac{d}{dt}f(t)$, $f'(t)$.

Upon writing:

$$df(t) = f'(t)\,dt,$$

instead of:

$$\frac{df(t)}{dt} = f'(t),$$

one can likewise say that $df(t)$ is the *differential* of $f(t)$. Therefore, the differential of $f(t)$ is the product of the derivative of $f(t)$ with the infinitesimal number dt.

The derivative of the derivative is further called the *second derivative*, the derivative of the second derivative will be called the *third derivative*, and so on. One sets:

$$\frac{d^2 f}{dt^2} = \frac{d^2}{dt^2}f = f'' = \frac{df'}{dt},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$\frac{d^n f}{dt^n} = \frac{d^n}{dt^n}f = f^{(n)} = \frac{df^{(n-1)}}{dt}.$$

Now, suppose that $A(t)$, $B(t)$ are geometric forms that have derivatives for any value of t that is being considered.

a ([19]). If A, B are forms of the same order then one will have:

$$d(A + B) = dA + dB \quad \text{or} \quad (A + B)' = A' + B'.$$

([19]) The rules of derivation that we give in propositions *a*-*d* are analogous to the ones in differential calculus.

b. If x is real number that is a function of t then one will have:

$$d(xA) = (dx)\,A + x(dA),$$

and for a constant number m:

$$d(mA) = m(dA).$$

c. $d(AB) = (dA)\,B + A\,(dB)$, however it is not generally permissible to change the order of the factors. More generally, for a non-zero integer n one can write:

$$(AB)^{(n)} = A^{(n)}B + nA^{(n-1)}\,B' + \frac{n(n-1)}{2}A^{(n-2)}\,B'' + \ldots + \frac{n(n-1)}{2}A''B^{(n-2)} + nA'B^{(n-1)} + AB^{(n)}.$$

d. The derivative of a constant form is zero. Conversely, $A(t)$ is constant in an interval when its derivative is zero for any value of t that is taken from the interval in question.

e. The derivative of a point is a vector, because if A is a point then one will have $A\omega = 1$, so $A'\omega = 0$; i.e., A' will be a vector.

f. The derivative of a vector is a vector, and consequently, the derivative of a bivector or a trivector is a bivector or trivector, resp. (*see* prop. *c*).

Let $\mathbf{A}$ be a vector in a given plane, and let φ be a real number that is a function of t, so:

$$d(e^{i\varphi}\,\mathbf{A}) = e^{i\varphi}\,d\mathbf{A} + i\,e^{i\varphi}\,\mathbf{A}\,d\varphi,$$

and in particular:

$$d(i\,\mathbf{A}) = i\,(d\mathbf{A}).$$

In the case where $\mathbf{A}$ is a constant vector, one will have:

$$d(e^{i\varphi}\,\mathbf{A}) = i\,e^{i\varphi}\,\mathbf{A}\,d\varphi,$$

and $d(e^{i\varphi}\,\mathbf{A})$ will be a vector that is perpendicular to the vector $e^{i\varphi}\,\mathbf{A}$.

Similarly, if $\mathbf{A}$ is a vector or bivector then one will have:

$$d(|\,\mathbf{A}) = |\,(d\mathbf{A}).$$

g. If $\mathbf{A}$ is a vector with constant non-zero modulus then the derivative of $\mathbf{A}$ will be a vector that is zero or perpendicular to the vector $\mathbf{A}$, since:

$$\mathbf{A}\,|\,\mathbf{A} = \text{const.},$$

so

$$\mathbf{A}\,|\,\mathbf{A}' + \mathbf{A}'\,|\,\mathbf{A} = 2\mathbf{A}\,|\,\mathbf{A}' = 0.$$

h. If $\mathbf{A}$ is a vector that is nowhere annulled in the interval of variation of t then the necessary and sufficient condition for the direction of $\mathbf{A}$ to be constant will be precisely

$\mathbf{A}\,(d\mathbf{A}) = 0$ or, more simply, $\mathbf{AA}' = 0$ for all values of t. Indeed, if one sets $\mathbf{B} = \mathbf{A}$ / mod $\mathbf{A}$ then one will likewise have:

$$\mathbf{A} = (\text{mod}\ \mathbf{A})\ \mathbf{B},$$

so

$$\mathbf{A}' = (\text{mod}\ \mathbf{A})'\ \mathbf{B} + (\text{mod}\ \mathbf{A})\ \mathbf{B}'$$

and

$$\mathbf{AA}' = (\text{mod}\ \mathbf{A})^2\ \mathbf{BB}'.$$

Now, mod $\mathbf{A} \neq 0$, by hypothesis, and the necessary and sufficient condition for $\mathbf{AA}' = 0$ is indeed $\mathbf{BB}' = 0$. However, since $\mathbf{B}$ is a unit vector, the condition $\mathbf{BB}' = 0$ (prop. *g*) is equivalent to $\mathbf{B}' = 0$ or $\mathbf{B}$ = const.; i.e., to posit $\mathbf{A}$ = const.

Under the same hypotheses as before, the necessary and sufficient condition for the vector $\mathbf{A}$ to be parallel to a fixed plane will be $\mathbf{AA}'\mathbf{A}'' = 0$ for all values of t.

If $\mathbf{A}$ is a non-zero bivector then $\mathbf{AA}' = 0$ (regressive product) will be the necessary and sufficient condition for $\mathbf{A}$ to be parallel to a fixed plane, while $\mathbf{AA}'\mathbf{A}'' = 0$ will be the necessary and sufficient condition for $\mathbf{A}$ to be parallel to a fixed line.

k. If $\mathbf{A}$ is a non-zero vector then:

$$(\text{mod}\ \mathbf{A})' = \frac{\mathbf{A}}{\text{mod}\,\mathbf{A}}\Big|\,\mathbf{A}'.$$

Indeed:

$$(\text{mod}\ \mathbf{A})^2 = \mathbf{A} \mid \mathbf{A},$$

and in turn:

$$(\text{mod}\ \mathbf{A})(\text{mod}\ \mathbf{A})' = \mathbf{A} \mid \mathbf{A}',$$

and on a plane:

$$(\text{mod}\ \mathbf{A})' = \frac{\mathbf{A}}{\text{mod}\,\mathbf{A}}\,i\mathbf{A}',$$

which proves the stated proposition, since mod $\mathbf{A} \neq 0$.

Examples. –

1. Let $A(t)$ be a point such that mod $A \neq 0$ for all values of t. We will prove (§ 2) that the line AA' is precisely the tangent at the point A to the curve that is described by that point.

Thus, if a point P in a given plane describes a circle whose center is at O then $(P - O)$ $i\ (P - O)$ = const. On the other hand, one can consider P to be a function of one numerical variable t, in such a way that:

$$(P - O)\ i\ P' + P'\ i\ (P - O) = 2(P - O)\ i\ P' = 0,$$

an expression that proves that the tangent is normal to the radius at the contact point. One also finds the same result by setting $P = O + r\,e^{i\varphi}\,\mathbf{I}$.

2. The point $P = O + r\varphi\,\mathbf{I} + ri\,\mathbf{I} - r\,e^{i\varphi}\,i\,\mathbf{I}$ (*see* no. 17) describes a cycloid. We then have:

$$\frac{dP}{d\varphi} = P' = r\mathbf{I} - r\,e^{i\varphi}\,i\,\mathbf{I}$$

or

$$i\,P' = ri\,\mathbf{I} - r\,e^{i\varphi}\,i\,\mathbf{I} = M - P.$$

Therefore, the normal at a point P of the cycloid is the line that joins the point P to the contact point of the moving circle with the fixed line.

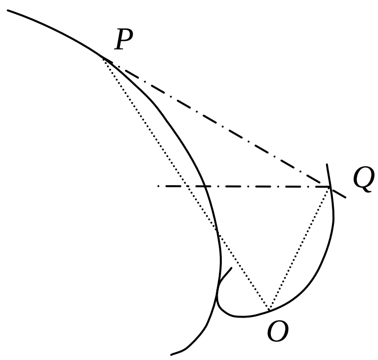

Figure 3.

3. Let O be a fixed point and let $P(t)$ be a point in a plane (*fig*. 3). Let $Q(t)$ represent the foot of the perpendicular that is drawn through O to the line PP'. The point Q will describe a *podaire* (?) of the locus of P. That being the case, one will have:

$$(Q - P)\,i(Q - O) = 0 \quad \text{or} \quad (Q' - P')\,i(Q - O) + (Q' - P')\,i\,Q' = 0.$$

However:

$$P'\,i(Q - O) = 0,$$

which proves that the normal at the point Q of the curve that is described by Q will pass the middle of the segment OP.

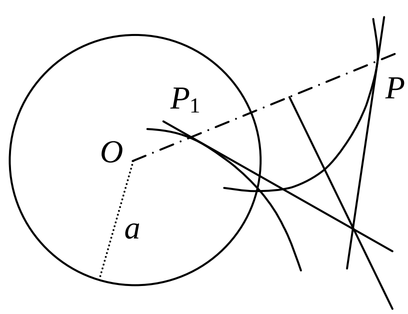

Figure 4.

4. With the same points, O fixed and $P(t)$ (*fig.* 4), consider the point $P_1(t) = O + \frac{a^2}{(P-O)^2}(P - O)$, into which a enters as a positive real number. P_1 will then generate the inverse curve to the one that was described by P under the transformation by reciprocal vector radii whose circle of inversion has O for its center and radius a. One has:

$$(P - O)(P_1 - O) = 0, \qquad (P - O)\, i\, (P_1 - O) = a^2,$$

or, upon differentiating:

$$P'(P_1 - O) = P_1'(P - O), \qquad P'\, i\, (P_1 - O) = -P_1'\, i\, (P - O),$$

and after dividing the two sides of these equations, one will get:

$$\tan(P', P - O) = -\tan(P_1', P - O),$$

which proves that the tangents to the points P and P_1 and the perpendicular to the line OP that is drawn through the middle of PP_1 will agree at the same point.

5. Let $P(t)$ and $Q(t)$ be two points in a plane such that for every value of t the line PQ is parallel to a fixed line. If m and n are two real numbers such that $m + n \neq 0$ then upon setting:

$$R = \frac{mP + nQ}{m + n}$$

one will have:

$$(m + n)^2\, RR' = m^2 PP' + n^2\, QQ' + mn\, PQ' + mn\, QP'.$$

Moreover, if the vector $P - Q$ has a constant direction then one can write:

$$(P - O)(P' - Q') = 0, \qquad \text{so} \qquad PP' + QQ' = PQ' + QP',$$

or finally:

$$RR' = \frac{mPP' + nQQ'}{m + n}.$$

Therefore: *The tangents at the points P, Q, R to their respective curves will pass through the same point.*

6. If $P(t)$ is a point, and the variable t that enters in is time then P' and P'' will be the velocity and acceleration, respectively, of the point P, and will represent their magnitude, direction, and sense. If m is the (constant) mass of the point P, which is assumed to be free, then mP'' will represent the magnitude, direction, and sense of the force, while the inner product $mP'' \mid dP$ will provide the elementary work that is done by the force. On the other hand, the inner product $P' \mid P' = (P')^2$ will be the square of the velocity, and $\frac{1}{2}m(P')^2$ will be the *vis viva*. Now, we have $d\left[\frac{1}{2}m(P')^2\right] = mP'' \mid dP$, and that formula

expresses the idea that *the increase in the vis viva will be equal to the elementary work that is done.*

In the case of a central force, the acceleration P'' will pass through a fixed point O; i.e., $OPP'' = 0$. Meanwhile, $(OPP')' = OPP''$, so $OPP' = \text{const.}$, which shows that under the action of a central force, motion will always be planar. If we set $\alpha = (P - O, P')$, $v = \text{mod}\, P'$, and if d is the distance from the point O to the line PP' then we will have:

$$d = [\text{mod}(P - O)] \sin \alpha, \qquad OPP' = \tfrac{1}{2}[v\, \text{mod}(P - O)] \sin \alpha,$$

so

$$vd = 2OPP' = \text{const.},$$

which proves the well-known property: *In the motion that results from the action of a central force, the product of the magnitude of the velocity with the distance from the center to the tangent to the variable point will be constant.*

7. Let F_1, F_2 be foci for an ellipse or a hyperbola, and let P be a point of the curve:

$$\text{mod}(P - F_1) \pm \text{mod}(P - F_2) = \text{const.}$$

One can consider P to be a function of a numerical variable t, and upon taking the derivative (prop. k), one will have:

$$\left[\frac{P - F_1}{\text{mod}(P - F_1)} \pm \frac{P - F_2}{\text{mod}(P - F_2)}\right] iP' = 0,$$

which expresses the idea that the tangent at P is one of the bisectors of the angles that are defined by the two lines PF_1, PF_2 .

For a Cassini oval, we will have:

$$\text{mod}(P - F_1)\, \text{mod}(P - F_2) = \text{const.},$$

so

$$\left[\text{mod}(P - F_2)\frac{P - F_1}{\text{mod}(P - F_1)} + \text{mod}(P - F_1)\frac{P - F_2}{\text{mod}(P - F_2)}\right] iP' = 0,$$

which provides a very simple construction of the normal at P.

The reader will have no difficulty in solving a host of questions by the method that we just presented.

38. Mean forms. – If u is a set of real numbers then we will say that the number x is the mean of the numbers in u when x is equal to the least upper bound of the u and the greatest lower bound of the u.

Let U be a set of first-order forms, and let α be a third-order form. We let the notation $U\alpha$ denote the set of numbers that are the product of each form in U by the form

α. We likewise agree to say that the first-order form X is the mean of the forms in the set U when for any form α the number $X\alpha$ is the mean of the numbers in the set $U\alpha$. In a completely similar way, we define the second and third-order forms that are the means of the second and third-order forms of a given set.

a. If $A_1, A_2, \ldots, A_n$ are forms of the same order, and $m_1, m_2, \ldots, m_n$ are positive numbers then the form:

$$\frac{m_1 A_1 + m_2 A_2 + \cdots + m_n A_n}{m_1 + m_2 + \cdots + m_n}$$

will be the mean of the forms $A_1, A_2, \ldots, A_n$. Indeed, suppose that $x_1, x_2, \ldots, x_n$ are numbers such that:

$$x_1 \leq x_2 \leq \ldots \leq x_n ,$$

so one will have:

$$(m_1 + m_2 + \ldots + m_n)\, x_n \geq m_1\, x_1 + m_2\, x_2 + \ldots + m_n\, x_n \geq (m_1 + m_2 + \ldots + m_n)\, x_1.$$

As a result, the number $\dfrac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}$ will indeed by the mean of the numbers $x_1, x_2, \ldots, x_n$.

b. If **I** and **J** are vectors, and **K** is a first-order form that is the mean of the vectors **I**, **J** then **K** will be a vector, and the vectors **I**, **J**, **K** will be coplanar (or **IJK** = 0). Indeed, we know that $\mathbf{K}\omega$ will be the mean of $\mathbf{I}\omega$ and $\mathbf{J}\omega$, and since $\mathbf{I}\omega = \mathbf{J}\omega = 0$, one will have $\mathbf{K}\omega = 0$; i.e., **K** will be a vector. Furthermore, if α is a triangle then the number $\mathbf{K}\alpha$ will be the mean of the numbers $\mathbf{I}\alpha$ and $\mathbf{J}\alpha$, and if $\mathbf{I}\alpha = \mathbf{J}\alpha = 0$ then either **I**, **J** will be parallel to the plane α or $\mathbf{K}\alpha = 0$, and **K** will be parallel to the plane α.

c. If $A_1, A_2, \ldots, A_n$ are points then a mean form for $A_1, A_2, \ldots, A_n$ will have a position that consists of a point that belongs to the smallest convex field that encloses the points $A_1, A_2, \ldots, A_n$.

39. Taylor's formula.

THEOREM I. – *If $f(t)$ is a geometric form that has derivatives up to order n for some value of t then one will have:*

$$f(t + h) = f(t) + hf'(t) + f''(t) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(t) + [f^{(n)}(t) + F], \tag{1}$$

for that value of t, where F is a form of the same order as f that is a function of h (and the value of t that is considered) and satisfies $\lim_{h=0} F = 0$, *in addition.*

Proof. – For example, let $f(t)$ be a first-order form and let α be a third-order form. Set:

$$\varphi(t) = f(t)\ \alpha.$$

$\varphi(t)$ is a numerical function of t that has derivatives up to order n [because $\varphi^{(n)}(t) = f^{(n)}(t)\ \alpha$]. For that function, Taylor's formula ([20]) gives:

$$\varphi(t + h) = \varphi(t) + h\ \varphi'(t) + \ldots + \frac{h^n}{n!}[\varphi^{(n)}(t) + F_1], \tag{2}$$

where F_1 is a number that is a function of h such that $\lim_{h=0} F_1 = 0$. If one replaces $\varphi(t)$ with $f(t)\ \alpha$ in formula (2) and sets $F_1 = F\alpha$ then it will become formula (1), in which F will indeed satisfy $\lim_{h=0} F = 0$.

THEOREM II. – *If $f(t)$ is a geometric form that has n successive derivatives in the interval from t to t + h then one can write:*

$$f(t + h) = f(t) + h\,f'(t) + \frac{h^2}{2!}\,f''(t) + \ldots + \frac{h^{n-1}}{(n-1)!}\,f^{(n)} + F,$$

where F is a form of the same order as f that is the mean of the forms $f^{(n)}(t)$ that one obtains by making t vary from t to t + h.

Proof. – For example, let $f(t)$ be a first-order form, and let α be a third-order form α. Moreover, set:

$$\varphi(t) = f(t)\ a.$$

$\varphi(t)$ is a numerical function of t that has derivatives up to order n in the interval from t to $t + h$, and Taylor's formula, with Lagrange's remainder, gives:

$$\varphi(t + h) = \varphi(t) + h\ \varphi'(t) + \ldots + \frac{h^{n-1}}{(n-1)!}[\varphi^{(n-1)}(t) + \frac{h^n}{n!}\Phi,$$

where Φ is a number that is the mean of the values of $\varphi^{(n)}(t)$ or $f^{(n)}(t)\ \alpha$. Consequently, the number $F\alpha$ is the mean of the numbers $f^{(n)}(t)\ \alpha$, no matter what the third-order form α is; i.e., F is a mean form for the forms $f^{(n)}(t)$.

40. Continuous forms. – Let $f(t)$ be a geometric form. We say that $f(t)$ is continuous in the interval of variation for t when for any number t_0 in that interval, one has:

([20]) See: *Lezioni di Analisi infinitesimale del Prof.* G. PEANO, Turin, 1893, and *Calcolo geometrico*, *loc. cit.*

$$\lim_{t=t_0} f(t) = f(t_0) \quad \text{or} \qquad \lim_{h=0} f(t_0 + h) = f(t_0).$$

For example, let t_1, t_2 be two values of t in the interval considered, and let $\varphi(t_1, t_2)$ be a form or a numerical function of t_1 and t_2 . If we make t_1 and t_2 tend to the same value t in the interval, and in an arbitrary manner – without, for example, imagining that one has $t_1 = t_2$ and that the function $\dfrac{\varphi(t_1,t_2)}{t_2-t_1}$ has a well-defined limit – then we will denote that limit by the notation:

$$\lim_{t_1=t,t_2=t} \frac{\varphi(t_1,t_2)}{t_2-t_1},$$

or, more simply:

$$\lim \frac{\varphi(t_1,t_2)}{t_2-t_1}.$$

In an entirely analogous manner, we will define:

$$\lim \frac{\varphi(t_1,t_2,t_3)}{(t_2-t_1)(t_3-t_1)(t_3-t_2)} \ldots$$

THEOREM I. – *If $f(t)$ is a form with a well-defined derivative in the interval of variation for t then $f(t)$ will be a continuous function.*

Proof. – Taylor's theorem tells us that:

$$f(t + h) = f(t) + h[f'(t) + F],$$

where F is a form such that $\lim_{h=0} F = 0$. Consequently:

$$\lim_{h=0} f(t + h) = f(t),$$

which proves the theorem.

THEOREM II. – *If $f(t)$ is a geometric form such that the derived form $f'(t)$ is a continuous function in the interval of variation for t then one will have:*

$$\lim \frac{f(t_2)-f(t_1)}{t_2-t_1} = f'(t).$$

Proof. – If we set $n = 1$, $t = t_2$, $h = t_2 - t_1$ in Taylor's formula (Theorem II) then it will become:

$$f(t_2) = f(t_1) + (t_2 - t_1)\, F$$

or

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} = F.$$

However, f' is assumed to be a continuous function; i.e.:

$$\lim F = f'(t),$$

because F is the mean form of the values of $f'(t)$.

THEOREM III. – *If $f(t)$ is a first-order form, a second-order form with zero invariant, or a third-order form, and if $f'(t)$ is a continuous function in the interval of variation of t then one will have:*

$$\lim \frac{f(t_1)\, f(t_2)}{t_2 - t_1} = f(t)\, f'(t).$$

Proof. – We know that:

$$\frac{f(t_1)\, f(t_2)}{t_2 - t_1} = f(t_1)\, \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

However:

$$\lim f(t_1) = f(t), \qquad \lim \frac{f(t_2) - f(t_1)}{t_2 - t_1} = f'(t),$$

which was to be proved.

THEOREM IV. – *If $f(t)$ is a first or third-order form such that f' and f'' are well-defined forms, and $f''(t)$ is itself continuous for the values of t considered then one will have:*

$$\lim \frac{f(t_1)\, f(t_2)\, f(t_3)}{(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)} = f(t)\, f'(t)\, f''(t).$$

Proof. – Let t_1, t_2, t_3 be values of t such that:

$$t_3 > t_1 > t_2 .$$

Set:

$$f_1 = f(t_1), \qquad f_2 = f(t_2), \qquad f_3 = f(t_3).$$

Taylor's formula gives us:

$$(1) \qquad \begin{cases} f_2 = f_1 + (t_2 - t_1)\, f_1' + \dfrac{(t_2 - t_1)^2}{2}\, f_{1,2}'', \\[2ex] f_3 = f_1 + (t_3 - t_1)\, f_1' + \dfrac{(t_3 - t_1)^2}{2}\, f_{1,3}'', \end{cases}$$

where $f''_{1,2}$, $f''_{1,3}$ are mean forms of the forms $f''(t)$ when t varies from t_1 to t_2 or from t_1 to t_3. Moreover, when f is a first or third-order form, we know that $f_1 f_2 f_3 = f_1(f_2 - f_1)$, and in turn, that:

$$\frac{f_1 f_2 f_3}{(t_2 - t_1)(t_3 - t_1)} = f_1 \frac{f_2 - f_1}{t_2 - t_1} \frac{f_3 - f_1}{t_3 - t_1} = f_1 \frac{f_2 - f_1}{t_2 - t_1}\left(\frac{f_3 - f_1}{t_3 - t_1} - \frac{f_2 - f_1}{t_2 - t_1}\right).$$

Comparing this with formulas (1), the latter will become:

$$\text{(2)} \qquad \frac{f(t_1)\, f(t_2)\, f(t_3)}{(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)} = \frac{1}{2} f_1 \frac{f_2 - f_1}{t_2 - t_1} \frac{(t_3 - t_1)\, f''_{1,3} + (t_1 - t_2)\, f''_{1,2}}{t_3 - t_2}.$$

Moreover, $t_3 > t_1 > t_2$, and the last factor in the right-hand side of equation (2) is a mean form of $f''_{1,2}$, $f''_{1,3}$. In the limit, by virtue of Theorem III and the continuity hypothesis for the form $f''(t)$, one will indeed have Theorem IV.

§ 2. – LINES AND ENVELOPES.

41. Lines and envelopes of lines in a projective plane. – Suppose that $P(t)$, $p(t)$ are non-zero form on a projective plane that are of first and second order, respectively, and that they are defined, along with their derivatives of arbitrary order, in the interval of variation of t. If m is a non-zero whole number then $P^{(m)}(t)$, $p^{(m)}(t)$ will be forms in the fixed plane. Indeed, let α be a fixed, non-zero, third-order form such that $P(t)\ \alpha = 0$, $p(t)\ \alpha = 0$ for every value of t. As a result, $P^{(m)}(t)\ \alpha = 0$ and $p^{(m)}(t)\ \alpha = 0$, which was to be proved.

One calls the set P of projective points that are positions of forms $P(t)$ when t in the given interval a *line*.

One calls the set p of projective lines that are positions of forms $p(t)$ when t varies in the given interval an *envelope*.

These sets are projective elements that can be considered to be independent of the number t, and of the interval in which it can vary.

If R, R_1 are points of the line P, and the line RR_1 has a well-defined line r for its limit when R_1 varies on the line P in such a way as to tend to the point R then the line r will be called *tangent to the line P at the point R.*

If r, r_1 are lines in the envelope p and the point rr_1 has a well-defined point R for its limit when r_1 varies in the envelope p in such a way as to tend to the line r then the point R will be called *the characteristic of the envelope p on the line r.*

The normal to the line P at the point R is the perpendicular to the tangent at the point R (if it exists) that is raised in the plane of the curve.

If the point $P(t)$ is situated on a fixed line r for every value of t, and the line P does not reduce to just one point then the line r will be the tangent to the line P at any point in that line.

If the line $p(t)$ passes through a fixed point R for every value of t, and the envelope p does not reduce to just one line then the point R will be the characteristic of the envelope p on any line of that envelope.

THEOREM I. – *If, for a given value of t, m is the smallest non-zero, whole number x such that:*

$PP^{(x)} \neq 0$ *then the tangent to the line P at the point P will be the line* $PP^{(m)}$.

$pp^{(x)} \neq 0$ *then the characteristic of the envelope p on the line p will be the point* $pp^{(m)}$.

Proof. (for the statement on the left). – Upon setting:

$$P_1 = P(t+h),$$

Taylor's formula will give:

$$P_1 = P + hP' + \frac{h^2}{2!}P'' + \ldots + \frac{h^{m-1}}{(m-1)!}P^{(m-1)} + \frac{h^m}{m!}(P^{(m)} + Q),$$

where Q is a first-order form such that $\lim_{h=0} Q = 0$. Moreover, by hypothesis, $PP^{(x)} = 0$ for $x = 1, 2, \ldots, m-1$, and as a result:

$$PP_1 = \frac{h^m}{m!}(PP^{(m)} + PQ).$$

However:

$$\begin{aligned} \lim_{h=0} \text{posit } PP_1 &= \lim_{h=0} \text{posit}(PP^{(m)} + PQ) \\ &= \text{posit } \lim_{h=0} (PP^{(m)} + PQ) = \text{posit } PP^{(m)}, \end{aligned}$$

which proves the theorem.

The principle of duality for the plane will provide the theorem on the right.

THEOREM II.

If $PP'P'' \neq 0$ *for every value of t then the line P will be the locus of characteristics of the envelope whose lines are the tangents to the line P.*

If $pp'p'' \neq 0$ *for every value of t then the envelope p will be the set of tangents to the line whose points are the characteristics of the envelope p.*

Proof. (on the left). – To that effect, set $a = PP'$. a will then be a non-zero function of t, and the envelope a will admit the tangents to the line P for its lines. We will have:

$$a' = P'P' + PP'' = PP'',$$

and from the regressive product aa', one will see that:

$$aa' = PP' \,.\, PP'' = PPP'' \,.\, P' - P'PP'' .P = PP'P'' \,.\, P.$$

One can also express the left-hand side of Theorem II by saying: *If R is a point on the line P, and r is the tangent to that line at the point R then the point R will be the limiting position of the point at which the line r meets the tangent to the line P at another point R_1 when the point R_1 tends to be displaced on the line P in order to make it coincide with the point R.* One can give an analogous interpretation to the theorem on the right.

THEOREM III.

If $P(t)$ is a non-zero first-order form on a fixed plane, and $P'(t)$ is a continuous form such that one has $P(t_0)\ P'(t_0) \neq 0$ for some value t_0 of t then the tangent to the line P at the point $P(t_0)$ will be the limiting position of the line $P(t_1)P(t_2)$ when t_1 and t_2 tend to the value t_0 .

If $p(t)$ is a non-zero, second-order for on a fixed plane, and $p'(t)$ is a continuous form such that one has $p(t_0)\ p'(t_0) \neq 0$ for some value t_0 of t then the characteristic of the envelope p on the line $p(t_0)$ will be the limiting position of the point $p(t_1)p(t_2)$ when t_1 and t_2 tend to the value t_0 .

Proof. (on the left). – One has:

$$\text{posit } P(t_1)P(t_2) = \text{posit } \frac{P(t_1)P(t_2)}{t_2 - t_1},$$

and upon applying Theorem III of no. 40, one will get:

$$\lim \text{posit } P(t_1)P(t_2) = \text{posit } P(t_1)P'(t_2),$$

which proves the theorem.

Examples. –

1. Let $y = f(x)$ be the Cartesian equation of a curve. One has:

$$P = O + (x + iy)\,\mathbf{I},$$

where O is a fixed point, and $\mathbf{I}$ is a fixed unit vector in the plane of the curve. If x is the independent variable then one will have:

$$P' = (1 + iy')\,\mathbf{I},$$

and if y has a finite and well-defined derivative in the interval in which x varies then one must have that $P' \neq 0$, and the tangent to the point P will be the line PP'.

If θ is the angle that is defined by the x-axis and the tangent at the point P, or if $\theta = (\mathbf{I}, P')$ then we will have:

$$\tan\theta = \frac{\mathbf{I}\,P'}{\mathbf{I}\,i\,P'} = \frac{y\,\mathbf{I}\,i\,\mathbf{I}}{\mathbf{I}\,i\,\mathbf{I}} = y',$$

which gives the ordinary geometric significance of the number y'.

2. If $\rho = f(\varphi)$ is the equation of the curve in polar coordinates then:

$$P = O + \rho\, e^{i\varphi}\,\mathbf{I},$$

so:

$$P' = (\rho + i\rho)\, e^{i\varphi}\,\mathbf{I}.$$

If we set $\theta = (P - O, P')$ then we will get:

$$\tan\theta = \frac{\rho}{\rho'},$$

which is the usual formula from analytic geometry.

3. Let $P(t)$ be a first-order form, and let $PP^{(m)}$ be the tangent to the point P. The normal will be the line $P\ i\ (PP^{(m)}.\omega)$. Therefore, if $P(t)$ is a point then the normal to the point P will be the line $PiP^{(m)}$, since $P^{(m)}$ will be a vector that is parallel to the tangent at the point P.

4. Let $P(t)$ be a point such that $PP'P'' \neq 0$ for a given value of t, and let A be a non-zero, first-order form that belongs to the plane of the line P such that the point A is not situated on the tangent to the curve at the point P, in addition.

We say that the line P has its *concavity* at the point P turned towards the point A when the points of the line P are on the same side of the tangent at P as the point A in some neighborhood of the point A. In other words, the line turns its concavity towards the point A at the point P if one can fix a number ε such that the triangles:

$$P(t)\,P'(t)\,P(t+h), \qquad P(t)\,P'(t)\,A$$

have the same sense for any h, but are nevertheless smaller than ε in absolute value.

Under these condition, one can say that: *The concavity at the point P is turned towards the point A when the number* $\dfrac{PP'P''}{PP'A}$ *is positive.*

Indeed:

$$P(t+h) = P + hP' + \frac{h^2}{2}(P'' + Q)$$

and

$$PP'P(t+h) = \frac{h^2}{2}(PP'P'' + PP'Q).$$

Since $\lim_{h=0} Q = 0$, we can determine a number ε such that if h is smaller than ε in absolute value then the number $PP'Q$ will have the same sign as that of the number $PP'P''$. Moreover, for these values of h, the number $PP'P(t+h)$ will obviously have the same sign as the number $PP'P''$, because $h^2 / 2$ will always be positive.

Relative to Cartesian coordinates, one has:

$$PP'P'' = \tfrac{1}{2}y'', \qquad PP'\,i\mathbf{I} = \tfrac{1}{2}, \qquad PP'O = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 0 & 1 & y' \\ 1 & 0 & 0 \end{vmatrix} = \tfrac{1}{2}(xy' - y),$$

and at the point P the curve turns its concavity in the position of the y-axis, or even presents its concavity at the origin when $y'' > 0$ or $(xy' - y)\,y'' > 0$.

An entirely similar application can be made in polar coordinates, for which one will have:

$$P' = \rho'\,e^{i\varphi}\,\mathbf{I} + \rho\,e^{i\varphi}\;i\mathbf{I}, \qquad P'' = (\rho'' - \rho)\,e^{i\varphi}\,\mathbf{I} + 2\rho' e^{i\varphi}\;i\mathbf{I},$$

and thus:

$$PP'P'' = \frac{1}{2}\begin{vmatrix} 1 & \rho & 0 \\ 0 & \rho' & \rho \\ 0 & \rho'' - \rho & 2\rho' \end{vmatrix} = \tfrac{1}{2}(2\rho'^2 - \rho\rho'' + \rho^2),$$

$$PP'O = \frac{1}{2}\begin{vmatrix} 1 & \rho & 0 \\ 0 & \rho' & \rho \\ 1 & 0 & 0 \end{vmatrix} = \tfrac{1}{2}\,\rho^2.$$

5. Let $P_1(t)$, $P_2(t)$ be two points in the plane such that for every value of t:

$$P_1P_2 \neq 0, \qquad P_1' \neq 0, \qquad P_2' \neq 0,$$

and

$$P_1P_2\,P_1' \neq P_1P_2\,P_2'.$$

The position of the form $p = P_1P_2$ describes an envelope whose characteristic is described by the position of the first-order form:

$$pp' = P_1P_2\;(P_1'P_2 + P_1\,P_2') = P_1P_2\,P_2' \,.\, P_1 - P_1P_2\,P_1' \,.\, P_1 \,.$$

Therefore, the characteristic of the envelope p on the line p is the barycenter of the points P_1, P_2, which have masses that are equal to the distances – with a definite sign – from the points $P_2 + P_2'$, $P_1 + P_1'$ to the line P_1P_2.

For example, if:

$$P_1 = O + t\mathbf{I}, \quad P_2 = O + \sqrt{a^2 - t^2}\ i\,\mathbf{I},$$

then the line P_1P_2 will envelop an *asteroid*. The distances from the points $P_1 + P_1'$ and $P_2 + P_2'$ to the line P_1P_2 will be:

$$\frac{\sqrt{a^2-t^2}}{a} \qquad \text{and} \qquad -\frac{t^2}{a\sqrt{a^2-t^2}}.$$

As a result, the masses of the points P_1, P_2 will be:

$$t^2 \qquad \text{and} \qquad a^2 - t^2,$$

and the characteristic P of the envelope P_1P_2 on the line P_1P_2 will be the foot of the perpendicular that issues from the point $P_1 + P_2 - O$ to the line P_1P_2 .

42. Skew curves and envelopes of planes. – Let $P(t)$, $\pi(t)$ be non-zero forms that are of first and third order, respectively, and are defined, along with the derivatives in a given interval.

One calls the set of projective points that are positions of forms $P(t)$ when t varies in a given interval a *line*.

If the points of the line P are situated on the same line or on a fixed plane for any value of t then we will say that the line is a *straight* or *planar line*, respectively.

If it is impossible to determine an interval, inside of which one varies t, relative to which the line P is a straight or planar line then one will say that the line P is a *skew curve*.

If R, R_1 are projective points on the line P, and the line that joins R and R_1 has a well-defined line for its limit when R_1 varies on the line P and tends to R then that line will be called *tangent to the line P at the point R*.

One calls the set of projective planes that are positions of the forms $\pi(t)$ when t varies in the given interval an *envelope*.

If the planes of the envelope π pass through the same line or through a fixed point for any value of t then we will say that the envelope π is an *axial* or *conical envelope*, respectively.

If it is impossible to determine an interval, inside of which one varies t, relative to which the envelope π is an axial or conical envelope then we will say that the envelop π is a *skew envelope*.

If ρ, ρ_1 are projective planes of the envelope π, and the line that is common to ρ and ρ_1 has a well-defined line for its limit when ρ_1 varies on the envelope π and tends to ρ then that line will be called the *characteristic of the envelope π in the plane ρ.*

If the envelope π is an axial envelope and does not reduce to a unique plane then the characteristic of ρ will be the line through which all of the planes of the envelope π will pass. That line can be called the *axis of the envelope*. If that envelope π is a conical envelope then the characteristic of ρ – if it exists, moreover – will pass through the point that is common to all of the planes of the envelope π. That point can be called the *summit of the envelope*.

If R, R_1 are projective points of the line P and if the projective line r is the tangent to the line P at R, while the plane that passes through r and R_1 has the a well-defined plane for its limit when R_1 varies on P and tends to R then that plane will be called the *osculating plane of the line P at the point R*.

Finally, we shall call the envelope whose planes are the osculating planes for the line P the *osculating envelope of the line P*.

If ρ, ρ_1 are projective planes of the envelope π, and if the projective line r is the characteristic on ρ of the envelope π, while the point that is common to r and ρ_1 has a well-defined point for its limit when ρ_1 varies on the envelope π and tends to ρ then that point will be called the *point of regression of the envelope π on the plane ρ*.

We shall call the line whose points are points of regression for the envelope π the *line of regression for the envelope π.*

Theorem I. – *If, for some value of t, m is the smallest of the non-zero, whole numbers x such that:*

$PP^{(x)} \neq 0$ *then the tangent to the line P at the point P will be the line $PP^{(m)}$,*

$\pi\pi^{(x)} \neq 0$ *then the characteristic of the envelope π on the plane π will be the line $\pi\pi^{(m)}$,*

and this is proved in a way that is identical to that of Theorem I of no. 41. In particular:

If $PP' \neq 0$ then the tangent to the line P at the point P will be the line PP'.

If $\pi\pi' \neq 0$ then the characteristic of the envelope π at the plane π will be the line $\pi\pi'$.

THEOREM II. – *If, for a given value of t, m and n are the smallest non-zero, whole numbers x, y* ($x < y$) *such that:*

$PP^{(x)} \neq 0$ *then the osculating plane to the line P at the point P will be the plane $PP^{(m)}P^{(n)}$.*

$\pi\pi^{(x)} \neq 0$ *then the point of regression of the envelope π on the plane π will be the point $\pi\pi^{(m)}\pi^{(n)}$.*

Proof. (on the left). – To that effect, set $P_1 = P(t + h)$ and apply Taylor's formula. One gets:

$$P_1 = P + hP' + \ldots + \frac{h^m}{m!}P^{(m)} + \ldots + \frac{h^{n-1}}{(n-1)!}P^{(n-1)} + \frac{h^n}{n!}(P^{(n)} + Q),$$

where Q is a first-order form such that $\lim_{h=0} Q = 0$. Moreover, one has, by hypothesis:

$$PP^{(m)}P^{(y)} = 0 \quad \text{for} \qquad y = m + 1, \ldots, n - 1.$$

Therefore:

$$PP^{(m)}P_1 = \frac{h^n}{n!}(PP^{(m)}P^{(n)} + PP^{(m)}Q),$$

and in turn:

$$\lim_{h=0} PP^{(m)}P_1 = \text{posit } PP^{(m)}P^{(n)}.$$

However, the line $PP^{(m)}$ is tangent to the line P at the point P, and consequently, $\lim_{h=0}$ posit $PP^{(m)}P_1$ is the osculating plane to the point P.

If $PP'P'' \neq 0$ then the osculating plane to the line P at the point P will be the plane $PP'P''$.

If $\pi\pi'\pi'' \neq 0$ then the point of regression of the envelope π on the plane π will be the point $\pi\pi'\pi''$.

In the case where the line P is a straight line, the osculating plane at each of its points will be indeterminate, because if P describes a straight line $P = xA + yB$, where A, B are constant forms and x, y are functions of t, but then $P^{(m)} = x^{(m)}A + y^{(m)}B$, and $PP^{(m)}P^{(n)}$ will be always be zero. The same could be said for an axial envelope.

If the line P is only a planar line then its osculating plane will be the same as the plane of the curve at each of the points where the tangent is defined. Indeed, the plane $PP^{(m)}P_1$ will have a constant position that is nothing but that of the plane of the curve. One can deduce analogous conclusions for a conical envelope.

THEOREM III.

If one has $PP'P''P''' \neq 0$ for every value of t then:

a. The line P will be a skew curve.

b. The tangents to the skew curve will be the characteristics of the osculating envelope of the curve P.

c. The curve P will be the line of regression of the osculating envelope for the curve P ([21]).

If one has $\pi\pi'\pi''\pi''' \neq 0$ for every value of t then:

a. The envelope π will be a skew envelope.

b. The characteristics of the skew envelope π will be tangents to the line of regression of the envelope.

c. The envelope π will be the osculating

([21]) Let $P(t)$ be a point. For a given value of t, imagine that m, n, p are the smallest positive, whole numbers x, y, z such that $x < y < z$, and $P\,P^{(x)}\,P^{(y)}\,P^{(z)} \neq 0$. If one sets $P_1 = P(t + h)$ then Taylor's formula will give, successively:

$$PP_1 = \frac{h^m}{m!}\,(PP^{(m)} + Q_1),$$

envelope of the line of regression of the envelope π.

Proof. (of the left-hand statements). If the line P is a planar line then one will have $P = xA + yB + zC$, where A, B, C are constant forms, and x, y, z are functions of t. For any m, n, p ($m < n < p$), one must then have $PP^{(m)}P^{(n)}P^{(p)} = 0$. The stated part of the theorem will then result when one has $PP'P''P''' \neq 0$ for every value of t.

Now, suppose that $\alpha = PP'P''$. Since, by hypothesis, $PP'P'' \neq 0$ for every value of t, the envelope α will then be the osculating envelope of the skew curve P. One will have:

$$\alpha' = PP'P''', \quad \alpha'' = PP''P''' + PP'P^{\mathrm{IV}},$$

and for the regressive products $\alpha\alpha'$, $\alpha\alpha'\alpha''$, one will have:

$$\alpha\alpha' = PP'P''\mathbf{.}\, PP'P''' = P''PP'P''\mathbf{.}\, PP' = PP'P''P'''\mathbf{.}\, PP',$$
$$\alpha\alpha'\alpha'' = PP'P''P'''\mathbf{.}\, PP'\mathbf{.}\, (PP'P''' + PP'P^{\mathrm{IV}}) = -(PP'P''P''')^2\mathbf{.}\, P,$$

from which, one can infer the equalities:

$$\text{posit } \alpha\alpha' = \text{posit } PP', \qquad \text{posit } \alpha\alpha'\alpha'' = \text{posit } P,$$

which will prove parts b, c of the theorem.

THEOREM IV

If $P(t)$ is a first-order form, and $P'(t)$ is *If $\pi(t)$ is a third-order form, and $\pi'(t)$ is*

$$PP^{(m)}P_1 = \frac{h^n}{n!}\,(PP^{(m)}P^{(n)} + Q_2),$$
$$P\,P^{(m)}P^{(n)}P_1 = \frac{h^p}{p!}\,(PP^{(m)}P^{(n)}\,P^{(p)} + Q_3),$$

where Q_1, Q_2, Q_3 will be first-order forms that are subject to the condition that $\lim Q_1 = \lim Q_2 = \lim Q_3 = 0$. Upon supposing that A is the infinitely small principal, one sees that the distances from the point P_1 to the point P, to the tangent to the point P, and to the osculating plane at that point P, will have the infinitesimal orders m, n, p, respectively. There is a geometric interpretation for the numbers m, n, p that gives the type of singularity at the point P. We would not like to study the singular points of curves in detail here. Here is an example: If φ varies from $-\pi/2$ to $\pi/2$ then the point:

$$P = O + \cos^2\varphi\,\mathbf{I} + \sin\varphi\cos\varphi\,\mathbf{J} + \sin\varphi\,\mathbf{K}$$

will describe a *Viviani window* on the spherical surface with center O and unit radius. One will have:

$$P'P''P''' = \cos\varphi,$$

and only the point $P(\pi/2) = O + \mathbf{K}$ will be a singular point of the curve. At the point $O + \mathbf{K}$, the tangent will be the line $(O + \mathbf{K})\,\mathbf{J}$, and the osculating plane will be the plane $(O + \mathbf{K})\,\mathbf{J}\,(2\mathbf{I} - \mathbf{K})$. One will have a singularity, because $P'(\pi/2)$ is parallel to the vector $P'''(\pi/2)$.

a continuous form such that $P(t_0)P'(t_0) \neq 0$ *for every value of* t *then the tangent to the line* P *at the at the point* $P(t_0)$ *will be the limiting position of the line* $P(t_1)P(t_2)$ *when* t_1, t_2 *tend to the value* t_0 .

a continuous form such that $\pi(t_0)\pi'(t_0) \neq 0$ *for any value of* t *then the characteristic of the envelope* π *on the plane* $\pi(t_0)$ *will be the limiting position of the line* $\pi(t_1)\pi(t_2)$ *when* t_1, t_2 *tend to the value* t_0.

This is proved as the theorem in no. 41 was.

THEOREM V.

If $P(t)$ *is a first-order form that is defined, along with* $P'(t)$ *and* $P''(t)$, *and if* $P''(t)$ *is, in addition, a continuous form such that:*

$$P(t_0)\, P'(t_0)\, P''(t_0) \neq 0$$

for some value t_0 *of* t *then the osculating plane to the line* P *at the point* $P(t_0)$ *will be the limiting position of the plane:*

$$P(t_1)\, P(t_2)\, P(t_3)$$

when t_1, t_2, t_3 *tend to the value* t_0 .

If $\Pi(t)$ *is a third-order form that is defined, along with* $\Pi'(t)$ *and* $\Pi''(t)$, *and if* $\Pi''(t)$ *is, in addition, a continuous function such that:*

$$\Pi(t_0)\, \Pi'(t_0)\, \Pi''(t_0) \neq 0$$

for some value t_0 *of* t *then the point of regression of the envelope* Π *in the plane* $\Pi(t_0)$ *will be the limiting position of the point:*

$$\Pi(t_1)\, \Pi(t_2)\, \Pi(t_3)$$

when t_1, t_2, t_3 *tend to the value* t_0 .

Proof (of the left-hand statement). – One has:

$$\text{posit } P(t_1)\, P(t_2)\, P(t_3) = \text{posit } \frac{P(t_1)P(t_2)P(t_3)}{(t_2-t_1)(t_3-t_1)(t_3-t_2)}.$$

However,:

$$\lim \frac{P(t_1)P(t_2)P(t_3)}{(t_2-t_1)(t_3-t_1)(t_3-t_2)} = \tfrac{1}{2}\, P(t_0)\, P'(t_0)\, P''(t_0),$$

which proves the theorem.

a. One calls the plane that passes through the point P and is perpendicular to the tangent line at the point P the *normal plane* to the line P at the point P. One further calls the plane that is drawn through the tangent to the point P and is perpendicular to the osculating plane at that point the *rectifying plane*. For example, if the plane $PP^{(m)}P^{(n)}$ osculates the point P then the planes $P \mid (PP^{(m)}\mathbf{.}\ \omega)$ and $PP^{(m)} \mid (PP^{(m)}P^{(n)}\mathbf{.}\ \omega)$ will be the normal and rectifying plane at that point, respectively.

One calls the line that is common to the osculating and normal planes at a point P the *principal normal* to the line P at the point P, and the intersection of the normal and rectifying planes is the *binormal*. If the plane $PP^{(m)}P^{(n)}$ osculates at the point P, and if one supposes that $P(t)$ is a point then $P^{(m)}$ and $P^{(n)}$ will be vectors, and the planes $P \mid P^{(m)}$, $P\, P^{(m)} \mid P^{(m)}\, P^{(n)}$ will be the normal and rectifying planes, respectively, at the point P. The binormal will be the line $P \mid P^{(m)}P^{(n)}$, which will be the principal normal $(P \mid P^{(m)})$. $PP^{(m)}P^{(n)}$.

b. Let O, **I**, **J**, **K** be reference elements of a Cartesian coordinate system. If we set:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K},$$

where x, y, z are functions of t, then the point will describe a curve. If $PP^{(m)}P^{(n)}$ is the osculating plane at the point P then upon setting:

$$Q = O + X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K},$$

one will see that the point Q will be a point of the line $PP^{(m)}$ when the vectors $Q - P$, $P^{(m)}$ are parallel, and Q will be a point of the plane $PP^{(m)}P^{(n)}$ when the vectors $Q - P$, $P^{(m)}$, $P^{(n)}$ are coplanar. It will then result that the equation of the tangent to the point P will be:

$$\frac{X-x}{x^{(m)}} = \frac{Y-y}{y^{(m)}} = \frac{Z-z}{z^{(m)}},$$

and that of the osculating plane will be:

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x^{(m)} & y^{(m)} & z^{(m)} \\ x^{(n)} & y^{(n)} & z^{(n)} \end{vmatrix} = 0.$$

In the case of rectangular coordinates, the plane:

$$PP^{(m)} \mid P^{(m)}P^{(n)}$$

(viz., the rectifying plane) has the equation:

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x^{(m)} & y^{(m)} & z^{(m)} \\ \begin{vmatrix} y^{(m)} & z^{(m)} \\ y^{(n)} & z^{(n)} \end{vmatrix} & \begin{vmatrix} z^{(m)} & x^{(m)} \\ z^{(n)} & x^{(n)} \end{vmatrix} & \begin{vmatrix} x^{(m)} & y^{(m)} \\ x^{(n)} & y^{(n)} \end{vmatrix} \end{vmatrix} = 0,$$

and the normal plane is finally:

$$(X - x)\, x^{(m)} + (Y - y)\, y^{(m)} + (Z - z)\, z^{(m)} = 0.$$

§ 3. – RULED SURFACES.

43. Ruled surfaces in general. – Let $a(t)$ be a non-zero, second-order form with zero invariant that is defined, along with its derivatives of arbitrary order, on a given interval.

One calls the set of projective points that are situated on the line $a(t)$ when t varies in the interval considered the *ruled surface a*, or simply, the *surface a*; in particular, the lines $a(t)$ are called the *generators* of the surface a.

If $P(t)$, $Q(t)$ are first-order forms such that $PQ \neq 0$, $Pa = Qa = 0$ for every value of t then the point $P + uQ$ will describe the line a when u varies from $-\infty$ to $+\infty$. As a result, every point on the surface can be considered to be a function of two variables, which justifies the terminology *surface* that we employed (*see* Note I). Moreover, we have no need to consider the points of the surface a to be functions of two variables, and we can develop the theory of ruled surfaces independently of the general theory of arbitrary surfaces.

Since a is, by hypothesis, a second-order form with a zero invariant, one will have:

$$aa = 0 \tag{1}$$

for every value of t, a formula that will give $aa' + a'a = 0$. However, $aa' = a'a$, so $2aa' = 0$, or:

$$aa' = 0 \tag{2}$$

for any t.

Moreover, if m is a whole number that is greater than unity then one will have:

$$(aa)^{(m)} = aa^{(m)} + \binom{m}{1} a'a^{(m-1)} + \ldots + \binom{m}{1} a^{(m-1)} a' + a^{(m)}a \,;$$

i.e.:

$$(aa)^{(m)} = \sum_{r,s} h_{r,s} a^{(r)} a^{(s)} \,, \tag{3}$$

where r, s are positive whole numbers such that $r + s = m$ and $r < s$, while the $h_{r,s}$ are whole numbers that are functions of r and s.

We will make frequent use of formulas (1), (2), (3).

44. For example, let r, r_1 be generators of the surface a, and let R be a projective point of r, and suppose that the plane that contains R and r_1 had a well-defined plane for its limit when r_1 varies while remaining on the surface a and tends to r. This plane will be called the *tangent plane* ([22]) *to the ruled surface a at the point R.*

One calls the perpendicular to the tangent plane to a surface a at a point R that issues from R the *normal to the surface a at the point R.*

THEOREM I. – *If, at a point S on a line that is traced on the ruled surface a, the tangent s to the curve and the tangent plane σ to the surface a are well-defined then the line s will be contained in the plane σ.*

([22]) If R is a point at a finite distance then the definition that we just gave will be a logical consequence of that of the tangent plane to a surface, in general, that we gave in Note II.

Proof. – If we let S_1 be a point on the line, and let r_1 be the generator of the surface a that passes through S_1 then the line that joins S and S_1 will be contained in the plane determined by S and r_1 entirely, which proves that, in the limit, s will be contained in σ.

THEOREM II. – *If P is a non-zero, first-order form that has a point on the line a* (i.e., $Pa = 0$) *for some well-defined value of t, and if m is the smallest of the non-zero whole numbers x such that* $Pa^{(x)} \neq 0$ *then the tangent plane to the point P of the ruled a will be precisely the plane $Pa^{(m)}$.*

Proof. – Set $a_1 = a(t + h)$, and apply Taylor's theorem:

$$a_1 = a + ha' + \ldots + \frac{h^{m-1}}{(m-1)!}a^{(m-1)} + \frac{h^m}{m!}(a^{(m)} + q),$$

where q is a second-order form such that $\lim\limits_{h=0} q = 0$. Moreover, by hypothesis, $Pa^{(x)} = 0$ for $x = 1, 2, \ldots, m - 1$, so:

$$Pa_1 = \frac{h^m}{m!}(Pa^{(m)} + Pq).$$

However:

$$\text{posit } Pa_1 = \text{posit } (Pa^{(m)} + Pq),$$

so

$$\lim_{h=0} \text{ posit } Pa_1 = \text{posit } Pa^{(m)},$$

an equality that proves the theorem, since the tangent plane to the point P is nothing but the limiting position of the plane Pa_1 .

THEOREM III. – *While preserving the hypotheses of theorem II, one sees that the tangent plane to the point P will contain the line a (i.e., it contains the generator that passes through the point P).*

Proof. – Upon developing the regressive product $Pa^{(m)}\mathbf{.}\ a$, one will have:

$$Pa^{(m)}\mathbf{.}\ a = -\,Pa\mathbf{.}\ a^{(m)} + a^{(m)}a\mathbf{.}\ P = aa^{(m)}\mathbf{.}\ P,$$

but formula (2) of no. 43 proves that $aa^{(m)}\mathbf{.}\ P$ will be the sum of products of the form $a^{(r)}a^{(s)}\mathbf{.}\ P = a^{(r)}P\mathbf{.}\ a^{(s)} + a^{(s)}P\mathbf{.}\ a^{(s)}$. Therefore, since $r < m$, $s < m$, one will have:

$$Pa^{(r)} = Pa^{(s)} = 0, \qquad \text{so} \qquad Pa^{(m)}\mathbf{.}\ a = 0,$$

which proves the theorem.

45. From now on, the following conventions will always be implicit in the theorems that we shall state: The non-zero, first-order forms P, Q, R, S have their positions on the line a (i.e., $PA = Qa = Ra = Sa = 0$). When we say that *the plane tangent to the point P of the ruled surface is the plane $Pa^{(m)}$*, we intend that to mean that *there exist positive, whole numbers x such that $Pa^{(x)} \neq 0$, where m is the smallest of these numbers.* Finally, when we say that *the plane tangent to the point P is indeterminate*, we will be supposing that, in other words, *for any positive, whole number x, one will have $Pa^{(x)} = 0$.*

THEOREM I. – *If the plane tangent to the point P is $Pa^{(m)}$ then one and only one of the following properties will always be verified:*

a. *At any point R that is distinct from P, the tangent plane will be $Ra^{(n)}$, with $n < m$, and that plane, as well as the number n, will remain fixed when the point R varies on the line a.*

b. *At any point R that is distinct from P, the tangent plane will be $Ra^{(m)}$, which will coincide with the plane $Pa^{(m)}$, except for a point S where the tangent plane will be either the plane $Sa^{(n)}$, with $n > m$, or indeterminate.*

c. *At any point R, the tangent plane will be the plane $Ra^{(m)}$, and the tangent planes at two distinct points of the line a will be different.*

Proof. – Indeed, let Q be an arbitrary point on the line a that is distinct from P. The tangent plane at the point Q will be the plane $Qa^{(n)}$, with $n < m$, $n = m$, or $n > m$, or it might even be indeterminate. These different cases will imply properties *a*, *b*, *c*, resp., as we shall show.

For any R, we have:

(1) $$R = xP + yQ,$$

where x, y are real numbers. Therefore, if the plane $Q^{(n)}$ ($n < m$) is tangent to the point Q then, knowing that $Pa^{(n)} = 0$, formula (1) will give us that $Ra^{(n)} = yQa^{(n)}$. If $RP \neq 0$ then one will have $y \neq 0$, and $Ra^{(n)} \neq 0$; i.e., n will be precisely the smallest positive, whole number x such that $Ra^{(x)} \neq 0$, or even that the plane $Pa^{(n)}$ will be tangent to the point R. However, by virtue of the equality $Ra^{(n)} = y\,Qa^{(n)}$, that plane will be identical to $Qa^{(n)}$, which will succeed in establishing property *a*.

The first part of property *b* results from the fact that if the plane $Qa^{(n)}$ ($n > m$) is tangent to Q then one will have $Ra^{(m)} = x\,Pa^{(m)}$, a formula that again persists if the tangent plane to the point Q is indeterminate.

If the tangent plane at Q is $Qa^{(m)}$ then one will have:

(2) $$Ra^{(m)} = x\,Pa^{(m)} + y\,Qa^{(m)}.$$

One can deduce that $Qa^{(m)} = h\,Pa^{(m)}$ for the plane $Pa^{(m)}$ = plane $Qa^{(m)}$, where h is a non-zero, real number, and formula (2) will become $Ra^{(m)} = (x + hy)\,Pa^{(m)}$. When $x / y \neq -h$, the plane $Ra^{(m)}$ = plane $Pa^{(m)}$ will be the tangent plane to the point R, while if $x / y = -h$, the tangent plane to the point S = point$(Q - hP)$ will be $Sa^{(m)}$, with $n > m$, or even indeterminate; one thus completes the proof of property *b*. If plane $Pa^{(m)} \neq$ plane $Qa^{(m)}$ then the form $Ra^{(m)}$ cannot be zero, and $Ra^{(m)}$ will be the tangent plane at R. The tangent planes at two distinct points R and S will not coincide, moreover, because if:

$$S = x_1P + y_1Q$$

then one will have:

$$Ra^{(m)}.\, Sa^{(m)} = \begin{vmatrix} x & y \\ x_1 & y_1 \end{vmatrix} Pa^{(m)}.\, Qa^{(m)},$$

and in turn, one must have:

$$\begin{vmatrix} x & y \\ x_1 & y_1 \end{vmatrix} = 0 \qquad \text{or} \qquad Pa^{(m)}.\, Qa^{(m)} = 0,$$

which finally establishes property *c*.

THEOREM II. – *If $Pa^{(m)}$ is the tangent plane at each point P of the line a – except for at most one point – then $a^{(m)}\, a^{(m)} = 0$ will be the necessary and sufficient condition for the tangent planes at distinct points to coincide.*

Proof. – Indeed, one has upon developing the regressive product $Qa^{(m)}.\, Ra^{(m)}$:

$$Qa^{(m)}.\, Ra^{(m)} = (Q.\, Ra^{(m)})\, a^{(m)} + (Ra^{(m)}.\, a^{(m)})\, Q.$$

However:

$$Q.\, Ra^{(m)} = 0,$$

because the plane $Ra^{(m)}$ contains (no. 44, Theorem III) the line a and, in turn, the point Q. Then:

(1) $$Qa^{(m)}.\, Ra^{(m)} = (Ra^{(m)}.\, a^{(m)})\, Q.$$

The regressive product $Ra^{(m)}.\, a^{(m)}$ further provides:

$$Ra^{(m)}.\, a^{(m)} = -\, Ra^{(m)}.\, a^{(m)} + a^{(m)}\, a^{(m)}.\, R,$$

or

$$Ra^{(m)}.\, a^{(m)} = \tfrac{1}{2}\, a^{(m)}\, a^{(m)}.\, R,$$

and formula (1) becomes:

$$Qa^{(m)}.\, Ra^{(m)} = \tfrac{1}{2}\, a^{(m)}\, a^{(m)}.\, RQ.$$

Furthermore, since $QR \neq 0$, $a^{(m)}\, a^{(m)}$ will indeed be the necessary and sufficient condition for one to have $Qa^{(m)}.\, Ra^{(m)} = 0$, which is precisely what we had to establish.

THEOREM III. – *If the tangent planes at two arbitrary, but distinct points of the line a are different then if $Pa^{(m)}$ is, for example, the tangent plane to one of them P, and α is a non-zero, third-order form that contains the form a* (i.e., $a\alpha = 0$) *then the plane a will be tangent to the ruled surface at the point $\alpha\, a^{(m)}$.*

Proof. – $\alpha\, a^{(m)}.\, a^{(m)}$ is the tangent plane to the point $\alpha\, a^{(m)}$, but:

$$a^{(m)}\alpha\, .\, a^{(m)} = -\, a^{(m)}a^{(m)}.\ \alpha - \alpha\, a^{(m)}.\ a^{(m)},$$

or

$$\alpha\, a^{(m)}.\ a^{(m)} = -\tfrac{1}{2}\, a^{(m)}a^{(m)}.\ \alpha\, .$$

Theorem II gives $a^{(m)}a^{(m)} \neq 0$; i.e.:

$$\text{plane } \alpha\, a^{(m)}.\ a^{(m)} = \text{plane } \alpha,$$

which proves the theorem.

THEOREM IV. – *Preserving the hypothesis of Theorem III, if α varies then the sheaf of planes α will be projective at the contact points of these planes with the ruled surface* (i.e., *the points $\alpha a^{(m)}$*).

Proof. – Two different planes α will correspond to the distinct points $\alpha a^{(m)}$, and conversely. Therefore, the correspondence between the planes α and the points $\alpha a^{(m)}$ will be single-valued and reciprocal.

Then, set:

$$P_1 = a_1\, Q + y_1\, R, \qquad P_2 = x_2\, Q + y_2\, R.$$

The double ratio of the sequence of points Q, R, P_1, P_2 (which depends solely upon the positions of the forms Q, R, P_1, P_2) is the number:

$$\frac{QP_1}{RP_1}\frac{RP_2}{QP_2} = \frac{y_1 QP_1}{x_1 RP_1}\frac{x_2 RP_2}{y_2 QP_2} = \frac{y_1 x_2}{x_1 y_2}\, .$$

However, we know that:

$$P_1\, a^{(m)} = x_1\, Q\, a^{(m)} + y_1\, R\, a^{(m)}, \qquad P_2\, a^{(m)} = x_2\, Q\, a^{(m)} + y_2\, R\, a^{(m)},$$

and, in turn, the double ratio of the sequence of planes $Q\, a^{(m)}$, $R\, a^{(m)}$, $P_1\, a^{(m)}$, $P_2\, a^{(m)}$ is again that same number $\dfrac{y_1 x_2}{x_1 y_2}$.

46. Skew ruled surfaces. – We shall say that the ruled surface a is a *skew ruled surface* when, for any value of t, the tangent planes (if they exist) at two distinct points of the line a do not coincide, except perhaps for the lines a that correspond to a system of isolated values of t in the interval in question; such generators will then be called *singular generators.*

If we suppose that Pa' is the tangent plane at each point P of an arbitrary non-singular generator then Theorem II of no. 45 will show that $a'a' \neq 0$ for any value of t that corresponds to a non-singular generator; i.e., that a' is not reducible to a line segment or a bivector, but is always the sum of a line segment and a bivector. Similarly, if α is a third-order form with $\alpha a = 0$ then Theorem III of no. 45 will prove that the plane α touches the skew ruled surface a at the point $a\alpha'$.

a. In the case where the form a is not a bivector, for a given value of t, one calls the plane that is tangent to the surface at a point at infinity on the line a – i.e., the tangent plane to the point $a\omega$, precisely – the *asymptotic plane of the surface or the generator a*. The asymptotic plane for the generator a will then be the plane $a\omega.\, a'$ or the plane $a'\omega.\, a$, because since $aa' = 0$ (cf., pp. ?), one will have:

$$a\omega.\, a' = aa'.\, \omega - \omega a'.\, a = -\, a'\omega.\, a.$$

The plane that passes through the line a and is perpendicular to the asymptotic plane is called the *central plane of the surface for the generator a*, and its contact point with the surface is called the *central point of the generator*. The central plane for the line a is then the plane $a \mid (a\omega.\, a'\omega)$, since the asymptotic plane $a'\omega.\, a$ possesses the same orientation as the bivector $a\omega.\, a'\omega$. Likewise, $[a \mid (a\omega.\, a'\omega)]\, a'$ is the central point of the line a.

One calls the locus of central points on the lines a the *line of striction of the skew ruled surface*.

b. If a_1 is a generator of the skew ruled surface a then the asymptotic plane for the line a is the limiting position of the plane that passes through the line a and is parallel to the line a_1 when it tends to coincide with the line a as one varies on the surface.

Set $a_1 = a(t + h)$. The plane that is parallel to a_1 and passes through the line a is the plane $a.\, a_1\omega$, and upon developing a_1 in a Taylor series, one will have:

$$a.\, a_1\omega = a.\, [a\omega + h(a' + q)\omega] = h(a.\, a'\omega + a.\, q\omega),$$

where q satisfies $\lim_{h=0} q = 0$. Therefore:

$$\lim \text{plane } a.\, a_1\omega = \text{plane } a.\, a'\omega,$$

which proves the proposition.

If mod PP_1 represents the shortest distance between the two lines a and a_1 then the central point for the line a will be the limiting position of the point P_1 when a_1 tends to a while remaining on the surface. Indeed, the point P_1 is nothing but the point $[a \mid (a\omega.\, a\omega)]\, a_1$. Upon developing a_1 using Taylor's formula and passing to the limit, one will obtain the point $[a \mid (a\omega.\, a'\omega)]\, a'$, which is the central point of the line a.

c. If b is a non-zero, second-order form with zero invariant then one can easily prove that the line $Pa'.\, b$ will describe a hyperboloid or a paraboloid of agreement with the surface all along the line a when the point P is displaced on the same generator a. Similarly, since:

$$\lim_{h=0} Pa'.\, a_1 = Pa'.\, a'',$$

the line $Pa' \mathbf{.}\, a''$ will indeed describe the osculating hyperboloid to the surface all along the line a.

d. Let $P(t)$ be a point (that is not at infinity), and let $\mathbf{I}(t)$ be a unit vector. Set $a = P\mathbf{I}$, so the line a will describe a ruled surface when t varies in the interval considered. That surface will be skew if:

$$a'a' = (\mathbf{P}'\mathbf{I} + P\mathbf{I}')(\mathbf{P}'\mathbf{I} + P\mathbf{I}') = 2\, P\mathbf{P}'\mathbf{I}\,\mathbf{I}' \neq 0\ ;$$

i.e., if the vectors $\mathbf{P}'$, $\mathbf{I}'$ are non-zero then the vectors $\mathbf{P}'$, $\mathbf{I}$, $\mathbf{I}'$ will not be coplanar. $\mathbf{P}'\mathbf{I}\,\mathbf{I}'$ is then the asymptotic plane for the line $P\mathbf{I}$. Since $\mathbf{I}'$ is perpendicular to the vector $\mathbf{I}$, the point P will describe the line of striction of the surface when the vectors $\mathbf{P}'$, $\mathbf{I}'$, $|\,\mathbf{I}\,\mathbf{I}'$ are coplanar, and conversely.

Now, suppose that the line P is the line of striction of for the skew ruled surface $P\mathbf{I}$. The fact that the vectors $\mathbf{P}'$, $\mathbf{I}$, $|\,\mathbf{I}\,\mathbf{I}'$ are coplanar gives:

$$\mathbf{P}' = h\mathbf{I} + k\,|\,\mathbf{I}\,\mathbf{I}',$$

where h, k are real numbers ($k \neq 0$, because $P\mathbf{P}'\mathbf{I}\,\mathbf{I}'$ is not zero). If Q is a point of the line $P\mathbf{I}$ then one will have:

$$Q = P + x\mathbf{I},$$

where x is a real number, and since $Q(P\mathbf{I})'$ is the tangent plane to the point Q, one can write:

$$Q(P\mathbf{I})' = (P + x\mathbf{I})(P'\mathbf{I} + P\mathbf{I}') = (P + x\mathbf{I})(k\mathbf{I}\,|\,\mathbf{I}\,\mathbf{I}' + P\mathbf{I}') = P(k\mathbf{I}\,|\,\mathbf{I}\,\mathbf{I}' - x\,\mathbf{I}\,\mathbf{I}').$$

However, the bivectors $\mathbf{I}\,|\,\mathbf{I}\mathbf{I}'$, $\mathbf{I}\mathbf{I}'$ have the same modulus and are rectangular, in such a way that if θ is one of the angles that the tangent plane at Q forms with the central plane then one will have:

$$\tan\theta = \frac{x}{k},$$

a formula that is due to Chasles. The number k is called the *distribution parameter for the generator* $P\mathbf{I}$.

47. Developable surfaces. – We say that the ruled surface a is a *developable surface* when for any value of t the tangent plane – if it exists – at each point of the line a – except at most one of them – coincides with a fixed plane, which one calls the *tangent plane along the line a.*

Therefore, suppose that the tangent plane is Pa' at each point P of a generator, except at the point Q. Theorem I of no. 45 proves that the tangent plane at the point Q will be Qa'', or Qa''', …, or even less if it is indeterminate. Likewise, Theorem II of no. 45 is applicable, and will permit us to write $a'a' = 0$ for every value of t; i.e., a' will be a line segment or a bivector. Moreover, the line a' will be contained in the plane Pa', which is tangent along the generator a, precisely, because $Pa' . a' = - Pa' . a' + a'a' . P$, so $Pa' . a' = 0$.

a. Let O be a constant, first-order form that is non-zero, moreover, and let $A(t)$ be a non-zero form of the same order. The second-order form OA generates a *cone* whose summit is O. In the case where the point O is at infinity, the surface OA is further called a *cylinder*. On the other hand, one knows that $(OA)' = OA'$ – i.e., $(OA)'(OA)' = 0$ – which proves that the cone is a developable surface, just as the tangent plane to the point O is indeterminate, because for any positive, whole number x, one will have $O(OA)^{(x)} = O . OA^{(x)} = 0$. The tangent plane along the line OA will be OAA'.

b. If $a\omega \neq 0$ and $a . a'\omega . a''\omega \neq 0$ for every value of t then one can state the following theorems:

THEOREM I. – *The tangent plane along the line a is $a\omega . a'$, or also the plane $a'\omega . a$, which is identical to the first one.*

Proof. – One has $a\omega . a' = a\, a' . \omega - a'\omega . a = - a'\omega . a$, and since $a . a'\omega \neq 0$, the tangent plane to the point at infinity $a\omega$ is precisely the plane $a\omega . a'$ = plane $a'\omega . a$, while Theorem I of no. 45 implies that plane $a\omega . a'$ is tangent along the line a.

THEOREM II. – *The envelope whose planes are tangent to the surface a has the lines a for its characteristics.*

Proof. – Upon setting $\alpha = a'\omega . a$, we will have:

$$a' = a''\omega . a + a'\omega . a' = a''\omega . a,$$

because a' is a line segment, and $a'\omega . a' = 0$. It will then suffice to take the regressive product $\alpha\alpha'$ in order to see that:

$$\alpha\alpha' = (a'\omega . a)(a''\omega . a) = [a'\omega(a''\omega . a)]\, a + [a(a''\omega . a)]\, a'\omega = (a . a'\omega . a''\omega)\, a,$$

which proves the theorem.

Definition. – One calls the line of regression of the envelope of tangent planes of a developable surface *the edge of regression.*

THEOREM III. – *The edge of regression of the surface a can be regarded as being generated by the point $a(a' . a''\omega)$.*

Proof. – Let $\alpha = a' . a''\omega$. One has:

$$\alpha' = a''\omega . a \qquad \text{and} \qquad \alpha'' = \omega . a + a''\omega . a',$$

so

$$\alpha\alpha'\alpha'' = (a.\ a'\omega.\ a''\omega)\,[a(a''\omega.\ a) + a''\omega.\ a'] = (a.\ a'\omega.\ a''\omega)\ a(a'a''.\ \omega),$$

which establishes the theorem.

THEOREM IV. – *The point of the edge of regression that belongs to the line a is given by the regressive product aa', when it is developed on the tangent plane along the line a.*

Proof. – Let $P(t)$ be a point such that $PP'P'' \neq 0$ for any value of t. Since $(PP')'\ (PP')' = PP''PP'' = 0$, the line PP' will describe a developable surface that admits the osculating planes $PP'P''$ to the curve P for its tangent planes, because:

$$(PP'.\ \omega)(PP')' = P'(PP'') = -\,PP'P''.$$

That developable surface PP' is called the *osculating developable to the curve P*.

§ 4. – FRENET FORMULAS.

48. Arcs. – Let $P(t)$ be a point that is a continuous function of t. Let a, b be two particular values of t, and let $t_1, t_2, \ldots, t_{n-1}$ be a sequence of values such that $a \leq t_1 \leq t_2 \leq \ldots \leq t_{n-1} \leq b$, or even $a \geq t_1 \geq t_2 \geq \ldots \geq t_{n-1} \geq b$. The upper limit of the set of numbers that one can deduce from:

$$\text{(1)} \qquad \text{mod } P(a)\ P(t_1) + \text{mod } P(t_1)\ P(t_2) + \ldots + \text{mod } P(t_{n-1})\ P(b),$$

by varying the sequence $t_1, t_2, \ldots, t_{n-1}$ is called the *length of the arc (on the line P) that is bounded by the points* $P(a)$, $P(b)$, and is represented by the notation arc $P(a)\ P(b)$. The line that is bounded by the points $P(a)$, $P(b)$ that has length mod $P(a)\ P(b)$ is called the *chord* of arc $P(a)\ P(b)$.

Under these conditions, if there exists an upper number to any value of (1) then arc $P(a)\ P(b)$ will be a well-defined real number, since otherwise arc $P(a)\ P(b) = \infty$. For example, one will always have:

$$\text{mod } P(a)\ P(b) \leq \text{arc } P(a)\ P(b) \qquad \text{and} \qquad \text{arc } P(a)\ P(a) = 0.$$

THEOREM I. – *If $P(t)$ is a point, and $P'(t)$ is a (vector) form that is a continuous function of t, and if one has $P'(t') \neq 0$ for some particular value t' of t then the ratio of an arc of the line P to its chord will have the limit unity when the extremities of the arc tend to the point $P(t')$.*

Proof. – It results from the hypotheses and Theorem IV of no. 42 that the tangent to the line P at $P(t')$ will be the limiting position of the line that joins two arbitrary points of P when these two points tend to coincide at the point $P(t')$. One can thus determine two different values a and b for t such that $a \leq t \leq b$, and furthermore, such that the line that joins the two arbitrary points of arc $P(a)$ $P(b)$ will make an angle with the line $P(a)$ $P(b)$ that is less than a given angle θ that lies between 0 and $\pi / 2$.

If $t_0 = a$, t_1, t_2, …, t_{n-1}, $t_n = b$ have the same significance as before then if one sets $P_r = P(t_r)$ for $r = 0, 1, …, n$ then one can write:

$$(P_n - P_0)^2 = (P_1 - P_0) \mid (P_n - P_0) + (P_2 - P_1) \mid (P_n - P_0) + \ldots + (P_n - P_{n-1}) \mid (P_n - P_0),$$

and dividing by:

$$\text{mod } (P_n - P_0) = \text{mod } P_0 \, P_n$$

will give:

$$\text{mod } P_0 \, P_n = \text{mod } P_0 \, P_n \cos \varphi_1 + \ldots + \text{mod } P_{n-1} \, P_n \cos \varphi_n ,$$

where

$$\varphi_r = (P_r - P_{r-1}, P_n - P_0) \qquad \text{for } r = 1, 2, \ldots, n.$$

One then deduces that:

$$\text{(2)} \qquad \text{mod } P_0 \, P_n = (\text{mod } P_0 \, P_1 + \ldots + \text{mod } P_{n-1} \, P_n) \cos \varphi .$$

In this formula, φ represents a mean angle of the angles $\varphi_1 , \varphi_2 , …, \varphi_n$. It notably results from equation (2) that the number (1) will always be less than mod $\dfrac{P_0 P_n}{\cos\theta}$, and consequently, arc P_0 P_n will be a well-defined real number. Formula (2) will therefore persist in the upper limit; i.e.:

$$\text{mod } P_0 \, P_n = \text{arc } P_0 \, P_n \cos \psi,$$

where ψ is a mean angle of the angles that P_0 P_n makes with the lines that join two arbitrary points of the arc P_0 P_n . If we make $P(a)$ and $P(b)$ tend to the point $P(t')$ from now on then ψ will tend to zero, from which one deduces that:

$$\lim \frac{\text{arc } P(a)P(b)}{\text{mod } P(a)P(b)} = \lim_{\psi=0} \frac{1}{\cos\psi} = 1,$$

which is precisely what we had to prove.

THEOREM II. – *If* $P(t)$ *is a point, and* $\mathbf{P}'(t)$ *is a non-zero vector that is a continuous function of t, and if, in addition, a and b are two values of t such that* $a \leq b$ *then one will have:*

$$\text{arc } P(a) \, P(b) = \int_a^b \text{mod } dP .$$

Proof. – Upon agreeing to let Δ arc $P(a)$ $P(t)$ denote the increment in the function arc $P(a)$ $P(t)$ when t passes from the value t to the value $t + h$, one will obviously have:

$$\Delta \text{ arc } P(a) \, P(t) = \text{arc } P(a) \, P(t + h) = \frac{\text{arc } P(t)P(t+h)}{\text{mod } P(t)P(t+h)} \text{mod}[P(t + h) - P(t)],$$

and upon applying Theorem I in order to simultaneously pass to the limit $h = 0$, one will get:

$$d \text{ arc } P(a)\, P(t) = \text{mod } dP,$$

a formula that proves the theorem.

49. Let $P(t)$ be a point, and let $\mathbf{P}'(t)$ be a continuous, non-zero function of t. We will let $s(t)$ – or, more simply, s – represent any function of t such that $s + c$ remains well-defined by the equation:

$$ds = \text{mod } dP, \tag{1}$$

in which the numerical constant c has been chosen arbitrarily, moreover. Since dt is positive, that equation can be written:

$$\frac{ds}{dt} = \text{mod } \frac{dP}{dt}. \tag{1$'$}$$

We then let s denote the arc of the line P, and if a, b ($a \le b$) are two arbitrary values of t then one will have:

$$\text{arc } P(a)\, P(b) = s(b) - s(a).$$

We further call the point $P(a)$ for which $s(a) = 0$ the *origin of the arc on the line P*.

When t varies in the interval considered so as to tend to a value t_0, but on the condition that $\lim\limits_{t=t_0} P$ be a well-define projective point P_0 , we will set:

$$\text{arc } P(a)\, P_0 = \lim_{t=t_0} \int_a^t \text{mod } dP = [s(t) - s(a)].$$

Examples. –

1. Upon considering orthogonal, Cartesian coordinates, in particular:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K},$$

with three functions x, y, z of t, one will have:

$$dP = dx\,\mathbf{I} + dy\,\mathbf{J} + dz\,\mathbf{K},$$

so

$$ds = \sqrt{dx^2 + dy^2 + dz^2}\,,$$

which is the usual formula of analytic geometry.

If one still desires them, then:

$$\cos(dP, \mathbf{I}) = \frac{dx}{ds}, \qquad \cos(dP, \mathbf{J}) = \frac{dy}{ds}, \qquad \cos(dP, \mathbf{K}) = \frac{dz}{ds},$$

will give the cosines of the angles that the tangent to the point P makes with the coordinate axes.

2. In polar coordinates on the plane, one will have:

$$P = O + \rho\, e^{i\varphi}\, \mathbf{I}$$

or

$$dP = (d\rho + i\rho\, d\varphi)\, e^{i\varphi}\, \mathbf{I},$$

and

$$ds = \sqrt{d\rho^2 + \rho^2 d\varphi^2}\ .$$

3. The point $P = O + e^{a\varphi}\ e^{i\varphi}\, \mathbf{I}$ ($a \neq 0$) describes a logarithmic spiral when φ varies from $-\infty$ to $+\infty$. If one takes φ precisely to be the independent variable then:

$$P' = (a + i)\, e^{a\varphi}\ e^{i\varphi}\, \mathbf{I},$$

and therefore

$$\text{mod } P' = \sqrt{1+a^2}\ e^{a\varphi},$$

or furthermore:

$$ds = \sqrt{1+a^2}\ e^{a\varphi}\ d\varphi.$$

If φ_0 and φ_1 are two particular values of φ then:

$$\text{arc } P(\varphi_0)\, P(\varphi_1) = \sqrt{1-a^2}\int_{\varphi_0}^{\varphi_1} e^{a\varphi} d\varphi = \frac{\sqrt{1+a^2}}{a}(e^{a\varphi_1} - e^{a\varphi_0}),$$

and one will have:

$$\lim_{\varphi_0=-\infty} \text{arc } P(\varphi_0)\, P(\varphi_1) = \frac{\sqrt{1+a^2}}{a} e^{a\varphi_1}\ .$$

Thus, upon taking the asymptotic point O to be the origin of the arc of the curve, one will have for any φ:

$$s = \frac{\sqrt{1+a^2}}{a} e^{a\varphi}\ .$$

50. Curvature and radius of curvature. – Once more, let $P(t)$ be a point such that the vectors $P'(t)$, $P''(t)$ are well-defined for any value of t, and in addition, suppose that $P'(t) \neq 0$.

If s represents the arc-length of the line P then we can consider P to be a function of the variable s, and the formula $ds = \text{mod } dP$ will permit us to obtain the derivatives of P with respect to s by observing that $\dfrac{dP}{ds} = \dfrac{dP}{dt}\dfrac{dt}{ds}$.

We set:

(1) $$\mathbf{T}(s) = \frac{dP}{ds},$$

where **T** is a unit vector that is parallel to the tangent to the line P at the point P, because $\frac{dP}{ds} = \frac{dP}{\operatorname{mod} dP}$ will result from the fact that mod $P \neq 0$. The vector $\frac{d\mathbf{T}}{ds} = \frac{d^2P}{ds^2}$ is well-defined, since P'' itself is also well-defined, and if $\mathbf{T}\frac{d\mathbf{T}}{ds} \neq 0$ then the plane $P\mathbf{T}\frac{d\mathbf{T}}{ds}$ will osculate at P.

Upon setting:

(2) $$\frac{1}{\rho} = \operatorname{mod} \frac{d\mathbf{T}}{ds},$$

we will call the number $1 / \rho$ the *curvature of the line P at the point P*. The inverse (viz., ρ) of the curvature is further called the *radius of curvature at the point P*.

THEOREM. – *In order for the line P to be a straight line, it is necessary and sufficient that the curvature be zero at any point P.*

Proof. – Indeed, if the point P describes a straight line then $P = O + s\mathbf{I}$, where O is a fixed point of the line P, and **I** is a constant unit vector. One will then have $\mathbf{T} = \mathbf{I}$ and $d\mathbf{T} / ds = 0$, which will indeed imply that $1 / \rho = 0$ for every value of s; the condition is therefore necessary. Now, suppose that one has $1 / \rho = 0$ for any value of s; i.e., $d\mathbf{T} / ds$ is constantly zero. It will then result that **T** is a constant vector. Now, $dP = \mathbf{T}\, ds$, and in turn, $d(P - s\mathbf{T}) = 0$; $P - s\mathbf{T} = O$, where O is a fixed point; the stated condition is therefore also sufficient.

51. For any t, $1 / \rho \neq 0$, we set:

(3) $$\mathbf{N}(s) = \rho \frac{d\mathbf{T}}{ds};$$

i.e., [formula (2)]:

$$\mathbf{N}(s) = \frac{d\mathbf{T} / ds}{\operatorname{mod} d\mathbf{T} / ds}.$$

The vector $d\mathbf{T} / ds$ is not zero, since, by hypothesis, $1 / \rho \neq 0$, so **N** will be a unit vector, just like **T**, to which it is perpendicular. Therefore, **N**, which is parallel to the osculating plane at the point P, will be the vector that is parallel to the principal normal at the point P ([23]).

While preserving the preceding hypotheses, one sees that: $P\mathbf{T}$ is the tangent, $P\mathbf{N}$ is the principal normal, and $P\mathbf{TN}$ is the osculating plane at the point P.

([23]) If $1 / \rho = 0$ for some value of s, and the osculating lane at the point P is well-defined then only the sense of the unit vector that is parallel to the principal normal at the point P will remain indeterminate.

The point $A = P + \rho\mathbf{N}$ that is situated on the principal normal at the point P will be called the *center of curvature at the point.*

Examples. –

1. In orthogonal, Cartesian coordinates:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K},$$

so:

$$\frac{d^2P}{ds^2} = \frac{d\mathbf{T}}{ds} = \frac{d^2x}{ds^2}\mathbf{I} + \frac{d^2y}{ds^2}\mathbf{J} + \frac{d^2z}{ds^2}\mathbf{K},$$

and

$$\frac{1}{\rho} = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}\ .$$

Since d^2P / ds^2 is parallel to the vector $\mathbf{N}$, moreover, one will have:

$$\cos(\mathbf{N}, \mathbf{I}) = \rho\,\frac{d^2x}{ds^2}, \qquad \cos(\mathbf{N}, \mathbf{J}) = \rho\,\frac{d^2y}{ds^2}, \qquad \cos(\mathbf{N}, \mathbf{J}) = \rho\,\frac{d^2z}{ds^2},$$

which are equalities that will give the cosines of the angles that the principal normal makes with the coordinate axes.

The center of curvature Q will likewise have the coordinates:

$$X = x + \rho^2\,\frac{d^2x}{ds^2}, \qquad Y = y + \rho^2\frac{d^2y}{ds^2}, \qquad Z = z + \rho^2\,\frac{d^2z}{ds^2}.$$

2. Set $v = ds / dt$. (If the variable t represents time then v will be the magnitude of the velocity at the point P.) $\mathbf{P}' = v\mathbf{T}$, and if one takes the derivative with respect to time then one will have:

$$\mathbf{P}'' = v'\mathbf{T} + v\,\frac{d\mathbf{T}}{ds}\frac{ds}{dt} = v'\mathbf{T} + \frac{v^2}{\rho}\mathbf{N},$$

where $v^2 / \rho\ \mathbf{N}$ is the normal component of the vector $\mathbf{P}''$ (viz., the acceleration). Therefore, if one sets norm.comp.$\mathbf{P}'' = v^2 / \rho\,\mathbf{N}$ then one will get the equation:

$$\rho = \frac{v^2}{\operatorname{mod}\text{norm.comp.}\mathbf{P}''},$$

which will provide a very simple construction of the center of curvature when one knows the vectors $\mathbf{P}'$, $\mathbf{P}''$. Through the points $P + \mathbf{P}'$, $P + \mathbf{P}''$ (*fig.* 5), draw (in the osculating plane at P) parallels to the vector $\mathbf{N}$, $\mathbf{P}'$ that intersect at M. The perpendicular to PM that issues from the point $P + \mathbf{P}'$ will meet the principal normal $P\mathbf{N}$ at the center of curvature Q, and this will result immediately from considering the similar triangles whose summits are the points P, $P + \mathbf{P}'$, Q and $P + \mathbf{P}'$, M, P.

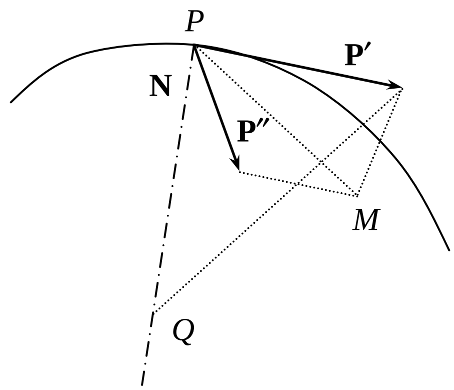

Figure 5.

3. The point $P = O + h\, e^{i\varphi}\, \mathbf{I} + k\, e^{-i\varphi}\, \mathbf{I}$ describes an ellipse whose semi-diameters are $h + k$ and $h - k$. One then has:

$$\mathbf{P}' = h\, e^{i\varphi}\, i\, \mathbf{I} - k\, e^{-i\varphi}\, i\, \mathbf{I}, \qquad \mathbf{P}'' = -\, h\, e^{i\varphi}\, \mathbf{I} - k\, e^{-i\varphi}\, \mathbf{I},$$

so

$$\mathbf{P}'' = O - P.$$

However:

$$O - \mathbf{P}' = P\left(\varphi + \frac{\pi}{2}\right),$$

and (*example* 2) in order to obtain the center of curvature at the point P (*fig.* 6), it will suffice to trace out the parallelogram that circumscribes the ellipse and to carry out the construction that is indicated by the figure.

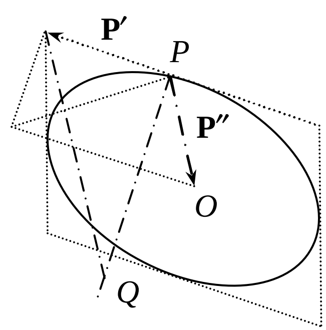

Figure 6.

4. For the cycloid (*see* no. 17), one likewise has:

$$\mathbf{P}' = r\, \mathbf{I} - r\, e^{-i\varphi}\, \mathbf{I} = i(M - P),$$

$$\mathbf{P}'' = r\, e^{-i\varphi}\ i\,\mathbf{I} \qquad = C - P.$$

It results from this that the modulus of the normal component of $\mathbf{P}'$ is equal to $\frac{1}{2}\mathrm{mod}(M - P)$, and consequently:

$$\rho = 2 \bmod (M - P);$$

i.e., the center of curvature $Q = P + \rho\mathbf{N}$ is such that $P + Q = 2M$, and the expression for the point Q is:

$$Q = M + (M - P) = O + r\varphi\,\mathbf{I} - ri\,\mathbf{I} + r^{-i\varphi}\ i\mathbf{I}\ .$$

If we then let O denote the center of curvature of the cycloid at the point $P(\pi)$ then we will see that:

$$O_1 = O + r\pi\,\mathbf{I} - 2ri\mathbf{I},$$

so it easily results for the point Q that:

$$Q = O_1 + r\,(\varphi - \pi)\,\mathbf{I} + ri\,\mathbf{I} - r\,e^{-i(\varphi-\pi)}\ i\,\mathbf{I},$$

and Q describes a cycloid that is equal to the cycloid of the point P that one can deduce from the latter by a translation whose vector $r\pi\ \mathbf{I} - 2ri\ \mathbf{I}$ gives the magnitude, the direction, and the sense.

52. Now, suppose that the line P is planar (i.e., not straight). The vector $\mathbf{N}$ is parallel to the vector $i\mathbf{T}$, and we can give a sign to the curvature $1 / \rho$ such that one will have, for any value of s:

(1) $$\frac{d\mathbf{T}}{ds} = \frac{1}{\rho}\, i\mathbf{T}.$$

a. The locus of the center of curvature of the line P is the locus of the characteristics of the envelope of the normals to that line, because, upon setting $a = Pi\mathbf{T}$, one will see that:

$$\frac{da}{ds} = \mathbf{T}i\mathbf{T} - \frac{1}{\rho} P\mathbf{T},$$

and the development of the regressive product $a\ da / ds$ will give precisely:

$$a\frac{da}{ds} = P\mathbf{T}i\mathbf{T}.\ i\mathbf{T} + \frac{1}{\rho} P\mathbf{T}i\mathbf{T}.\ P = \frac{1}{2\rho}(P + \rho i\mathbf{T}),$$

a relation that was to be established.

b. Set $\alpha = (\mathbf{i}, \mathbf{T})$, and take $\mathbf{I}$ to be a fixed unit vector in the plane of the curve P. α is a function of s such that:

$$\frac{d\alpha}{ds} = \frac{1}{\rho},$$

Indeed:

$$\cos \alpha = \mathbf{I}\; i\mathbf{T}$$

and

$$-\sin \alpha \frac{d\alpha}{ds} = \frac{1}{\rho} \mathbf{I}\; i(i\mathbf{T}) = -\frac{1}{\rho} \mathbf{IT} = -\frac{1}{\rho} \sin \alpha.$$

Therefore, if $\sin \alpha \neq 0$ then:

$$\frac{d\alpha}{ds} = \frac{1}{\rho}.$$

However, in the case where one has $\sin \alpha = 0$ for some certain value of t, if h is a constant such that:

$$\sin(\alpha + h) \neq 0$$

then one will have ([24]):

$$\frac{d(\alpha+h)}{ds} = \frac{1}{\rho} \qquad \text{or} \qquad \frac{d\alpha}{ds} = \frac{1}{\rho}.$$

c. If one gives the number r as a function of s then the line P will be defined, except for its position plane.

Indeed, let $\mathbf{T}_0$ be a constant unit vector. Set:

(2) $$\mathbf{T} = \left(e^{i\int \frac{ds}{\rho}} \right) \mathbf{T}_0 .$$

One will have:

$$\frac{d\mathbf{T}}{ds} = \frac{1}{\rho} i\mathbf{T},$$

which is nothing but formula (1), and since:

$$\frac{dP}{ds} = \mathbf{T},$$

one will have:

(3) $$P = P_0 + \left(\int e^{i\int \frac{ds}{\rho}} ds \right) \mathbf{T}_0 ,$$

a formula in which P_0 represents an arbitrary point of the plane, and the quadratures are performed by starting with a definite value of s. Upon introducing the angle α that was defined in *b*, formula (3) will take the form:

([24]) One will arrive at the same result by considering the spherical indicatrix of the curve P (*see* no. 55).

(4) $$P = P_0 + \left(\int e^{i\alpha} ds\right) \mathbf{T}_0 .$$

The locus of points P passes through P_0, where it admits the line $P_0\mathbf{T}_0$ for its tangent.

For example, if r is constant then the point P will describe a circle, because upon setting $s = \rho\varphi$, formula (4) will give:

$$P = P_0 + \mathbf{T}_0 = P_0 - \rho\, i\, (e^{i\varphi} - 1)\, \mathbf{T}_0 = (P_0 + \rho\, i\mathbf{T}_0) - \rho\, e^{i\varphi}\, i\mathbf{T}_0 ,$$

and it is obvious that the point P will describe a circle whose center is $P_0 + \rho i\mathbf{T}$ and whose radius is ρ.

The reader can, by way of exercise, determine the expression for the point P in the following cases:

Development of the circle……………………….	$\rho^2 = 2as$,
Epicycloid, hypocycloid, cycloid………………..	$\dfrac{s^2}{a^2} + \dfrac{\rho^2}{b^2} = 1$,
Logarithmic spiral……………………………….	$\rho = as$,
Clotoyde (?)……………………………………..	$\rho s = a^2$,
Tractrix…………………………………………..	$\rho = a\sqrt{e^{2s/a} - 1}$,
Catenary………………………………………….	$r = a + \dfrac{s^2}{a}$.

d. Upon supposing that the point P describes a skew curve, an arbitrary point Q of the developable ruled surface that is described by $P\mathbf{T}$ will have the expression $Q = P + u\,\mathbf{T}$, where u is a function of s. In addition, let $P_1(s)$ be a point that describes a planar line of arc length s whose curvature at each point is the same as that of the curve P at the corresponding point. One can represent the points of the developable surface $P\mathbf{T}$ on the plane of the line P_1 by making the point Q correspond to the point:

$$Q_1 = P_1 + u\,\mathbf{T}_1 ,$$

and since:

$$\text{mod } dQ = \text{mod } dQ_1 = \sqrt{\left(1 + \frac{du}{ds}\right)^2 + \frac{u^2}{\rho^2}}\; ds,$$

the correspondence considered will preserve the magnitude of the arcs that are traced on the developable. In general, we will express that property by saying that one can develop the surface $P\mathbf{T}$ on a plane, or also by saying, more simply, that the surface $P\mathbf{T}$ is developable.

Examples. –

1. If the curve P_1 rolls without slipping on another curve P_1 then the trajectory of a point Q that is invariably linked to P_1 is called a *roulette*. Therefore, let O be a fixed

point of the curve P (*fig.* 7), and let Q_0 be the position of Q when the curve P_1 touches P at the point O. If one has:

$$\text{arc } OP = \text{arc } OP_1$$

then one can consider the points P and P_1 to be functions of the same variable s, which is the arc-length that is common to the two curves. Upon letting $\mathbf{T}_1$, ρ_1 be the elements that we called $\mathbf{T}$, ρ when they related to P, but now they relate to P_1, and supposing that the point P takes the position P_1 after rolling through an angle φ (which is a function of s), one will have:

(5) $$\mathbf{T} = e^{i\varphi}\mathbf{T}_1 , \qquad Q - P = e^{i\varphi}(Q_0 - P_1).$$

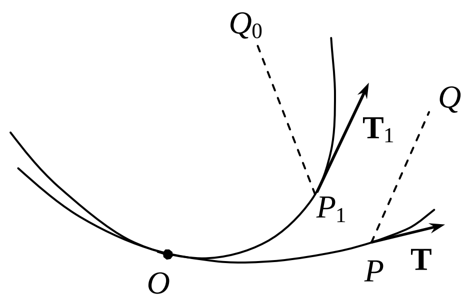

Figure 7.

Take the derivative of the first equality. From formula (1), that will give:

$$\frac{1}{\rho} i\mathbf{T} = \frac{1}{\rho_1} e^{i\varphi}\; i\,\mathbf{T}_1 + e^{i\varphi}\; i\,\mathbf{T}_1 ,$$

or, for the first of formulas (5):

(6) $$\frac{1}{\rho} - \frac{1}{\rho_1} = \frac{d\varphi}{ds},$$

and one thus recovers a formula that is due to Savary. If we now derive the second formula (5), then that will give:

$$\frac{dQ}{ds} - \mathbf{T} = - e^{i\varphi}\,\mathbf{T}_1 + e^{i\varphi}\, i\,(Q_0 - P_1),$$

or

$$\frac{dQ}{ds} = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) i\,(Q - P),$$

which proves that in the case where $\frac{1}{\rho} \neq \frac{1}{\rho_1}$ the normal to the point Q of the roulette that is described by Q will be the line that joins the point Q at the contact point of the moving curve with the fixed curve.

2. If the point P_1 describes the locus of centers of curvature of the curve P then one will have $P_1 = P + \rho\, i\mathbf{T}$, and if one lets s_1 and $1 / \rho_1$ denote the arc-length and curvature, resp., at P_1 of the curve P_1 with $\mathbf{T}_1 = dP_1 / ds$ then one will have $\frac{dP_1}{ds} = \frac{d\rho}{ds}\, i\,\mathbf{T}$. If ρ is an increasing function in the interval considered then one will have:

$$ds_1 = d\rho, \tag{1}$$

$$\mathbf{T}_1 = i\,\mathbf{T}. \tag{2}$$

Therefore, let a and b be two values of s ($a < b$). Formula (1) gives:

$$s_1(b) - s_1(a) = \rho(b) - \rho(a),$$

which shows that *arc* $P_1(a)\,P_2(b)$ *is equal to the difference between the radii of curvature at the points* $P(a)$ *and* $P(b)$. On the other hand, one can deduce from formulas (1) and (2) that:

$$\frac{d\mathbf{T}_1}{ds_1} = \frac{d\mathbf{T}_1}{ds}\frac{ds}{ds_1} = -\frac{1}{\rho}\frac{d\rho}{ds}\mathbf{T} = \frac{1}{\rho}\frac{d\rho}{ds}i\mathbf{T}_1\,,$$

and since $d\rho$ is positive:

$$\rho_1 = \rho\,\frac{d\rho}{ds}.$$

3. Now, let a be a constant number. Set:

$$P_1 = P + a\mathbf{T} \quad \text{and} \quad P_2 = P + \rho\, i\mathbf{T}\,.$$

One has:

$$\frac{dP_1}{ds} = \mathbf{T} + \frac{a}{\rho}\, i\mathbf{T},$$

and in turn:

$$(P_2 - P_1)\, i\frac{dP_1}{ds} = (\rho i\mathbf{T} - a\mathbf{T})\, i\left(\mathbf{T} + \frac{a}{\rho} i\mathbf{T}\right) = \rho\frac{a}{\rho} - a = 0,$$

a relation that proves that *the binormal to the point* P_1 *to the line* P_1 *passes through the center of curvature at the corresponding point* P *of the line* P ([25]).

([25]) Similarly, if the point P describes a skew curve, and if $P_1 = P + a\mathbf{T}$, $P_2 = P + \rho\mathbf{N}$ then one will have:

$$(P_2 - P_1) \mid dP_1 = 0,$$

4. Once again, let $P_1 = P + u_1\,\mathbf{T}$ (*fig.* 8), where u_1 is a function of the arc-length s of the locus of the point P. Set:

$$\alpha_1 = \left(\mathbf{T}, \frac{dP_1}{ds}\right), \qquad M = P + \rho i\mathbf{T}, \qquad Q_1 = P + u_1 \cot \alpha_1\, i\,\mathbf{T}.$$

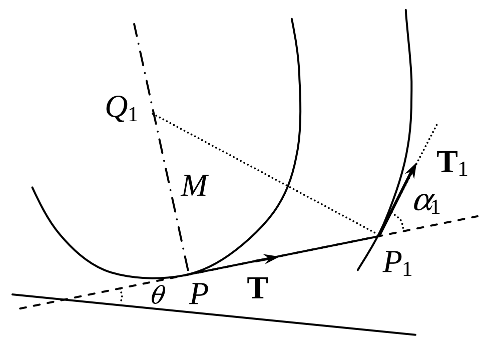

Figure 8.

Since:

$$\frac{dP_1}{ds} = \left(1 + \frac{du_1}{ds}\right)\mathbf{T} + \frac{u_1}{\rho} i\mathbf{T},$$

one will have:

$$\mathbf{T}\frac{dP_1}{ds} = \frac{u_1}{\rho}\,\mathbf{T}i\mathbf{T} \qquad \text{or} \qquad \left(\operatorname{mod}\frac{dP_1}{ds}\right)\sin\alpha_1 = \frac{u_1}{\rho}.$$

However, if s_1 is the arc-length of the curve that is described by the point P_1 then one will have:

$$ds_1 = \left(\operatorname{mod}\frac{dP_1}{ds}\right) ds, \qquad \text{so} \qquad ds_1 = \frac{u_1}{\sin\alpha_1}\frac{ds}{\rho}$$

and

(1) $$ds_1 = \frac{u_1}{\sin\alpha_1} d\theta,$$

in which θ is the angle that the vector $\mathbf{T}$ makes with a fixed vector in the plane (*see* this number, part b).

Since $\left.\frac{dP_1}{ds}\right| \mathbf{T} = 1 + \frac{du_1}{ds}$, one will have:

$$\left(\operatorname{mod}\frac{dP_1}{ds}\right)\cos\alpha_1 = 1 + \frac{du_1}{ds},$$

---

which shows that the normal plane at the point P_1 will pass through the center of curvature P_2 at the point P.

(if u_1 if an increasing function of s), and by virtue of the formulas that we just used in order to prove formula (1), one can also write:

$$(2) \qquad du_1 = (u_1 \cot \alpha_1 - \rho)\, d\theta .$$

Upon observing that that Q_1 is the point at which the normal to the point P meets the normal at P_1 and the fact that M is the center of curvature at the point P, one will have the geometric interpretation of formulas (1), (2), since $\dfrac{u_1}{\sin \alpha_1}$, $u_1 \cot \alpha_1 - \rho$ are the magnitudes of the line segments P_1Q_1, MQ_1 . Formulas (1), (2) are due to Mannheim in a treatise (*Cours de Géométrie descriptive*) to which we shall refer the reader for the applications.

5. We shall always suppose that a is a constant number, and set $P_1 = P + a\mathbf{T}$. What is the necessary and sufficient condition for the point P_1 to describe a straight line? If $1 / \rho_1$ is the curvature at the point P_1 then the locus of points P_1 will be a straight line when one has $1 / \rho_1 = 0$ for all values of s. One can arrive at that condition by using the expression for $d\mathbf{T}_1$, but one finds the result more simply by observing that the point P_1 will describe a straight line when the vector dP_1 has a constant direction; i.e. (*see* no. 37 *h*) when $\dfrac{dP_1}{ds}\dfrac{d^2P_1}{ds^2} = 0$ for all values of s. Now, since we suppose that $1 / \rho \neq 0$, we will have:

$$\frac{dP_1}{ds} = \mathbf{T} + i\frac{a}{\rho}\mathbf{T} = \frac{1}{\rho}(\rho\mathbf{T} + ai\,\mathbf{T}),$$

$$\frac{d^2P_1}{ds^2} = \frac{1}{\rho}i\mathbf{T} - \frac{a}{\rho^2}\mathbf{T} - \frac{a}{\rho^2}\frac{d\rho}{ds}i\mathbf{T} = \frac{1}{\rho^2}\left[-a\mathbf{T} + \left(\rho - a\frac{d\rho}{ds}\right)i\,\mathbf{T}\right],$$

$$\frac{dP_1}{ds}\frac{d^2P_1}{ds^2} = \frac{1}{\rho^3}\begin{vmatrix} \rho & a \\ -a & \rho - a\dfrac{d\rho}{ds}\end{vmatrix}\mathbf{T}\; i\;\mathbf{T} = \frac{1}{\rho^3}\left(\rho^2 + a^2 - a\rho\frac{d\rho}{ds}\right),$$

in such a way that the point P_1 will describe a straight line when, for any value of s:

$$(1) \qquad a\rho = \rho^2 + a^2\,\frac{d\rho}{ds},$$

or even:

$$\rho^2 + a^2 = a_0^2\, e^{2s/a},$$

where a_0 is an arbitrary, non-zero constant.

For any a_0, there exists a value of s for which $\rho = 0$, and since the preceding formulas will persist in the limit $\rho = 0$, if one takes the origin of the arc to be the point that corresponds to $\rho = 0$ then one will find $a_0 = a$, precisely, as the value of the constant, in such a fashion that the desired condition will take the form:

$$\rho^2 = a^2\left(e^{2s/a} - 1\right) . \tag{2}$$

The curve that is described by the point P_1 whose radius of curvature is given by formula (2) is a *tractrix*, and the locus of its centers of curvature is a *catenary*. Moreover, the third example will provide a very simply geometric construction of the tractrix when the corresponding catenary is known, and conversely.

For a catenary, let σ and $1 / \tau$ denote the arc-length and curvature at the point that is precisely the center of curvature of the tractrix P_1 at P_1 . The second example shows us that $d\sigma = d\rho$. However, for $\sigma = 0$, one will have $\rho = 0$, and upon simultaneously supposing that $\sigma = 0$, one can write:

$$\sigma = \rho. \tag{1}$$

Meanwhile, the same example will further give us that:

$$\tau = \rho\,\frac{d\rho}{ds},$$

along with formula (1):

$$\tau = a + \frac{\rho^2}{a},$$

an expression that, when compared to formula (3), will finally give:

$$\tau = a + \frac{\sigma^2}{a},$$

which is a relationship between the arc-length and the radius of curvature for a catenary.

53. Torsion and radius of torsion. – If the line that is described by the point P is not a straight line, and the function P satisfies the conditions that were stated in no. 51 then the unit vectors **T**, **N** will be defined, and upon setting:

$$\mathbf{B}(s) = |\,\mathbf{T}(s)\,\mathbf{N}(s) \tag{1}$$

for any value of s, the vector **B** that is thus defined will be parallel to the binormal at the point P.

Upon assuming that $d\mathbf{B} / ds$ is a well-defined function of s, $d\mathbf{B} / ds$ will be a normal vector to the vector **B**; i.e., it will be parallel to the osculating plane P**TN** at P, at least when the vector $d\mathbf{B} / ds$ is not zero. If one is further given that $\mathbf{B} \mid \mathbf{T} = 0$ then one will have:

$$\frac{d\mathbf{B}}{ds}\bigg|\,\mathbf{T}+\mathbf{B}\bigg|\frac{1}{\rho}\mathbf{N}=\frac{d\mathbf{B}}{ds}\bigg|\,\mathbf{T}=0 \qquad\qquad (\text{because } \mathbf{B}\mid\mathbf{N}=0),$$

and the vector $d\mathbf{B} / ds$ will be zero or parallel to the vector **N**.

We let $1 / \tau$ represent the real number – which can be positive, negative, or zero – such that:

$$\frac{d\mathbf{B}}{ds}=\frac{1}{\tau}\mathbf{N}. \tag{2}$$

The absolute value of the number $1 / \tau$ is then the modulus of the vector $d\mathbf{B} / ds$, and one calls the number $1 / \tau$ the *torsion of the curve P at the point P*, or one also calls the inverse of the torsion the *radius of torsion at the point P.*

THEOREM. – *In order for the line P to be a planar curve, it is necessary and sufficient that the torsion be zero for any value of s.*

Proof. – Indeed, if the curve is planar then **B** will be a vector that is perpendicular to the plane of the curve, and consequently:

$$\frac{d\mathbf{B}}{ds}=0 \qquad \text{or} \qquad \frac{1}{\tau}=0,$$

which shows that the stated condition is indeed necessary. On the other hand, suppose $1 / \tau = 0$ for any value of s. **B** is a constant vector, and if P_0 is a fixed point on the line P then one will have:

$$[(P-P_0)\mid\mathbf{B}]=\mathbf{T}\mid\mathbf{B}=0;$$

i.e., $[(P-P_0)\mid\mathbf{B}$ will be a constant number. However, if one makes P tend to P_0 then the vector $P-P_0$ will tend to a vector that is parallel to **T**, and in turn, $[(P-P_0)\mid\mathbf{B}=0$, which amounts to saying that the curve P will be traced on the plane $P_0\mid\mathbf{B}$, so the condition is indeed sufficient.

54. Frenet formulas. – We have the following formulas that relate to the vector **T**, **N**, **B**, and whose geometric significance is well-known:

$$\mathbf{T}^2=\mathbf{N}^2=\mathbf{B}^2=1, \tag{1}$$

$$\mathbf{N}\mid\mathbf{B}=\mathbf{B}\mid\mathbf{T}=\mathbf{T}\mid\mathbf{N}=0, \tag{2}$$

$$\mathbf{T}=\mid\mathbf{NB}, \qquad \mathbf{N}=\mid\mathbf{BT}, \qquad \mathbf{B}=\mid\mathbf{TN}, \tag{3}$$

which expresses the idea that **T**, **N**, **B** are unit vectors (1) that are mutually perpendicular (2). The trivector **TNB** is, moreover, positive and equal to $\frac{1}{6}\,\omega$, and since we suppose implicitly that:

$$\frac{dP}{ds} = \mathbf{T},$$

in order to define the sign of **T**, one will get ([26]):

$$(4) \qquad \begin{cases} \dfrac{d\mathbf{T}}{ds} = \dfrac{1}{\rho}\mathbf{N}, \\[2mm] \dfrac{d\mathbf{B}}{ds} = \dfrac{1}{\tau}\mathbf{N}, \\[2mm] \dfrac{d\mathbf{N}}{ds} = -\dfrac{1}{\rho}\mathbf{T} - \dfrac{1}{\tau}\mathbf{B}. \end{cases}$$

The first two of these formulas are known already [no. 51, formula (2) and no. 53, formula (2)], and in order to prove the third one, it will suffice to consider the expression $\mathbf{N} = |\,\mathbf{BT}$, from which one deduces immediately that:

$$\frac{d\mathbf{N}}{ds} = \left|\,\mathbf{B}\frac{1}{\rho}\mathbf{N} + \right|\,\frac{1}{\tau}\mathbf{NT} = \frac{1}{\rho}|\,\mathbf{B}\,\mathbf{N} + \frac{1}{\tau}|\,\mathbf{NT} = -\frac{1}{\rho}\mathbf{T} - \frac{1}{\tau}\mathbf{B}\,.$$

Upon representing three rectangular, unit vectors by **I**, **J**, **K**, one can write, by starting with formulas (4):

([26]) The linear equations (4) show that a curve is determined, up to position, as a function of the arc-length when one possesses expressions for the curvature and torsion – viz., $1 / \rho$ and $1 / \tau$, resp. – as functions of arc-length. Indeed, upon taking two rectangular unit vectors $\mathbf{T}_0$ and $\mathbf{N}_0$ and setting $\mathbf{B}_0 = |\,\mathbf{T}_0\,\mathbf{N}_0$, one will express **T**, **N**, **B** as functions of the constants $\mathbf{T}_0$, $\mathbf{N}_0$, $\mathbf{B}_0$ by developing them in a convergent series [*see* G. PEANO, "Integrazione per serie delle equazioni differenziali lineari," Atti Acc. Torino, 1887.] Therefore, if P_0 is a fixed point then one will have $P = P_0 + \int \mathbf{T}\, ds$, and the curve will be well-defined, up to position. It will pass through the point P_0, and the lines $P_0\mathbf{T}_0$, $P_0\mathbf{N}_0$, $P_0\mathbf{B}_0$ will be the tangent, principal normal, and binormal at that point, respectively.

$$(5)\qquad \left\{\begin{aligned}
&\frac{d}{ds}\cos(\mathbf{T},\mathbf{I}) = \frac{1}{\rho}\cos(\mathbf{N},\mathbf{I}),\\
&\frac{d}{ds}\cos(\mathbf{T},\mathbf{J}) = \frac{1}{\rho}\cos(\mathbf{N},\mathbf{J}),\\
&\frac{d}{ds}\cos(\mathbf{T},\mathbf{K}) = \frac{1}{\rho}\cos(\mathbf{N},\mathbf{K}),\\
&\frac{d}{ds}\cos(\mathbf{B},\mathbf{I}) = \frac{1}{\tau}\cos(\mathbf{N},\mathbf{I}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\frac{d}{ds}\cos(\mathbf{N},\mathbf{I}) = -\frac{1}{\rho}\cos(\mathbf{T},\mathbf{I}) - \frac{1}{\tau}\cos(\mathbf{B},\mathbf{I}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}\right.$$

because, for example:

$$\left.\frac{d\mathbf{T}}{ds}\right| \mathbf{I} = \frac{1}{\rho}\mathbf{N} \mid \mathbf{I},$$

or

$$\frac{d}{ds}(\mathbf{T} \mid \mathbf{I}) = \frac{1}{\rho}(\mathbf{N} \mid \mathbf{I}).$$

Formulas (4), when they are known in the form (5), are due to Frenet, although they are often called the *Serret formulas*.

55. Spherical indicatrix and contingency angle. – Let $\mathbf{I}(t)$ be a unit vector with a non-zero derivative in the interval considered, and let O be a fixed point. If one sets:

$$Q = O + \mathbf{I}$$

then the point Q will describe a curve Q on the spherical surface with center O and unit radius that one calls the *spherical indicatrix of the vector* **I**. In the particular case where the vector **I** is parallel to a fixed plane, the curve Q will be an arc of a great circle on the sphere. If one represents the arc-length that is described by the point Q by φ then one will have:

$$(1)\qquad d\varphi = \operatorname{mod} dP.$$

If **T**, **N**, **B** are the vectors that we already considered relative to the curve P, and if we set, in an analogous manner:

$$Q_1 = O + \mathbf{T}, \quad Q_2 = O + \mathbf{B}, \quad Q_3 = O + \mathbf{N},$$

then the points Q_1, Q_2, Q_3 will describe the *spherical indicatrices* for the *tangent, binormal,*and *principal normal,* respectively, of the curve *P*, which are curves whose arc-lengths φ_1, φ_2, φ_3, by virtue of formula (1) and the Frenet formulas, will be given by the equations:

$$d\varphi_1 = \frac{1}{\rho}\,ds,$$

$$d\varphi_2 = \frac{1}{\operatorname{mod}\tau}\,ds,$$

$$d\varphi_3 = \sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}}\,ds,$$

the first two of which provide a geometric interpretation for the curvature and modulus of the torsion in terms of the spherical indicatrices of the tangents and binormals to the curve *P*. For even more symmetry, we say that $1 / \rho$ and $1 / \tau$ are the *first* and *second curvature* of the curve *P* at the point *P*, which leads us naturally to let $1 / \lambda$ denote the *third curvature* (or *normal curvature*) when we set:

$$\frac{1}{\lambda} = \sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}},$$

with the choice of + sign in front of the radical. The latter of the three preceding formulas then expresses the idea that the elementary arc-length of the spherical indicatrix of the principal normal at the point that corresponds to the point *P* is equal to the product of the third curvature at *P* times *ds*.

We shall return shortly to the vector **I** in order to also call $d\varphi$ the *contingency angle* of the vector **I**, which one usually expresses by saying that the vector **I** makes the angle $d\varphi$ with the infinitely close vector $\mathbf{I}(t + dt)$. Moreover, the exact significance of these words is nothing but that which is expressed by formula (1), and the geometric interpretation is provided by the spherical indicatrix of the vector **I**. In particular, if the vector **I** is parallel to a fixed plane then $d\varphi$ will represent the angle that is defined between $d\mathbf{I}$ and a fixed vector in the plane.

We say that the contingency angle of the vector $\mathbf{J}(t)$ – which is assumed to be non-zero, along with its derivative, in the interval considered – is the contingency angle of the unit vector $\mathbf{J} / \operatorname{mod} \mathbf{J}$. If we represent the contingency angle of the vector **J** by $d\psi$ then we will have:

$$d\psi = \frac{\operatorname{mod}(\mathbf{J}\,d\mathbf{J})}{(\operatorname{mod}\mathbf{J})^2}. \tag{1}'$$

Indeed, the vector $d\mathbf{I}$ is perpendicular to the vector **I** and $\operatorname{mod}(\mathbf{I}\ d\mathbf{I}) = \operatorname{mod} d\mathbf{I}$. Formula (1) then gives:

$$d\varphi = \operatorname{mod}(\mathbf{I}\ d\mathbf{I}),$$

and upon setting $\mathbf{J} = (\mathrm{mod}\ \mathbf{J})\ \mathbf{I}$:

$$d\mathbf{J} = (\mathrm{mod}\ \mathbf{J})\ d\mathbf{I} + (d\ \mathrm{mod}\ \mathbf{J})\mathbf{I},$$

or

$$\mathbf{J}\ d\mathbf{J} = (\mathrm{mod}\ \mathbf{J})^2\ \mathbf{I}\ d\mathbf{I}.$$

Moreover, by definition, $d\psi = d\varphi$, which establishes formula (1)′, which we shall give some applications of in the following chapter.

CHAPTER III.

APPLICATIONS.

In this chapter, we shall show how the Frenet formulas can easily lend themselves to the study of the principal properties of curves and the ruled surfaces that relate to a curve.

The hypotheses and conventions that we shall retain in this chapter will be the following ones: The vector **T**, **N**, **B** are defined, along with their derivatives, at every point of the curve considered. The number $1 / \rho$ (viz., the curvature) is not annulled when the curve is not a straight line. Likewise, if the curve is not a planar curve then the number $1 / \tau$ (viz., torsion) is always non-zero ([27]). Finally, the notations $\mathbf{T}_1$, $\mathbf{N}_1$, $\mathbf{B}_1$, ρ_1, τ_1 will have the same significance at a point P_1 that they have at the point P without the primes.

§ 1. – HELIX.

56. If the point P describes a planar curve then the vector **B**, which is parallel to the normal, will always be perpendicular to the plane of the curve, and consequently, the line $P\mathbf{B}$ will describe a cylindrical surface such that the line P will be a cross-section. An arbitrary point P_1 of the cylindrical surface that is generated by the line $P\mathbf{B}$ will be given by the relation $P_1 = P + u\mathbf{B}$. The numbers s, u that determine the position of P_1 on the surface will be called the *coordinates* of P_1, upon taking the curve P itself to be the coordinate axis and the point P_0 ($s_0 = 0$) for the coordinate origin. The number s is the *abscissa* and the number u is the *ordinate* of the point P_1. In particular, if u is a function of s then the point P_1 will describe a curve on the cylindrical surface when s varies.

Upon considering a planar rectangular coordinate system, we can make the point P_1 of the cylinder correspond to the point of the plane whose coordinates are s and u, and conversely, and the fact that a similar correspondence is established by saying that one develops the cylindrical surface onto the plane ([28]).

([27]) Our goal is not to study singular points. With the restrictions that we just stated, we are excluding the singular points on the curve P exactly.

([28]) Let O be a fixed point, and let $\mathbf{I}(t)$ be a unit vector whose derivative is not zero. An arbitrary point P of the conical surface that is generated by the line $O\mathbf{I}$ will be given by the relation $P = O + u\mathbf{I}$, and if u is a function of t then the point P will describe a curve on the conical surface when t varies. If O is a fixed point, and $\mathbf{J}$ is a constant unit vector in a fixed plane then upon setting:

$$P_1 = O_1 + u\, e^{i\varphi}\,\mathbf{J} \qquad \text{with} \qquad d\varphi = \operatorname{mod} d\mathbf{I} \qquad (\textit{see}\text{ no. 55}),$$

we can represent the cone $O\mathbf{I}$ in the fixed plane, and, as for the cylinder, we will say that one develops the cone onto the plane, because:

$$\operatorname{mod} dP = \operatorname{mod} dP_1 = \sqrt{du^2 + (u \operatorname{mod} d\mathbf{I})^2}\ .$$

57. One says *helix* to refer to any curve that is traced upon a cylinder and cuts the generators of that cylinder with a constant angle, except for the very particular cases in which that angle is zero or equal to $\pi / 2$. The development of the cylinder onto the plane then transforms a helix into a line.

THEOREM I. – *The ordinate of an arbitrary point of a helix that is traced upon the cylinder P***B** *is proportional to its abscissa. Conversely, any curve that is traced upon the cylinder P***B** *is a helix if the ordinate of an arbitrary point is proportional to its abscissa.*

Proof. – We have said that if u is a function of s then the point:

$$P_1 = P + u\mathbf{B}$$

will describe a line on the cylinder. By taking the derivative of the two sides of the equation, in which **B** will figure as a constant vector, one gets:

$$\frac{dP_1}{ds} = \mathbf{T} + \frac{du}{ds}\mathbf{B}.$$

If φ is the constant angle by which the curve P_1 cuts the generators of the cylinder then:

$$\tan \varphi = \frac{\operatorname{mod}(dP_1)\mathbf{B}}{dP_1 \mid \mathbf{B}} = \frac{\operatorname{mod}\mathbf{BT}}{du} ds = \frac{ds}{du}$$

or

$$du = \cot \varphi \, ds.$$

Upon supposing that the point P_1 coincides with the point P_0 at the origin (where $s = 0$), one will then have the equation:

$$u = s \cot \varphi,$$

which indeed shows that the ordinate (viz., u) is proportional to the abscissa (viz., s) at any point of the helix that is described by P_1.

Conversely, if $u = as$ ($a \neq 0$) then the point P will describe a curve that cuts the generators at an angle whose cotangent is a; i.e., the point P_1 will describe a helix precisely.

Remark. – Let P_1 be a point of the helix that cuts the generators of the cylinder P**B** at a constant angle φ. One has:

(1) $$P_1 = P + s \cot \varphi \, \mathbf{B},$$

so

(2) $$\frac{dP_1}{ds} = \mathbf{T} + \cot \varphi \, \mathbf{B};$$

however, upon letting s_1 denote the arc-length of the curve P_1, so:

$$ds_1 = \text{mod } dP_1 = \sqrt{1+\cot^2\varphi}\, ds$$

and

(3) $$ds = \sin\varphi\, ds_1 ,$$

one will have $s = s_1 \sin \varphi$, which permits us to construct s_1 when we are given s and φ.

THEOREM II. – *At any point of the helix P_1, the ratio of the curvature to the torsion is constant, and conversely, if the ratio of the curvature to the torsion is constant at each point of a curve then that curve will be a helix.*

Proof. – Indeed, one has:

$$\mathbf{T}_1 = \frac{dP_1}{ds_1} = \frac{dP_1}{ds}\frac{ds}{ds_1},$$

and by virtue of formulas (2), (3):

(4) $$\mathbf{T}_1 = \sin\varphi\,\mathbf{T} + \cos\varphi\,\mathbf{B}.$$

Now, $d\mathbf{T}_1 / ds = \sin \varphi / \rho\, \mathbf{N}$ implies that $d\mathbf{T}_1 / ds_1 = \sin2 \varphi / \rho$, and if ρ_1 is the radius of curvature at the point P_1 then:

(3) $$\rho_1 = \frac{\rho}{\sin^2\varphi}, \qquad \text{since} \quad \mathbf{N}_1 = \mathbf{N}.$$

For the vector $\mathbf{B}_1$, one will have:

$$\mathbf{B}_1 = |\,\mathbf{T}_1\,\mathbf{N}_1 = |\,(\sin\varphi\,\mathbf{T} + \cos\varphi\,\mathbf{B})\,\mathbf{N} = \sin\varphi\,|\,\mathbf{TN} + \cos\varphi\,|\,\mathbf{BN},$$

so

(6) $$\mathbf{B}_1 = \sin\varphi\,\mathbf{B} - \cos\varphi\,\mathbf{T}.$$

Now, the derivative of $\mathbf{B}_1$ with respect to s_1 is:

$$\frac{d\mathbf{B}_1}{ds_1} = -\frac{\sin\varphi\cos\varphi}{\rho}\,\mathbf{N},$$

and if one takes into account the definition of $1 / \tau_1$ and the second of formulas (5) then one will get:

(7) $$\tau_1 = -\frac{\rho}{\sin\varphi\cos\varphi},$$

a formula that will indeed prove that $\rho_1 / \tau_1 = -\cot \varphi$ when one compares it to (5); i.e., the ratio of the curvature to the torsion will be constant at every point P_1 of the curve P_1 .

Conversely, if:

$$\frac{\rho_1}{\tau_1} = a \qquad (a \text{ constant})$$

then the first two Frenet formulas will give:

$$a\frac{d\mathbf{T}_1}{ds_1} = \frac{d\mathbf{B}_1}{ds_1};$$

i.e.:

$$a\,\mathbf{T}_1 - \mathbf{B}_1 = \mathbf{K},$$

where **K** is a well-defined constant vector of modulus $\sqrt{1+a^2}$. One deduces $\mathbf{K} \mid \mathbf{T}_1 = a$ from this, which expresses the constancy of the angle ($\mathbf{T}_1$, **K**), and in turn, the fact that the curve P_1 is a helix that is traced upon the cylinder that is described by the line $P_1\mathbf{K}$.

THEOREM III. – *At every point P_1 of a helix that is traced on the cylinder $P\mathbf{B}$, the binormal will be normal to the cylinder at P_1, and conversely, if the binormal to a curve P_1 on the cylinder $P\mathbf{B}$ at a point P_1 is the normal to the cylinder at P_1 then the curve P_1 will be a helix.*

Proof. – If the curve P_1 is a helix on the cylinder P_1 then the normal $P_1\mathbf{N}_1$ at the point P_1 will be parallel to the vector **N**; i.e., it will be perpendicular to the tangent plane to the cylinder at P_1.

Conversely, if the vector $\mathbf{N}_1$ at each point P_1 of a curve P_1 that is traced on the cylinder is parallel to the normal to the cylinder at P_1 then it will be perpendicular to **B**; i.e., $\mathbf{N}_1 \mid \mathbf{B} = 0$, or $1 / \rho_1\ \mathbf{N}_1 \mid \mathbf{B} = 0$, or by virtue of the first Frenet formula, $\left.\frac{d\mathbf{T}_1}{ds_1}\right| \mathbf{B} = 0$; however, $\frac{d}{ds_1}(\mathbf{T}_1 \mid \mathbf{B}) = \left.\frac{d\mathbf{T}_1}{ds_1}\right| \mathbf{B}$, so $\mathbf{T}_1 \mid \mathbf{B} = \text{const.}$; i.e., the vector $\mathbf{T}_1$ makes a constant angle with the vector **B**, precisely, so the curve P_1 will be a helix, since it cuts the generators of the cylinder at a constant angle.

58. A helix is called *ordinary* or *circular* when the cylinder on which it is traced is a surface of revolution.

THEOREM. – *The ordinary helix is the only skew curve with constant curvature and torsion* (Puiseux's theorem).

Proof. – If the point P describes a circle then ρ will be constant, and formulas (5), (7) of the preceding number will indeed show that *the ordinary helix is a skew curve whose curvature and torsion are constant*.

Conversely, if ρ_1 and τ_1 are constants for a curve P_1 then the ratio ρ_1 / τ_1 will be likewise constant, and the point P_1 will describe a helix. Formula (5) will then give ρ = const.; i.e. (*see* no. 52, *c*), the point P will describe a circle, and the helix P_1 will be an ordinary helix.

a. Let O be a fixed point, let $\mathbf{I}$ be a fixed unit vector in the plane, and let r be a constant number. If $P = O + r\,e^{i\theta}\,\mathbf{I}$ then the line $P\mathbf{B}$ will describe a cylinder of revolution that has the line $O\mathbf{B}$ for its axis, and a circle with center O and radius r for its cross section. Since $s = r\theta$, the point:

$$P_1 = O + r\,e^{i\theta}\,\mathbf{I} + r\theta\cot\varphi\,\mathbf{B}$$

will describe an ordinary helix.

The points $P_1(0)$, $P_1(2\pi)$ (where θ is the independent variable) are situated on the generator of the cylinder that passes through the point $P_1(0)$, and the distance between them is called the *step* of the helix. Now, we have:

$$P_1(0) = O + r\mathbf{I},$$
$$P_1(2\pi) = O + r\mathbf{I} + 2\pi r\cot\varphi\,\mathbf{B},$$

and

$$\text{mod}\,P_1(0)\,P_1(2\pi) = \pm\,2\pi r\cot\varphi,$$

according to whether $\varphi < \pi / 2$ or $\varphi > \pi / 2$. The number $\pm\,r\cot\varphi$ is called the *reduced step*, and one obtains it by dividing the step by 2π.

b. The derivative of P_1 with respect to θ:

$$P_1' = r\,e^{i\theta}\,i\,\mathbf{I} + r\cot j\,\mathbf{B}$$

gives a very simple construction of the tangent to the point P_1 by making use of the reduced step $\pm\,r\cot\varphi$.

c. The line P_1P_1' describes a developable surface that one calls the *ordinary developable helicoid*. If a is a constant number then:

$$P_2 = P_1 + aP_1'$$

will describe a curve that is traced on the helicoidal surface P_1P_1' and, if d is its distance from the line $O\mathbf{B}$ then one will have:

$$\tfrac{1}{2}\,\text{mod}(O\mathbf{B}).\ d = \text{mod}\,P_2O\mathbf{B} \qquad \text{or} \qquad d = 2\,\text{mod}\,P_2\,O\mathbf{B}.$$

However:

$$P_2O\mathbf{B} = P_1O\mathbf{B} + aP_1'\,O\mathbf{B} = rO\mathbf{B}(e^{i\theta}\,\mathbf{I}) + arO\mathbf{B}(e^{i\theta}\,i\mathbf{I}),$$

so

$$d = 2r\,\sqrt{1+a^2}\,.$$

Consequently, the point P_2 will describe a curve that is traced on a cylinder of revolution with axis $O\mathbf{B}$ and whose cross-sectional radius will be $r\sqrt{1+a^2}$. One will have:

$$P_2' = P_1' + aP_1''.$$

However, $P_1' \mid \mathbf{B} = r \cot \varphi$, $P_1'' \mid \mathbf{B} = 0$, and in turn:

$$P_2' \mid \mathbf{B} = r \cot \varphi ;$$

i.e., the point P_2 will describe a helix.

The step of the helix at the point P_2 will be the modulus of the vector $P_2(2\pi) - P_2(0)$, and since:

$$P_1'(0) = P_1'(2\pi),$$

one will have:

$$P_2(2\pi) - P_2(0) = P_1(2\pi) - P_1(0),$$

which will prove the equality of the steps that are described by P_1 and P_2 .

d. If one sets $Q = O + (a + r\theta \cot \varphi)\, \mathbf{B}$ then the line QP will describe an ordinary skew helicoid whose line $O\mathbf{B}$ will be the line of striction. It will have a director plane if $a = 0$, and a director cone if $a \neq 0$.

On the other hand, $P_1 - Q = r\, e^{i\theta}\, \mathbf{I} - a\, \mathbf{B}$ or mod $QP_1 = \sqrt{r^2 + a^2}$, and the point $P_2 = Q + b(P_1 - Q)$, where b is a constant number, will describe a helix that is traced on a cylinder with axis $O\mathbf{B}$, and whose step will be equal to that of the helix that is described by the point P_1.

§ 2. – RULED SURFACES THAT RELATE TO A CURVE.

When t varies, the planes $P\mathbf{NB}$, $P\mathbf{BT}$ will be tangents to two developable surfaces that one calls the *polar surface* and the *rectifying surface* of the curve P. Similarly, the lines $P\mathbf{N}$, $P\mathbf{B}$ will generate ruled surfaces that one calls the *surface of principal normals* and the *surface of binormals* of the curve P, respectively. As we have already seen, the line $P\mathbf{T}$ describes the osculating developable of the curve P, which is again the envelope of the plane $P\mathbf{TN}$.

These are precisely the surfaces that we presently propose to study.

59. Polar surface. – Set:

$$\alpha = P\mathbf{NB},$$

and take the derivatives (cf., the Frenet formulas):

$$\frac{d\alpha}{ds} = \mathbf{TNB} - P\left(\frac{1}{\rho}\mathbf{T} + \frac{1}{\tau}\mathbf{B}\right)\mathbf{B} + \frac{1}{\tau}P\mathbf{NN} = \mathbf{TNB} + \frac{1}{\rho}P\mathbf{BT},$$

or

(1) $$\frac{d\alpha}{ds} = \frac{1}{\rho}(P + \rho\mathbf{N})\,\mathbf{BT}\,;$$

i.e., the plane $d\alpha / ds$ is parallel to the rectifying plane at P.

The characteristic of the envelope α in the plane α, or the generator of the polar surface that corresponds to the point P, is the line $\alpha\, d\alpha / ds$, or even, from formula (1), the line $(P + \rho\mathbf{N})\,\mathbf{B}$.

Therefore:

The generator of the polar surface that corresponds to the point P passes through the center of the curvature $P + \rho\mathbf{N}$ and is parallel to the binormal.

In order to determine the edge of regression of the polar surface, one can consider that surface as being generated by the line:

$$a = (P + \rho\mathbf{N})\,\mathbf{B},$$

and develop the regressive product $a\, da / ds$ in the plane $P\mathbf{NB}$. Now:

$$\frac{da}{ds} = \frac{1}{\rho}(P + \rho\mathbf{N})\,\mathbf{N} + \left(\mathbf{T} - \mathbf{T} - \frac{\rho}{\tau}\mathbf{B} + \frac{d\rho}{ds}\mathbf{N}\right)\mathbf{B} = \frac{1}{\tau}P\mathbf{N} - \frac{d\rho}{ds}\mathbf{BN},$$

and if the curve P is skew (viz., $1 / \tau \neq 0$) then:

$$\frac{da}{ds} = \frac{1}{\tau}\left(P - \tau\frac{d\rho}{ds}\mathbf{B}\right)\mathbf{N}\,.$$

The line da / ds will then be parallel to the principal normal to the point P, its distance to the line $P\mathbf{N}$ will be $-\tau\, d\rho / ds$, and consequently, the point $P + \rho\mathbf{N} - \tau\, d\rho / ds\,\mathbf{B}$ will be common to the lines a and da / ds; i.e., *the edge of regression of the polar surface will be described by the point $P + \rho\mathbf{N} - \tau\, d\rho / ds\,\mathbf{B}$.*

b. If the curve P is planar then the polar surface will be a cylinder. The cross-section is the locus of centers of curvature $P + \rho\mathbf{N}$ of the curve. It is the limiting position of the sphere that is determined by four points of the curve that tend to P. By analogy, the circle with center $P + \rho\mathbf{N}$ (viz., the center of curvature) that passes through the point P will be called the *osculating circle* to the curve P at the point P.

c. We say that the curve P is a *spherical curve* when it is traced on a sphere. In order for the curve P to be a spherical curve, it is necessary and sufficient that the point $P + \rho\mathbf{N} - \tau\, d\rho / ds$ be a fixed point, a condition that one can write as:

$$\frac{d}{ds}\left(P + \rho\mathbf{N} - \tau\frac{d\rho}{ds}\mathbf{B}\right) = 0,$$

or, in an equivalent manner:

$$\frac{\rho}{\tau} + \frac{d}{ds}\left(\tau\frac{d\rho}{ds}\right) = 0.$$

60. Rectifying surface. – Set:

$$\alpha = P\mathbf{BT}.$$

The Frenet formulas give:

$$\frac{d\alpha}{ds} = -\frac{1}{\rho\tau}P(\rho\,\mathbf{TN} + \tau\,\mathbf{NB}).$$

The line $\alpha\, d\alpha / ds$ is the generator of the rectifying surface that corresponds to the point P, which is, moreover, situated in the plane α, $d\alpha / ds$; i.e., *the rectifying surface contains the curve P*.

In order to determine a second-order form whose position describes the rectifying surface, it then suffices to determne the regressive product of the bivectors of the forms α, $-\, d\alpha / ds$. These bivectors **BT**, $\rho\,$**TN** + $\tau\,$**NB** are such that:

$$\mathbf{BT}(\rho\,\mathbf{TN} + \tau\,\mathbf{NB}) = \mathbf{TNB}\,(\rho\,\mathbf{T} - \tau\,\mathbf{B}),$$

and if α is a second-order form that generates the rectifying surface then one can set:

$$a = P(\rho\,\mathbf{T} - \tau\,\mathbf{B}).$$

The determinant:

$$\delta = \begin{vmatrix} \rho & \tau \\ \dfrac{d\rho}{ds} & \dfrac{d\tau}{ds} \end{vmatrix}$$

will be annulled when ρ / τ is a constant, and conversely, if the conditions that we just imposed on ρ and τ are satisfied.

We will have $(\rho\,\mathbf{T} - \tau\,\mathbf{B})\dfrac{d}{ds}\,(\rho\,\mathbf{T} - \tau\,\mathbf{B}) = -\,\delta\mathbf{TB}$, and since the condition $(\rho\,\mathbf{T} - \tau\,\mathbf{B})\dfrac{d}{ds}\,(\rho\,\mathbf{T} - \tau\,\mathbf{B}) = 0$, or $d = 0$, or $\rho / \tau =$ const. for any s expresses the idea that the direction of the vector $\rho\,\mathbf{T} - \tau\,\mathbf{B}$ is constant, we will have:

The helix is the only skew curve that has a cylinder for its rectifying surface.

Upon developing the regressive product $\alpha\, d\alpha / ds$ on the plane P**BT**, one obtains a first-order form with the same position as the form:

$$\tau(\rho\,\mathbf{T} - \tau\,\mathbf{B}) + \delta P,$$

and consequently:

$$\tan\,\varphi = \pm\frac{\tau}{\rho},$$

so

(1) $$d\varphi = \pm\frac{\delta}{\rho^2+\tau^2}\,ds,$$

and if ψ is the contingency angle of the vector $\rho\,\mathbf{T} - \tau\,\mathbf{B}$ then:

(2) $$d\psi = \frac{\operatorname{mod}(\rho\mathbf{T}-\tau\mathbf{B})\,\dfrac{d}{ds}(\rho\mathbf{T}-\tau\mathbf{B})}{[\operatorname{mod}(\rho\mathbf{T}-\tau\mathbf{B})]^2}\;ds = \frac{\delta}{\rho^2+\tau^2}\,ds\,.$$

Upon developing the rectifying surface on a plane, the angle φ will not change, and ψ will become the angle that the transform of the generator makes with a fixed line in the plane. However, formulas (1), (2) give $d\psi = \pm\, d\varphi$ or $\psi \mp \varphi$ = const. Therefore:

Upon developing the rectifying surface onto a plane, the curve P will transform into a line ([29]) (Lancret's theorem)

61. Surface of principal normals. – If we set:

$$a = P\mathbf{N}$$

then we will get:

$$\frac{da}{ds} = \mathbf{TN} + P\frac{d\mathbf{N}}{ds} = -\,\frac{1}{\rho}\,P\mathbf{T} - \frac{1}{\tau}\,P\mathbf{B} + \mathbf{TN},$$

and in turn:

$$\frac{da}{ds}\,\frac{da}{ds} = 2\;\mathbf{TNB}\frac{d\mathbf{N}}{ds} = -\,\frac{2}{\tau}\,P\mathbf{TNB}.$$

The surface of principal normals of a skew curve is a skew ruled surface, and conversely.

The tangent plane to the surface of normals at the point P is the plane $P\;da\,/\,ds$ = plane $P\mathbf{TN}$. Consequently:

([29]) Indeed, if the point P describes a planar line then the point $P_1 = P + u\mathbf{T}$ will describe a straight line when the vector dP_1 makes a constant angle θ with a fixed vector $\mathbf{I}$ in the plane (no. 52, *c* and no. 50). Now, if θ is the angle that $\mathbf{T}$ makes with $\mathbf{I}$, and α is the angle between $\mathbf{T}$ and dP_1 then the condition $d\theta_1 = 0$ will be equivalent to $d\theta = \pm\, d\alpha$.

The osculating plane at P to the curve P is the tangent to the surface of normals at the point P, or, in other words, *the osculating developable to the curve P and the surface of principal normals agree along the curve P.*

Likewise, since the plane that is tangent to the surface of normals at the center of curvature $P + \rho\mathbf{N}$ is:

$$(P + \rho\mathbf{N})\ \frac{da}{ds} = \text{plane } P\mathbf{NB},$$

one will have:

The surface of normals and the polar surface agree along the locus of the centers of curvature for the curve P.

The asymptotic plane for the generator is nothing but the plane $a.\dfrac{da}{ds}\,\omega$. Now:

$$\frac{da}{ds}\,\omega = -\frac{1}{\rho}\ \mathbf{T} - \frac{1}{\tau}\mathbf{B},$$

so

$$a.\frac{da}{ds}\,\omega = P\left(\frac{1}{\rho}\mathbf{TN} - \frac{1}{\tau}\mathbf{NB}\right).$$

The bivector of the form $a.\dfrac{da}{ds}\,\omega$ is therefore $\dfrac{1}{\rho}\ \mathbf{T} - \dfrac{1}{\tau}\mathbf{B}$, whose index is the vector $\dfrac{1}{\rho}\ \mathbf{B} - \dfrac{1}{\tau}\mathbf{T}$, which is parallel to the vector $\rho\,\mathbf{T} - \tau\,\mathbf{B}$. Consequently:

The asymptotic plane for the generator of the surface of normals that passes through P is the perpendicular to the generator of the rectifying surface that passes through the same point P.

Since the central point of the generator a is:

$$\left[a\left|\left(a\omega.\frac{da}{ds}\,\omega\right)\right]\right.\frac{da}{ds},$$

if one develops the progressive and regressive product:

$$\left[a\left|\left(a\omega.\frac{da}{ds}\,\omega\right)\right]\right.\frac{da}{ds} = -\frac{1}{6}\left[\left(\frac{1}{\rho^2} + \frac{1}{\tau^2}\right)P + \frac{1}{\rho}\mathbf{N}\right],$$

then one will get:

The line of striction of the surface of normals can be considered to be described by the point $P + \lambda^2 / \rho\,\mathbf{N}$, *for which* $\frac{1}{\lambda} = \sqrt{\frac{1}{\rho^2} + \frac{1}{\tau^2}}$ *expresses its normal curvature.*

If we set $P_1 = P + P + \lambda^2 / \rho\,\mathbf{N}$ then we will have:

$$\frac{dP_1}{ds} = \left(\frac{d}{ds}\frac{\lambda^2}{\rho}\right)\mathbf{N} - \frac{\lambda^2}{\tau}\left(\frac{1}{\rho}\mathbf{B} - \frac{1}{\tau}\mathbf{T}\right),$$

or even:

$$\frac{dP_1}{ds} = \left(\frac{d}{ds}\frac{\lambda^2}{\rho}\right)\mathbf{N} - \frac{\lambda^2}{\tau}\Big|\,\mathbf{N}\frac{d\mathbf{N}}{ds},$$

so (*see* no. 46, *d*):

The distribution parameter of the generator P$\mathbf{N}$ *will be the number* λ^2 / τ.

62. Surface of binormals. – If we set:

$$a = P\mathbf{B}$$

then we will get:

$$\frac{da}{ds} = \mathbf{TB} + \frac{1}{\tau}P\mathbf{N},$$

and in turn:

$$\frac{da}{ds}\,\frac{da}{ds} = -\frac{2}{\tau}\,P\mathbf{TNB},$$

so:

The surface of binormals of a skew curve will be a skew ruled surface, and conversely.

Moreover, since the plane $P\ da / ds$ = plane $P\mathbf{TB}$ is tangent to the surface of binormals at P, one can further say that:

The surface of binormals and the rectifying surface of a curve P will agree along the curve P.

The asymptotic plane for the generator a is:

$$\text{plane } a.\frac{da}{ds}\,\omega = \text{plane } P\mathbf{B}\left(\frac{1}{\tau}\mathbf{N}\right) = \text{plane } P\mathbf{NB},$$

and in turn:

The asymptotic planes to the surface of binormals are planes that are normal to the curve P, or, in other words, *the surface of binormals and the polar surface agree along their curves at infinity.*

Since the plane P**BT** is also perpendicular to the plane P**NB**, one will have:

The line of striction of the surface of binormals will be precisely the curve P.

Since:

$$\frac{dP}{ds} = \mathbf{T} = -\tau \left| \mathbf{B}\frac{d\mathbf{B}}{ds} \right.,$$

one will have that:

*The distribution parameter of the generator P***B** *will be the number* τ.

63. Skew, ruled surfaces whose line of striction is given. – The surface of binormals of the skew curve P is not the only skew, ruled surface that admits the curve P for its line of striction, and we shall now propose to determine all of the skew, ruled surfaces whose curve P is precisely the line of striction.

To that effect, let **U**(s) be a unit vector that is further determined such that the skew surface P**U** will admit the curve P for its line of striction. We will have:

$$\frac{d}{ds}(P\mathbf{U})\frac{d}{ds}(P\mathbf{U}) = P\mathbf{T}\mathbf{U}\frac{d\mathbf{U}}{ds},$$

and the surface that is generated by the line P**U** will be skew if for any value of s:

$$\mathbf{T}\mathbf{U}\frac{d\mathbf{U}}{ds} \neq 0;$$

i.e., if **U** is not constant and the vector d**U** / ds is not coplanar with the vectors **T** and **U**. Under these conditions, by virtue of no. 46, the central point on the line P**U** will have a position of the form:

$$\left(P\mathbf{U}\left|\mathbf{U}\frac{d\mathbf{U}}{ds}\right.\right)\left(\mathbf{T}\mathbf{U} + P\frac{d\mathbf{U}}{ds}\right) = \left(P\mathbf{U}\frac{d\mathbf{U}}{ds}\left|\mathbf{U}\frac{d\mathbf{U}}{ds}\right.\right)P - \left(P\mathbf{T}\mathbf{U}\left|\mathbf{U}\frac{d\mathbf{U}}{ds}\right.\right)\mathbf{U};$$

i.e., the line of striction of the surface P**U** will be the curve P if one has:

$$(1) \qquad \mathbf{T}\mathbf{U}\left|\mathbf{U}\frac{d\mathbf{U}}{ds}\right. = 0 \qquad \text{and} \qquad \mathbf{U}\frac{d\mathbf{U}}{ds}\left|\mathbf{U}\frac{d\mathbf{U}}{ds}\right. \neq 0$$

for every value of s.

The second of these conditions will always be verified if the derivative of **U** is not zero, but the first one demands that the vector $d\mathbf{U} / ds$ not be coplanar with the vectors **T**, **U**. Consequently:

All of the skew, ruled surface whose line of striction is the curve P will be generated by a line P$\mathbf{U}$*, where* **U** *is a unit vector with non-zero derivative that is not parallel to the vector* **T***, and is defined by the differential equation:*

(2) $$\mathbf{TU} \left| \mathbf{U} \frac{d\mathbf{U}}{ds} \right. = 0.$$

Therefore, if we set:

$$\mathbf{U} = x\mathbf{T} + y\mathbf{N} + z\mathbf{B},$$

where x, y, z are functions of s such that $x^2 + y^2 + z^2 = 1$, then we can prove quite simply that the condition (2) is equivalent to the following one:

$$\frac{dx}{ds} = \frac{1}{\rho} y .$$

If one then supposes, for example, that y is known as a function of s then x will be well-defined. If $x^2 + y^2 < 1$ then z will likewise result, and the vector **U** will also be well-defined.

Now, suppose that one has $y = 0$ for every value of s. x and z will then be absolute constants, and one will have:

Every fixed line in the rectifying plane at the point P that does not coincide with the tangent at that point upon passing through P will describe a skew, ruled surface for which the curve P is the line of striction.

64. A developable, ruled surface that is described by a line whose position is fixed with respect to the tetrahedron *P*TNB. – If we set:

(1) $$a = xP\mathbf{T} + yP\mathbf{N} + zP\mathbf{B} + u\mathbf{NB} + v\mathbf{BT} + w\mathbf{TN},$$

where x, y, z, z, u, v, w are constant numbers such that:

(2) $$ux + vy + wz = 0,$$

then it will be clear that the line $a(s)$ will possess a position that is fixed with respect to the lines $P\mathbf{T}$, $P\mathbf{N}$, $P\mathbf{B}$. Under these conditions:

$$\frac{da}{ds} = -\frac{y}{\rho} P\mathbf{T} + \left(\frac{x}{\rho} + \frac{z}{\tau}\right) P\mathbf{N} - \frac{y}{\tau} P\mathbf{B} - \frac{v}{\rho}\mathbf{NB} + \left(\frac{u}{\rho} + \frac{w}{\tau} - z\right)\mathbf{BT} - \left(y - \frac{v}{\tau}\right)\mathbf{TN},$$

and the line a will describe a developable surface when:

$$\frac{da}{ds}\,\frac{da}{ds} = 0;$$

i.e., when:

$$\frac{y}{\rho}\frac{v}{\rho}+\left(\frac{x}{\rho}+\frac{z}{\tau}\right)\left(\frac{u}{\rho}+\frac{w}{\tau}-z\right)-\frac{y}{\tau}\left(y-\frac{v}{\tau}\right) = 0,$$

where, from formula (2), when:

$$-\frac{wz}{\rho^2}-\frac{ux}{\rho^2}+\frac{uz+wx}{\rho\tau} = \frac{xz}{\rho}+\frac{y^2+z^2}{\tau}. \tag{3}$$

If there exists no relation with constant coefficients between $1 / \rho$ and $1 / \tau$ then the line a can describe a developable surface only if $y = z = u = w = 0$; i.e., only when:

$$a = (P + h\mathbf{B})\,\mathbf{T}, \tag{4}$$

where h is a constant number that is, moreover, arbitrary. The line a will then be on the rectifying plane to the point P, and will be parallel to the tangent at P. Formula (4) will give:

$$\frac{da}{ds} = \frac{1}{\rho}(P+h\mathbf{B})\mathbf{N}+\frac{h}{\tau}\mathbf{NT} = \frac{1}{\rho}\left(P+h\mathbf{B}-h\frac{\rho}{\tau}\mathbf{T}\right)\mathbf{N},$$

and the point of intersection of the lines a and da / ds will be:

$$P_1 = P + h\mathbf{B} - h\,\frac{\rho}{\tau}\mathbf{T} = P - \frac{h}{\tau}(\rho\,\mathbf{T} - \tau\,\mathbf{B})\;;$$

i.e., the line of regression for the ruled surface a will be described by the generators of the rectifying surface that contains P.

Finally, if there exists a relation with constant coefficients between the numbers $1 / \rho$ and $1 / \tau$ that is a relation of the form (3), then there can exist lines a other than the ones that verify equation (4) in order to describe a developable surface. Upon following the method that we just pointed out, the reader will easily determine these lines in order to recover the results that were obtained already by Cesaro ([30]).

([30]) *Lezioni di Geometrica intrinseca*, Naples, 1896.

§ 3. – ORTHOGONAL TRAJECTORIES.

65. Orthogonal trajectories of the generators of a ruled surface. – A curve that is traced on a ruled surface that cuts the generators of the surface at a right angle will be called an *orthogonal trajectory to the generators of the surface*.

In general, we can consider a ruled surface to be generated by a line $P\mathbf{K}$, where $P(s)$ and $\mathbf{K}(s)$ take the form of a point and a unit vector, respectively. An arbitrary curve that is traced on the surface $P\mathbf{K}$ will then be described by the point:

$$P_1 = P + u\mathbf{K},$$

if one assumes that u is a function of s. We shall thus propose to determine u in such a way that P_1 will describe an orthogonal trajectory of the lines $P\mathbf{K}$. In order for this to be true, it is necessary and sufficient that:

$$\frac{dP_1}{ds}\,\bigg|\,\mathbf{K} = 0,$$

and by virtue of the fact that:

$$\frac{dP_1}{ds} = \mathbf{T} + \frac{du}{ds}\mathbf{K} + u\,\frac{d\mathbf{K}}{ds},$$

that condition becomes:

$$\mathbf{T} \mid \mathbf{K} + \frac{du}{ds} = 0.$$

Hence:

$$u = -\int (\mathbf{T} \mid \mathbf{K})\,ds,$$

which one can write in an equivalent fashion as:

$$u = -\int \cos(\mathbf{T}, \mathbf{K})\,ds.$$

Therefore, if one takes $s = 0$ to be a limit of the integral then one will see that the orthogonal trajectories to the generators of the surface $P\mathbf{K}$ will be described by the points:

$$P_1 = P - \left[\int_0 (\mathbf{T} \mid \mathbf{K})\,ds + c\right]\mathbf{K},$$

where c is an arbitrary constant.

Now, if the curve P is supposed to be an orthogonal trajectory then one will have $\mathbf{T} \mid \mathbf{K} = 0$, as well as $P_1 = P - c\,\mathbf{K}$, which shows that:

The distance between the points of two orthogonal trajectories that are situated on the same generator will be constant.

66. Developings. – One calls an orthogonal trajectory of the osculating developable of the curve P a *developing of the curve P*. One will thus obtain the point P_1 that describes a developing of the curve P by setting $\mathbf{K} = \mathbf{T}$ in the last formula of no. 65, and one will also get:

$$P_1 = P - (s + c)\,\mathbf{T}.$$

One will then have:

$$\frac{dP_1}{ds} = -\frac{s+c}{\rho}\mathbf{N},$$

so

The tangent at P of a developing of the curve P will be parallel to the principal normal at the corresponding point P.

Since:

$$\frac{d^2P_1}{ds^2} = -\left(\frac{d}{ds}\frac{s+c}{\rho}\right)\mathbf{N} + \frac{s+c}{\rho^2}\mathbf{T} + \frac{s+c}{\rho\tau}\mathbf{B},$$

one deduces that:

$$\left|\frac{dP_1}{ds}\frac{d^2P_1}{ds^2}\right. = -\frac{(s+c)^2}{\rho^2\tau}(\rho\,\mathbf{T} - \tau\,\mathbf{B}),$$

and:

The binormal at P_1 of a developing of the curve P is parallel to the generator of the rectifying surface that passes through the corresponding point P.

The curve P_1 can be planar only if the direction of the vector $\mathbf{B}_1$ is constant; i.e. (cf., the preceding proposition), when the rectifying surface of the curve P is a cylinder. Therefore:

The helix is the only skew curve whose developings are all planar curves,

and:

Every developing of a helix is located in a plane that is normal to the generators of the cylinder on which the helix is traced, and is a developing of the normal section itself of the cylinder that is made by that plane.

The plane normal to the point P_1 is the plane:

$$P_1\left|\frac{dP_1}{ds}\right. = \frac{s+c}{\rho}P_1\mathbf{TB} = \frac{s+c}{\rho}P\mathbf{TB}.$$

Therefore:

Every developing of the curve P on the polar surface will coincide with the rectifying surface of the curve P. One can also say: *The locus of the center of the osculating sphere* *to an arbitrary developing of the curve P will be the edge of regression of the rectifying surface of P.*

All of the curves P_1 whose polar surfaces coincide with the rectifying surface of the curve P are described by the point:

$$P_1 = P + x\mathbf{B} + y\mathbf{T},$$

such that the vector dP_1 / ds is parallel to the vector $\mathbf{N}$ ([31]). As a result, the numbers x, y, which are functions of s, will be subject to the conditions:

$$\frac{dx}{ds} = 0, \qquad \frac{dy}{ds} + 1 = 0,$$

which give:

(1) $$P_1 = P + u\mathbf{B} - (s + c)\,\mathbf{T},$$

where a and b are constants.

One easily obtains the curves (1) when the developings of P are known.

If $Q_1 = P + a_1\mathbf{B} - (s + c_1)\mathbf{T}$, and if m, n are numbers such that $m + n \neq 0$ then one will have:

([31]) In general, one can resolve the following question: *What are the curves $P_1(s)$ such that one of the lines $P_1\mathbf{T}_1$, $P_1\mathbf{N}$, $P_1\mathbf{B}_1$ is parallel to one of the lines $P\mathbf{T}$, $P\mathbf{N}$, $P\mathbf{B}$?* For all values of s, one will have:

(1) $\mathbf{T}_1\mathbf{N} = 0$ when $\dfrac{dP_1}{ds} = u\mathbf{N}$,

(2) $\mathbf{B}_1\mathbf{N} = 0$ when $\dfrac{dP_1}{ds} = u\lambda\left(\dfrac{1}{\rho}\mathbf{B} - \dfrac{1}{\tau}\mathbf{T}\right)$,

(3) $\mathbf{N}_1\mathbf{T} = 0$ when $\dfrac{dP_1}{ds} = u\,(\cos\varphi\,\mathbf{N} + \sin\varphi\,\mathbf{B})$ with $d\varphi = \dfrac{d\varphi}{\tau}$,

(4) $\mathbf{N}_1\mathbf{B} = 0$ when $\dfrac{dP_1}{ds} = u\,(\cos\varphi\,\mathbf{T} + \sin\varphi\,\mathbf{N})$ with $d\varphi = -\dfrac{d\varphi}{\tau}$,

(5) $\mathbf{N}_1\mathbf{N} = 0$ when $\dfrac{dP_1}{ds} = u\,(\cos\varphi\,\mathbf{B} + \sin\varphi\,\mathbf{T})$ with $d\varphi = 0$,

(6) $\mathbf{T}_1\mathbf{T} = 0$ or even $\mathbf{B}_1\mathbf{B} = 0$ when $\dfrac{dP_1}{ds} = u\mathbf{T}$,

(7) $\mathbf{T}_1\mathbf{B} = 0$ or even $\mathbf{B}_1\mathbf{T} = 0$ when $\dfrac{dP_1}{ds} = u\mathbf{B}$,

in which u is an arbitrary function of s.

One easily expresses the vectors $\mathbf{T}_1$, $\mathbf{N}_1$, $\mathbf{B}_1$, and the numbers ρ_1, τ_1 as functions of the vectors $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and the numbers ρ, τ, λ, u for the curves (1)-(7), and one obtains some very important properties.

Among the curves (1)-(7), there are some that have one of the surfaces $P_1\mathbf{N}_1\mathbf{B}_1$, $P_1\mathbf{B}_1\mathbf{T}_1$, $P_1\mathbf{T}_1\mathbf{N}_1$ coinciding with one of the surfaces $P\mathbf{NB}$, $P\mathbf{NB}$, $P\mathbf{TN}$, or one of the surfaces $P_1\mathbf{T}_1$, $P_1\mathbf{N}_1$, $P_1\mathbf{B}_1$ coinciding with one of the surfaces $P\mathbf{T}$, $P\mathbf{N}$, $P\mathbf{B}$. The reader can, by way of exercise, recover the *developings, the developments, and the Bertrand curves, etc.*

$$\frac{mP_1 + nQ_1}{m+n} = P + \frac{ma + na_1}{m+n}\mathbf{B} - \left(s + \frac{mc + nc_1}{m+n}\right)\mathbf{T},$$

and the point $\frac{mP_1 + nQ_1}{m+n}$ will describe a curve (1).

67. Developments. – Conversely, we shall say that the curve P_1 is one of the *developments* of the curve P if P is one of the developings of P_1 . We shall propose to determine all of the developments of a given curve P.

To that effect, if **K** represents a unit vector that is a function of s then the line P**K** will describe a developable surface only in the case where:

$$\frac{d(P\mathbf{K})}{ds}\frac{d(P\mathbf{K})}{ds} = 2\,P\mathbf{TK}\,\frac{d\mathbf{K}}{ds} = 0;$$

i.e., when:

$$\mathbf{TK}\frac{d\mathbf{K}}{ds} = 0 \tag{1}$$

for every value of s.

If **K** verifies condition (1) without being constant then the edge of regression P_1 of the surface P**K** will be a development of the curve P when **K** is parallel to the plane P**NB** (because the curve P must be an orthogonal trajectory of the generators of the surface P**K**), and the vector $d\mathbf{K} / ds$, which is parallel to the principal normal at the point P, will be parallel to the vector **T** (because the tangent at P to the developing is parallel to the principal normal at the point P_1 of the development). We can thus determine the vector **K** by choosing a number φ such that:

$$\mathbf{K} = \cos\varphi\,\mathbf{N} + \sin\varphi\,\mathbf{B}, \tag{2}$$

and $d\mathbf{K} / ds$ will be a vector that is parallel to **T**.

If one is given:

$$\frac{d\mathbf{K}}{ds} = -\frac{\cos\varphi}{\rho}\mathbf{T} + \left(\frac{1}{\tau} - \frac{d\varphi}{ds}\right)(\sin\varphi\,\mathbf{N} - \cos\varphi\,\mathbf{B}) \tag{2$'$}$$

then φ will be determined by the differential equation:

$$d\varphi = \frac{ds}{\tau}, \tag{3}$$

or

$$\varphi = \int_0 \frac{ds}{\tau} + \varphi_0\,, \tag{3$'$}$$

upon introducing an arbitrary constant φ_0 . Moreover, the edge of regression of the surface P**K** can be determined by:

$$\frac{d}{ds}(P\mathbf{K}) = \mathbf{TK} - \frac{\cos\varphi}{\rho}P\mathbf{T},$$

$$P\mathbf{K}.\frac{d}{ds}(P\mathbf{K}) = P\mathbf{TK}\left(\mathbf{K} + \frac{\cos\varphi}{\rho}P\right).$$

Therefore, when the point P_1 describes a development of the curve P, one will have:

$$P_1 = P + \frac{\rho}{\cos\varphi}\mathbf{K},$$

and, from formula (2):

(4) $$P_1 = P + \rho\,\mathbf{N} + \rho\tan\varphi\,\mathbf{B},$$

where φ is precisely the number that is provided by formula (3)′. Since the line $(P + \rho\mathbf{N})$ $\mathbf{B}$ generates the polar surface to the curve P, formula (1) will express the idea that:

The developments of the curve P are situated on the polar surface to P.

In the case where $1 / \tau = 0$, for all values of s, the curve P will be planar:

$$P_1 = P + \rho\,\mathbf{N} + \rho\tan\varphi_0\,\mathbf{B},$$

and one of the developments of P will be the curve that is described by the point $P + \rho\,\mathbf{N}$; i.e., the enveloping curve of normals to the curve P. The other developments are, moreover, skew curves that are traced on the cylinder whose cross-section is precisely the locus of the point $P + \rho\,\mathbf{N}$; on the other hand, dP_1 / ds is parallel to the vector $\mathbf{K}$, and the angle $\left(\frac{dP_1}{ds}, \mathbf{B}\right)$ must be constant. Therefore:

A planar curve has just one planar development and an infinitude of skew developments that are helices traced on the cylinder whose cross-section is precisely that planar development.

Formula (3) also shows that $d\varphi$ is the contingency angle of the vector $\mathbf{B}$, and in the development of the polar surface of the given curve onto a plane, the curve P_1 will transform into a line. In general, one can suppose that there exists a value s_0 of s such that:

$$\lim_{s=s_0}\rho(s) = 0,$$

so:

Upon developing the polar surface to the curve in question onto a plane, the developments of P will be transformed into lines that pass through a fixed point.

Let $d\psi_1$, $d\psi_2$, $d\psi_3$ be the contingency angles of the vectors $\mathbf{T}_1$, $\mathbf{N}_1$, $\mathbf{B}_1$. $\mathbf{T}_1$ will parallel to the vector **K**, just as $\mathbf{N}_1$ will be parallel to the vector **T**, and formulas (2) and (3) will give:

$$d\psi_1 = \pm \frac{\cos\varphi}{\rho}\,ds$$

and

$$d\psi_2 = \frac{1}{\rho}\,ds\,.$$

As a result, one will deduce from the fact that $\frac{1}{\rho^2} - \frac{\cos^2\varphi}{\rho^2} = \frac{\sin^2\varphi}{\rho^2}$ that:

$$d\psi_3 = \frac{\sin\varphi}{\rho}\,ds\,,$$

and upon observing that $\frac{\rho_1}{\tau_1} = \pm\frac{d\psi_2}{d\psi_1}$, one will get:

At each point of a development of the curve P, one will have:

$$\frac{\rho_1}{\tau_1} = \pm\tan\left(\int_0 \frac{ds}{\tau} + \varphi_0\right).$$

$\frac{\rho_1}{\tau_1}$ and $\int_0 \frac{ds}{\tau}$ must both be constants. However, $\int_0 \frac{ds}{\tau}$ is constant when $1/\tau = 0$ for any value of s. Therefore:

Planar curves have only planar curves or helices for their developments.

If P_1 is the edge of regression of the polar surface to the curve P then one will have (*see* no. 59):

$$\rho\tan\varphi = -\tau\frac{d\rho}{ds}\,,$$

or

$$\frac{d\rho}{\rho} = d\log\rho = -\tan\varphi\,\frac{ds}{\tau} = -\tan\varphi\,d\varphi = d\log\cos\varphi\,.$$

Consequently, $\rho/\cos\varphi$ will be a non-zero constant. Therefore:

The curves such that one of the developments coincides with the locus of the center of the osculating spheres will have their curvatures linked by the relation:

$$\rho = c \cos\left(\int_0 \frac{ds}{\tau} + \varphi_0\right),$$

where $c \neq 0$ and φ_0 enter in as arbitrary constants.

Similarly, one proves:

Only the planar curves have a development that coincides with the locus of the centers of curvature.

If the curve P_1 describes one of the developments of the curve P then by formulas (2), (2)′, (3), one will have that the rectifying plane to the point P_1 is parallel to the bivector $|\mathbf{T} = \mathbf{NB}$. Therefore, since plane $P_1\mathbf{NB}$ = plane $P\mathbf{NB}$ [formula (4)] is the rectifying plane to the point P_1, one will have:

The edge of regression of the rectifying surface of each development of P is the locus of the centers of the osculating spheres of the curve P.

Conversely, if the point P_1 describes a curve whose rectifying surface has an edge of regression that is the edge of regression for the polar surface to P then one must have:

$$P_1 = P + \rho\,\mathbf{N} + x\,\mathbf{B}, \tag{5}$$

with $\mathbf{N}_1$ parallel to the vector $\mathbf{T}$. One easily finds ([32]) that $\mathbf{N}_1$ is parallel to $\mathbf{T}$ only when dP_1 / ds is parallel to the vector:

$$\cos\varphi\,\mathbf{N} + \sin\varphi\,\mathbf{B} \qquad \text{with} \qquad d\varphi = \frac{ds}{\tau}.$$

Consequently, the number x in formula (5) is subject to the condition:

$$\frac{dx}{ds} = x\frac{\tan\varphi}{\tau} + \frac{\rho}{\tau} + \frac{d\rho}{ds}\tan\varphi.$$

However, $x = \rho\tan\varphi$ is a particular integral of that differential equation, and in turn, the general integral is:

$$x = \rho\tan\varphi + c\, e^{\int\frac{\tan\varphi}{\tau}ds},$$

or even, since $d\varphi = ds / \tau$:

$$x = \rho\tan\varphi + \frac{c}{\cos\varphi}.$$

From formula (5), one thus has:

([32]) Because if $\mathbf{N}, \mathbf{T} = 0$ then one will have $\mathbf{T}_1 = \cos\varphi\,\mathbf{N} + \sin\varphi\,\mathbf{B}$ (*see* the note on page 114).

$$P_1 = P + \rho\,\mathbf{N} + \left(\rho\tan\varphi + \frac{c}{\cos\varphi}\right)\mathbf{B},$$

which gives *all of the curves whose rectifying surfaces have the locus of centers of the osculating spheres of the curve P for their edges of regression.*

65. Orthogonal trajectories of the planes of an envelope. – Let π be a third-order form whose position has an envelope. One can, in a very general manner, set:

$$\pi = P\mathbf{IJ},$$

where P is a point, and **I** and **J** are two rectangular unit vectors that are functions of the arc-length s of the curve that is described by P. An arbitrary point P_1 of the plane π will be:

$$P_1 = P + x\mathbf{I} + y\mathbf{J},$$

and when the numbers x, y are functions of s, the point P_1 will describe a curve that one calls the *orthogonal trajectory* of the plane π, if the tangent at P_1 is constantly perpendicular to the plane π.

One then easily sees that the necessary and sufficient conditions for the point P_1 to describe an orthogonal trajectory of the plane p are:

$$(1) \qquad \frac{dP_1}{ds}\,\bigg|\,\mathbf{I} = 0, \qquad \frac{dP_1}{ds}\,\bigg|\,\mathbf{J} = 0.$$

However:

$$\frac{dP_1}{ds} = \mathbf{T} + \frac{dx}{ds}\mathbf{I} + x\frac{d\mathbf{I}}{ds} + \frac{dy}{ds}\mathbf{J} + y\frac{d\mathbf{J}}{ds},$$

and upon observing that:

$$\mathbf{I} \mid \mathbf{J} = 0, \qquad \mathbf{I}\,\bigg|\,\frac{d\mathbf{I}}{ds} = 0, \qquad \mathbf{J}\,\bigg|\,\frac{d\mathbf{J}}{ds} = 0,$$

the conditions (1) become:

$$(2) \qquad \mathbf{T} \mid \mathbf{I} + y\,\frac{d\mathbf{J}}{ds}\,\bigg|\,\mathbf{I} + \frac{dx}{ds} = 0, \qquad \mathbf{T} \mid \mathbf{J} + x\frac{d\mathbf{I}}{ds}\,\bigg|\,\mathbf{J} + \frac{dy}{ds} = 0.$$

These differential equations determine x and y ([33]), along with two arbitrary constants, and the orthogonal trajectories of the plane π define a doubly-indeterminate system.

a. Upon setting **I** = **N**, **J** = **B**, one can obtain the orthogonal trajectories of the plane P**NB**. In the expression:

([33]) *See* the note on page ?.

(3) $$P_1 = P + u\,(\cos\varphi\,\mathbf{N} + \sin\varphi\,\mathbf{B}),$$

one must determine the functions u and φ of s in such a way that the vector dP_1 / ds is parallel to the vector $\mathbf{T}$. For this, one equates the coefficients of $\mathbf{N}$ and $\mathbf{B}$ in the vector dP_1 / ds to zero (or, upon applying formulas (2), with $\mathbf{I} = \mathbf{N}$, $\mathbf{J} = \mathbf{B}$, $x = u\cos\varphi$, $y = u\sin\varphi$):

$$\frac{du}{ds}\cos\varphi - u\,\frac{d\varphi}{ds}\sin\varphi + \frac{u\sin\varphi}{\tau} = 0,$$

$$\frac{du}{ds}\sin\varphi + u\,\frac{d\varphi}{ds}\cos\varphi - \frac{u\cos\varphi}{\tau} = 0.$$

One thus infers that:

$$du = 0 \quad\text{and}\quad d\varphi = \frac{ds}{\tau},$$

and one sees that the point P_1 will describe an orthogonal trajectory to the planes $P\mathbf{NB}$ when u is a constant in formula (3) and $\varphi = \int ds / \tau$.

The line PP_1 describes the osculating developable of one of the developments (no. 67) of the curve P, and since P_1 is an orthogonal trajectory of the lines PP_1, one will have:

The orthogonal trajectories of the normal planes to the curve P are the developments of the developings of P, or one can also say that *they are the curves that the locus of centers of the osculating spheres* [*see*, note page 17, formula (6)] *has in common with the curve P.*

b. One can further obtain the orthogonal trajectories of the planes $P\mathbf{BT}$ by setting $\mathbf{I} = \mathbf{B}$, $\mathbf{J} = \mathbf{T}$. Equations (2) give:

$$\frac{dx}{ds} = 0, \qquad 1 + \frac{dy}{ds} = 0,$$

or

$$y = -(s + c), \qquad x = a,$$

with two arbitrary constants a and c. As a result:

$$P_1 = P - (s + c)\,\mathbf{T} + a\mathbf{B},$$

and one easily obtains the curves that are described by the point P_1 from the developments of P (*see* no. 68).

c. In order to obtain the orthogonal trajectories of the osculating planes of the curve P, it will suffice to set $\mathbf{I} = \mathbf{T}$, $\mathbf{J} = \mathbf{N}$. The curves P_1 are then such that the locus of centers of their osculating spheres is the curve P, and formulas (2) further give:

$$\frac{dx}{ds} = \frac{y}{\rho} - 1, \qquad \frac{dy}{ds} = -\frac{x}{\rho},$$

which are differential equations that always permit one to get expressions for x and y; for example, as convergent series ([34]).

§ 4. – BERTRAND CURVES.

69. We say that the curve P is a *Bertrand curve* if there exists a curve P that is different from P and has the same principal normals as it, and we then call P_1 one of the *conjugates* to the curve P.

If P is a Bertrand curve and the curve P_1 is one of the conjugates of P then the curves P, P_1 will be orthogonal trajectories of the surface of principal normals. The distance between the points P and P_1 must then be constant, i.e.:

$$P_1 = P + u\,\mathbf{N}, \tag{1}$$

where u is a (fixed) non-zero real number.

If P is a plane curve then, by virtue of the equation:

$$\frac{dP_1}{ds} = \left(1 - \frac{u}{\rho}\right)\mathbf{T},$$

the normals to the points P and P_1 will coincide.

Therefore:

Any planar curve is a Bertrand curve, and its conjugates are generated by the points $P + u\,\mathbf{N}$*, where* u *enters in as an arbitrary constant.*

70. From now on, we shall suppose that the skew curve P is a Bertrand curve and that P_1 is one of the conjugates of P. The vector $\mathbf{T}_1$, which is parallel to the tangent at P_1, is parallel to the plane $P\mathbf{TB}$, which permits us to set:

$$\mathbf{T}_1 = \cos\varphi\,\mathbf{T} + \sin\varphi\,\mathbf{B},$$

if we let φ denote a function of s such that $d\mathbf{T}_1 / ds$ (which is a vector that is parallel to $\mathbf{N}_1$) is a vector that is parallel to the vector $\mathbf{N}$, One therefore has:

$$\frac{d\mathbf{T}_1}{ds} = \left(\frac{\cos\varphi}{\rho} + \frac{\sin\varphi}{\tau}\right)\mathbf{N} + \frac{d\varphi}{ds}(\cos\varphi\,\mathbf{B} - \sin\varphi\,\mathbf{T}),$$

([34]) *See* the note on page 94.

and the equation $d\mathbf{T}_1 / ds\ \mathbf{N} = 0$ will be true for any s only if φ, which is the angle between the vectors $\mathbf{T}$ and $\mathbf{T}_1$, is a constant. Consequently:

The osculating plane to the point P of a Bertrand curve makes a constant angle with the osculating plane to the corresponding point of a curve that is conjugate to P.

Moreover, formula (1) gives:

$$\frac{dP_1}{ds} = \left(1 - \frac{u}{\rho}\right)\mathbf{T} - \frac{u}{\tau}\mathbf{B},$$

and if we set:

$$v = \operatorname{mod}\frac{dP_1}{ds} = \sqrt{\left(1-\frac{u}{\rho}\right)^2 + \left(\frac{u}{\tau}\right)^2}$$

then we will see that v is a non-zero number, and the const φ, which is such that $\pi > \varphi > 0$, and it satisfies the equation:

$$(2) \qquad 1 - \frac{u}{\rho} = v \cos \varphi, \qquad \frac{u}{\tau} = - v \sin \varphi$$

will be well-defined. Since $v \neq 0$ and $\sin \varphi \neq 0$, formulas (2) imply that:

$$(3) \qquad \frac{\sin\varphi}{\rho} - \frac{\cos\varphi}{\tau} = \frac{\sin\varphi}{u},$$

which expresses a relation with constant coefficients between the curvature and torsion at every point of the curve P, and is a necessary condition for the curve in question to be a Bertrand curve. Conversely, if the condition (3) is verified then the point $P_1 = P + u\,\mathbf{N}$ will essentially describe a curve that has the same normals as the curve P. Therefore:

In order for the skew curve P to be a Bertrand curve, it is necessary and sufficient that the curvature and torsion be coupled by a linear relation with constant coefficients of the form:

$$\frac{\sin\varphi}{\rho} - \frac{\cos\varphi}{\tau} = \frac{\sin\varphi}{u}$$

($\pi > \varphi > 0$ and $u \neq 0$) *at every point of P. Upon assuming that this condition is satisfied, the point* $P_1 = P + u\,\mathbf{N}$ *will describe a conjugate to the curve P, and* φ *will be the angle that the osculating plane to P makes with the osculating plane to the corresponding point* P_1 *of the conjugate curve.*

71. The curve P_1 that is conjugate to P is also a Bertrand curve. Therefore, if one observes that $\frac{ds_1}{ds} = v$, $\frac{ds}{ds_1} = \frac{1}{v}$, $P = P_1 - u\,\mathbf{N}$, and if φ is the angle that $\mathbf{T}$ makes with $\mathbf{T}_1$, as well as the angle between $\mathbf{T}$ and $\mathbf{T}_1$, then the curvature $1 / \rho_1$ and torsion $1 / \tau_1$ of the curve P_1 will be linked by formulas that are analogous to formulas (2), (3), namely:

$$1 - \frac{u}{\rho_1} = \frac{1}{v}\cos\varphi, \qquad \frac{u}{\tau_1} = -\frac{1}{v}\sin\varphi, \tag{2$'$}$$

$$\frac{\sin\varphi}{\rho} - \frac{\cos\varphi}{\tau} = \frac{\sin\varphi}{u}, \tag{3$'$}$$

or even the ones that one can deduce from them by changing u into $-u$, according to whether $\mathbf{N}_1 = -\mathbf{N}$ or $\mathbf{N}_1 = \mathbf{N}$. In order to determine the sign of u completely in formulas (2)′, (3)′, observe that the vector $\mathbf{B}_1$, which is parallel to the vector:

$$\left| \frac{dP_1}{ds} \mathbf{N} = \left(1 - \frac{u}{\rho}\right)\mathbf{B} + \frac{u}{\tau}\mathbf{T}, \right.$$

is given by the relation:

$$\mathbf{B}_1 = \pm(\cos\varphi\,\mathbf{B} - \sin\varphi\,\mathbf{T}),$$

according to whether $\mathbf{N}_1 = \pm\mathbf{N}$. By virtue of (3), the derivative:

$$\frac{d\mathbf{B}_1}{ds_1} = \frac{1}{v}\left(\frac{\cos\varphi}{\tau} - \frac{\sin\varphi}{\rho}\right)\mathbf{N}_1,$$

gives:

$$\frac{d\mathbf{B}_1}{ds_1} = -\frac{\sin\varphi}{uv}\mathbf{N}_1,$$

and in turn:

$$\frac{1}{\tau_1} = -\frac{\sin\varphi}{uv} \qquad \text{[second formula (2)$'$]},$$

if one considers the second Frenet formula.

It is thus indeed proved that formulas (2)′, (3)′ persist for ρ_1 and τ_1 with $\mathbf{N}_1 = -\mathbf{N}$.

72. Here are some consequences of formulas (2), (3), (2)′, (3)′:

From the second of formulas (2) and (2)′, it results that:

$$\frac{1}{\tau}\frac{1}{\tau_1} = \left(\frac{\sin\varphi}{u}\right)^2,$$

which proves that:

The product of the torsions at every point of a Bertrand curve and its conjugate has a constant value that is equal to the square of $\sin \varphi / u$.

If $1 / \rho$ is constant, and the same is not true for $1 / \tau$, then formula (3) will give $\varphi = \pi / 2$, $u = r$. Therefore, the preceding theorem and formula (3)′ prove that:

If the curvature of a skew, Bertrand curve is constant, but not the torsion, then that curve will admit the locus of the centers of curvature for its unique conjugate, or, what amounts to the same thing, the locus of the centers of the osculating spheres.

The angle between the osculating planes at a point of the curve and at the corresponding point of the conjugate curve will be a right angle.

The curve P and its conjugate will have the same curvature, and the product of the torsions at two corresponding points will be equal to the square of the curvature.

One easily deduces from formula (3) that:

A skew, Bertrand curve cannot have a constant torsion without the curvature being likewise constant, or in other words, *the only skew Bertrand curves that have constant torsion are the ordinary helices.*

Formula (3) further gives:

A skew, Bertrand curve that is not an ordinary helix possesses just one conjugate. On the contrary, an ordinary helix has an infinitude of conjugate curves that are described by the points $P_1 = P + u\mathbf{N}$, *where u enters in as an arbitrary constant.*

When P is an ordinary helix, one can determine u in such a way that $ds_1 = ds$; i.e., such that the curves P, P_1 have the same arc-length. One must have $v = 1$, or $u = 2\,\lambda^2 / \rho$, if $1 / \lambda$ represents the normal curvature. Therefore (no. 64):

If P is an ordinary helix and the central point for the line PP_1 *of the surface of principal normals of P is the mean of P and* P_1 *then the curves P,* P_1 *will have the same arc-length, and conversely.*

Let r be the double ratio of the sequence of points P, P_1, $P + \rho\,\mathbf{N}$, $P_1 - \rho_1\,\mathbf{N}$. Since $P_1 = P + u\,\mathbf{N}$, we will have $PP_1 = uP\mathbf{N}$ and $P_1P = -\,uP_1\,\mathbf{N}$. Therefore:

$$r = \frac{P(P+\rho\mathbf{N})}{P(P_1-\rho_1\mathbf{N})}\,\frac{P_1(P_1-\rho_1\mathbf{N})}{P_1(P+\rho\mathbf{N})} = \frac{\rho P\mathbf{N}}{uP\mathbf{N}-\rho_1 P\mathbf{N}}\,\frac{-\rho_1 P_1\mathbf{N}}{-uP_1\mathbf{N}+\rho P_1\mathbf{N}} = \frac{\rho}{u-\rho_1}\,\frac{\rho_1}{u-\rho} = \frac{1}{1-\dfrac{u}{\rho}}\,\frac{1}{1-\dfrac{u}{\rho_1}},$$

or, by virtue of (3) and (3)′:

$$r = \frac{1}{\cos^2 \varphi}.$$

An arbitrary point of P, its correspondent point P_1 on the conjugate curve, and the centers of curvature at the points P, P_1 will have a constant double ratio that is equal to $1 / \cos^2 \varphi$.

NOTES.

I.

Forms that are functions of two or more variables. – As in analysis, we shall represent a form that is a function of the variables u, v, or u, v, w, … by $f(u, v)$, $f(u, v, w)$, …, resp. Similarly:

$$\frac{d}{du} f(u, v), \qquad \frac{df(u,v)}{du}, \qquad \text{or} \qquad f'_u(u,v)$$

will be the partial derivative of $f(u, v)$ with respect to u.

Under the same restrictions as the ones that are introduced in analysis, we shall call, for example, the infinitesimal number $df(u, v)$ such that:

$$df(u, v) = \frac{df(u,v)}{du}du + \frac{df(u,v)}{dv}dv,$$

or in another form, such that:

$$df(u, v) = f'_u(u,v)\,du + f'_v(u,v)\,dv,$$

the *total differential of* $f(u, v)$.

Under these conditions, if $f(u, v)$ enters into consideration as a non-zero, continuous, first-order form then when u and v vary between some given limits:

$$\text{posit } f(u, v)$$

will generate a surface, and if u_0 and v_0 are particular values of u and v then the points $f(u_0, v)$, $f(u, v_0)$ will describe lines on that surface that one calls the *u-curves* and the *v-curves*, respectively, which will provide, if one so desires, a set of Gaussian coordinate lines on the surface.

Similarly, if $f(u, v)$ is a non-zero, second-order form with zero invariant then posit $f(u, v)$ will describe a congruence of lines, and finally, in the case where $f(u, v)$ is a third-order form, posit $f(u, v)$ will take form of a double infinitude of planes that are, in general tangent to a well-defined surface.

One can easily extend these considerations to the functions:

$$f(u, v, w), \ldots$$

II.

Tangent plane. – Suppose that $P(u, v)$ is a point that is a continuous function of u and v. We say that a plane π is *tangent at P to the surface P* if the line PP_1 makes a (signed) angle with the plane π that has the limit zero when the point P_1 tends to approach the point P indefinitely in an arbitrary manner, with the single condition that it constantly remain on the surface.

One calls the perpendicular to the plane π that passes through the point P the *normal to the surface at P*.

The definition of the tangent plane shows that: *If the tangent plane π to the surface is defined at the point P, along with the tangent r at P to an arbitrary curve that is traced on the surface when starting at the point P then the line r will necessarily be contained in the plane π.*

If the vectors dP / du and dP / dv are continuous functions, and if the bivector $\frac{dP}{du}\frac{dP}{dv}$ is not zero then the plane $P\frac{dP}{du}\frac{dP}{dv}$ will be tangent to the surface P at P.

Indeed, set: $P_1 = P(u + h, v + h)$, so one has:

$$P_1 = P + h\frac{dP}{du} + k\frac{dP}{dv} + Q,$$

where Q is a vector of infinitesimal order higher than unity when one takes $\sqrt{h^2+k^2}$ to be the infinitely small principal. One deduces from this that:

$$(P_1 - P)\frac{dP}{du}\frac{dP}{dv} = Q\frac{dP}{du}\frac{dP}{dv},$$

and consequently, when P_1 tends to P, the vector Q will tend to zero, along with the angle that is defined between the vector $P_1 - P$ and the plane $P\frac{dP}{du}\frac{dP}{dv}$.

If $z = f(x, y)$ is the Cartesian equation of the surface then one will have:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

and

$$\frac{dP}{dx}\frac{dP}{dy} = \left(\mathbf{I}+\frac{dz}{dx}\mathbf{K}\right)\left(\mathbf{J}+\frac{dz}{dy}\mathbf{K}\right) = -\frac{dz}{dx}\mathbf{JK} - \frac{dz}{dy}\mathbf{KI} + \mathbf{IJ};$$

i.e., the angular coefficients of the tangent plane to the point P will be:

$$-\frac{dz}{dx},\ -\frac{dz}{dy},\ 1,$$

respectively, and the equation of that tangent plane will be:

$$Z - z = (X - x)\,\frac{dz}{dx} + (Y - y)\,\frac{dz}{dy}.$$

One thus recovers the usual well-known expression.

III.

First-order differential parameters. – In the questions of mechanics or physics, one is frequently presented with a situation in which one has to consider a number u that is a position of a variable point P. In that case, if P is, for example, a function of its Cartesian coordinates x, y, z then the quantity u will likewise be a function of the variables x, y, z.

We shall say *differential parameter of* u, and denote it by ∇u, to mean a vector such that:

(1) $$du = \nabla u \mid dP.$$

If ∇u, ∇v are two differential parameters of u then from relation (1) one will have:

$$\nabla u \mid dP = \nabla' u \mid dP \qquad \text{or} \qquad (\nabla u - \nabla' u) \mid dP = 0.$$

As a result, if the differential parameter of u exists, and if dP is not zero then the vector ∇u will be defined in a unique fashion.

One has $du = 0$ if u = const., and the vectors ∇u and dP will be zero or rectangular. However, for u = const., the point P will describe a surface or even a line if P was already subject to being found on a surface and if ∇u and dP are well-defined without being zero then formula (1) will prove that the line $P\ \nabla u$ is the normal at P to the surface that is described by the point P or a normal to the curve that is described by P.

Let O be a fixed point, or the foot of the perpendicular that is based at the point P on a fixed line (or plane), in such a way that $OP \neq 0$. If we set:

$$u = \text{mod } OP$$

– i.e., if u essentially represents the distance from the variable point P to a fixed point, line, or plane – then we will have:

(2) $$\nabla u = \frac{\text{mod}(P - O)}{P - O}.$$

Indeed (*see* no. 37, *k*), we know that:

$$du = \frac{P - O}{\text{mod}(P - O)} \,\Bigg|\, (dP - dO).$$

However, one has that $dO = 0$, or that the vector $P - O$ is perpendicular to the vector dO, and consequently:

$$du = \frac{P-O}{\mathrm{mod}(P-O)} \,\Big|\, dP,$$

an expression that one can compare to formula (1) in order to obtain the required theorem.

Furthermore, let u and v be functions of P, and let $f(u, v)$ be a function of u¸v with well-defined partial derivatives. One has:

(3) $$\nabla f = \frac{df}{du}\nabla u + \frac{df}{dv}\nabla v\,.$$

Indeed:

$$df = \frac{df}{du}du + \frac{df}{dv}dv = \frac{df}{du}\nabla u \,\Big|\, dP + \frac{df}{dv}\nabla v \,\Big|\, dP = \left(\frac{df}{du}\nabla u + \frac{df}{dv}\nabla v\right)\Big|\, dP\,,$$

an equality that, along with formula (1), will establish the stated property.

If O is a fixed point, and **I**, **J**, **K** are rectangular unit vectors, and if, moreover:

$$P = O + x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$$

then one will have:

(4) $$\nabla u = \frac{du}{dx}\mathbf{I} + \frac{du}{dy}\mathbf{J} + \frac{du}{dz}\mathbf{K}\,.$$

Indeed, from formula (2):

$$\nabla x = \mathbf{I}, \qquad \nabla y = \mathbf{J}, \qquad \nabla z = \mathbf{K},$$

and in turn:

$$\begin{aligned} du &= \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz \\ &= \frac{du}{dx}\nabla x \,\Big|\, dP + \frac{du}{dy}\nabla y \,\Big|\, dP + \frac{du}{dz}\nabla z \,\Big|\, dP \\ &= \left(\frac{du}{dx}\mathbf{I} + \frac{du}{dy}\mathbf{J} + \frac{du}{dz}\mathbf{K}\right)\Big|\, dP, \end{aligned}$$

which, by virtue of formula (1), proves the theorem.

In general (cf., Lamé), one says *differential parameter of u* to refer to the number mod ∇u, or even $\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}$, and the consideration of the differential parameter as a vector is due to Hamilton.

IV.

Curvilinear coordinates. – Let $P(u, v)$ be a point that is a continuous function and admits derivatives with respect to u and v. If the variables u and v are coupled by an arbitrary relation then the point $P(u, v)$ will describe a curve on the surface $P(u, v)$. If we call the arc-length of that curve s, and suppose that all of the conditions that were stated in § 4 of chapter II are verified then we will have:

$$ds = \operatorname{mod} dP. \tag{1}$$

However, one has:

$$dP = \frac{dP}{du}du + \frac{dP}{dv}dv,$$

moreover, and if one sets:

$$E = \frac{dP}{du}\left|\frac{dP}{du}\right., \qquad F = \frac{dP}{du}\left|\frac{dP}{dv}\right., \qquad G = \frac{dP}{dv}\left|\frac{dP}{dv}\right. \tag{2}$$

then formula (1) will give:

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2, \tag{3}$$

which is a well-known conventional formula. From formulas (2), the numbers E, F, G that one usually considers for the Gaussian coordinates have a very simple geometrical significance.

The second of formulas (2) proves that in the case where dP / du and dP / dv are non-zero vectors, the coordinate lines u, v on the surface can intersect at a right angle only if $F = 0$.

One can further deduce from formulas (2) that:

$$EG - F^2 = \left(\operatorname{mod}\frac{dP}{du}.\operatorname{mod}\frac{dP}{dv}\right)^2 - \left[\operatorname{mod}\frac{dP}{du}.\operatorname{mod}\frac{dP}{dv}\cos\left(\frac{dP}{du},\frac{dP}{dv}\right)\right]^2$$

or

$$EG - F^2 = \left(\operatorname{mod}\frac{dP}{du}\frac{dP}{dv}\right)^2,$$

which proves that the discriminant of the differential quadratic form (2) is positive or zero.

If one further associates formulas (2) with the preceding then one will obtain the equation:

$$\sin\left(\frac{dP}{du},\frac{dP}{dv}\right) = \sqrt{\frac{EG - F^2}{EG}},$$

which will determine the angle between the coordinate lines that pass through the point P; of course, one must suppose that $\frac{dP}{du}\frac{dP}{dv}$ is non-zero.

If one gives a relation between u and v, and one lets θ denote the angle that is formed, for example, between the tangent at P to the curve that is described by the point P and the line v then one will have:

$$\sqrt{E}\cos\theta\, ds = \frac{dP}{du}\,\Big|\, dP,$$

or

$$\cos\theta = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right).$$

In a completely similar manner, one will have:

$$\sqrt{E}\ \sin\theta\, ds = \operatorname{mod}\left(\frac{dP}{du}dP\right) = \operatorname{mod}\left(\frac{dP}{du}\frac{dP}{dv}\right) dv,$$

which, from the preceding, one can write as:

$$\sin\theta = \frac{\sqrt{EG - F^2}}{\sqrt{E}}\frac{dv}{ds}.$$

If the bivector $\frac{dP}{du}\frac{dP}{dv}$ is not zero then upon setting:

$$\mathbf{K} = \left|\frac{\frac{dP}{du}\frac{dP}{dv}}{\operatorname{mod}\left(\frac{dP}{du}\frac{dP}{dv}\right)}\right.,$$

one will see that $\mathbf{K}$ is a unit vector that is perpendicular to the tangent plane to the surface at P; furthermore, $P\mathbf{K}$ is the normal to the surface at the same point. Moreover, set:

$$D = \frac{d^2P}{du^2}\,\Big|\,\mathbf{K}, \qquad D' = \frac{d^2P}{du\,dv}\,\Big|\,\mathbf{K}, \qquad D'' = \frac{d^2P}{dv^2}\,\Big|\,\mathbf{K}.$$

Since:

$$d^2P = \frac{d^2P}{du^2}du^2 + 2\frac{d^2P}{du\,dv}du\,dv + \frac{d^2P}{dv^2}dv^2,$$

one sees that:

$$d^2P \mid \mathbf{K} = D\,du^2 + 2D'\,du\,dv + D''\,dv^2,$$

and the right-hand side of this formula is generally called the the *second differential form* of the surface P. This differential form gives the magnitude of the normal component to the vector d^2P; i.e., the product of that differential form with the vector $\mathbf{K}$ is precisely the normal component of the vector d^2P. By analogy, the vectors $D\mathbf{K}$, $D'\mathbf{K}$, $D''\mathbf{K}$ are the normal components of the vectors $\frac{d^2P}{du^2}$, $\frac{d^2P}{du\,dv}$, $\frac{d^2P}{dv^2}$. One then sees quite easily the geometrical significance of the usual elements that are considered in the theory of curvilinear coordinates.

By applying the method of the elements that we just presented, one can easily prove the theorems of Meusnier, Dupin, Euler, etc., and obtain the lines of curvature, asymptotic lines, geodesic lines, etc. However, the limits that have been imposed upon us do not permit a more complete development of differential geometry.

FIN.