# On the foundations of Dimensional Analysis

by

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## I. Introduction

1. The *Dimensional Analysis*, called also the *Principle of Similitude*, is a computing method used in practical problems of physics, technics, natural philosophy and other disciplines. The method deals with *dimensions* of actual quantities. It often enables us to solve problems by means of quite elementary calculations. As an illustration, we are going to consider the following problem.

An engine of power $N$ turns a mixer. Its wings, of diameter $D$, move with angular velocity $V$ in a liquid, of viscosity $H$. How does the angular velocity $V$ depend on $N$, $D$ and $H$?

Using the Dimensional Analysis, we solve this problem as follows. We choose a system of units, for example cm, g, sec (CGS), and write down the dimensions of quantities $V, N, D, H$:

$$[V]=[\mathrm{sec}^{-1}],\quad [N]=[\mathrm{cm}^2\mathrm{g}\,\mathrm{sec}^{-3}],\quad [D]=[\mathrm{cm}],\quad [H]=[\mathrm{cm}^{-1}\,\mathrm{g}\,\mathrm{sec}^{-1}].$$

By eliminating from this equations the dimensions [cm], [g], [sec], we get

$$[V]=\left[\sqrt{\frac{N}{HD^3}}\right].$$

The quantities in brackets have the same dimension, and consequently, they differ by a constant numerical factor $a$ only. Hence

$$V=a\sqrt{\frac{N}{HD^3}}.$$

2. There is a large number of books and papers on Dimensional Analysis. Many applications of this method and various formulations of its theory are given in them. Some of those books and papers are quoted at the end of this paper, the list, however, is by no means complete.



The origin of the Dimensional Analysis goes back as far as Newton [9]. More attention has been paid to this method since the papers of O. Reynolds, who adjusted it succesfully to problems of hydrodynamics. However, several difficulties and paradoxes of this method are known [2], [6], [7], [10], [11], [12], [13].

3. The principal cause of those difficulties, in my opinion, lies in the fact that the primitive notions, axioms, sometimes even theorems of the Dimensional Analysis are not formulated clearly [2]. This makes it difficult, of course, to check the correctnes of proofs.

Some authors identify dimensional quantities with ordinary numbers, real or complex, [2], and, as a matter of fact, they do not introduce the notions of dimension, or of dimensional quantity into the Dimensional Analysis, although they formulate theorems on those very notions [8]. Sometimes, in the proofs of theorems, such mathematical tools are used as, for instance, differential equations. This seems very odd as regards the foundations of Dimensional Analysis. (On the other hand, it is known that differential equations yield an interesting field of applications).

4. The purpose of this paper is to construct the Dimensional Analysis by means of quite simple algebraic methods, namely using the theory of the linear space. All known theorems of the Dimensional Analysis can be formulated and proved easily and clearly in the language of linear space. Such a point of view makes a simple physical interpretation possible and removes the source of paradoxes. Moreover, it is intuitive to consider the dimensional quantities as elements of a space, different from ordinary numbers. The algebraical methods of the Dimensional Analysis can also be applied to non-physical problems, for instance to the theory of quality control by sampling, [5], or to obtain the particular solutions of some differential equations [12].

There are known another attempts of algebraic foundations of Dimensional Analysis, [1], [8], [14].

## II. The linear space

1. The *linear space* $\Sigma$ (over the field of real numbers) is usually defined by the following axioms ($A, B, C, X$ are elements of $\Sigma$ and $a, b$ are real numbers):

1. $A+B=B+A$;
2. $(A+B)+C=A+(B+C)$;
3. the solution $X$ of $A+X=B$ exists for any pair $A, B$ of elements of $\Sigma$;
4. $(a+b)A=aA+bA$;

5. $a(A+B)=aA+aB$;
6. $a(bA)=(ab)A$;
7. $1A=A$.

These axioms imply the existence, in $\Sigma$, of an element $0$ such that $0+A=A$ for any $A\epsilon\Sigma$; such an element $0$ is unique.

Any linear space $\Sigma^0$ contained in $\Sigma$ is said to be a *linear subspace* of $\Sigma$.

The elements $A_1,\ldots,A_m$ of $\Sigma$ will be called *linearly independent on* $\Sigma^0$ when the equality

$$a_1A_1+\ldots+a_mA_m=A^0,$$

where $A^0\epsilon\Sigma^0$, implies $a_1=\ldots=a_m=0$ and $A^0=0$. In the limit case, where $0$ is the unique element of $\Sigma^0$, the linear independence on $\Sigma^0$ coincides with the linear independence in the usual sense.

We shall suppose in what follows that $\Sigma$ is $n$-dimensional on $\Sigma^0$, *i. e.* that there exist, in $\Sigma$, $n$ elements linearly independent on $\Sigma^0$, but not $n+1$ such elements.

**2.** We shall consider the functions

$$\Phi(P_1,\ldots,P_r),$$

where the $P_1,\ldots,P_r$ and the value of $\Phi$ belong to $\Sigma$.

THEOREM I. *Suppose that*

$$\Phi(P_1+S_1^0,\ldots,P_r+S_r^0)-\Phi(P_1,\ldots,P_r)\epsilon\Sigma^0 \tag{1}$$

*for any* $S_1^0,\ldots,S_r^0\epsilon\Sigma^0$.

*If* $A_1,\ldots,A_n$ *are linearly independent on* $\Sigma^0$, $P_1^0,\ldots,P_r^0$ *are elements of* $\Sigma^0$, $p_{jk}$ *are real numbers and*

$$P_j=P_j^0+\sum_{k=1}^{n}p_{jk}A_k \qquad (j=1,2,\ldots,r),$$

*then*

$$\Phi(P_1,\ldots,P_r)=F^0+\sum_{k=1}^{n}f_kA_k, \tag{2}$$

*where* $F^0$ *is an element of* $\Sigma^0$ *and the coefficients* $f_k$ *are real numbers which do not depend on* $P_1^0,\ldots,P_r^0$.

Proof. Let $P_1^0,\ldots,P_r^0$ and $Q_1^0,\ldots,Q_r^0$ be any two sets of elements of $\Sigma^0$. If

$$Q_j=Q_j^0+\sum_{k=1}^{n}p_{jk}A_k \qquad (j=1,2,\ldots,r),$$



we can write

$$\Phi(Q_1,\ldots,Q_r)=G^0+\sum_{k=1}^{n}g_kA_k, \tag{3}$$

where $G^0\epsilon\Sigma^0$ and the $g_k$ are real numbers.

To have the theorem, it suffices to prove that

$$f_k=g_k \quad \text{for} \quad 1\leqslant k\leqslant n.$$

In view of (1), we have

$$\Phi(P_1,\ldots,P_r)-\Phi(Q_1,\ldots,Q_r)=H^0,$$

where $H^0\epsilon\Sigma^0$. Substituting here (2) and (3), we are led to the equality

$$\sum_{k=1}^{n}(f_k-g_k)A_k=H^0-F^0+G^0,$$

which implies $f_k-g_k=0$ $(k=1,\ldots,n)$, for the elements $A_1,\ldots,A_n$ are linearly independent on $\Sigma^0$.

**3.** *A transformation* $\Theta A$, mapping $\Sigma$ into itself, is *linear*, when

$$\Theta(A+B)=\Theta A+\Theta B$$

and

$$\Theta(aA)=a\Theta A.$$

It is *one-to-one* if, and only if, $\Theta A=0$ implies $A=0$. We shall reserve, in this paragraph, the symbol $\Theta$ to denote such linear one-to-one transformations which are identical on $\Sigma^0$, *i. e.* such that $\Theta A^0=A^0$ for $A^0\epsilon\Sigma^0$.

THEOREM II. *Suppose that*

$$\Theta\Phi(P_1,\ldots,P_r)=\Phi(\Theta P_1,\ldots,\Theta P_r) \tag{4}$$

*for any transformation* $\Theta$.

*If* $A_1,\ldots,A_m$ $(m\leqslant n)$ *are linearly independent on* $\Sigma^0$, $P_j$ *are elements of* $\Sigma^0$, $p_{jk}$ *are real numbers and*

$$P_j=P_j^0+\sum_{k=1}^{m}p_{jk}A_k \qquad (j=1,2,\ldots,r),$$

*then*

$$\Phi(P_1,\ldots,P_r)=F^0+\sum_{k=1}^{m}f_kA_k,$$

*where the element* $F^0$ $(F^0\epsilon\Sigma^0)$ *and the coefficients* $f_n$ *(real numbers) do not depend on* $A_1,\ldots,A_m$.

Proof. Let $A_1,\ldots,A_m$ and $B_1,\ldots,B_m$ be any two sets of elements of $\Sigma$, linearly independent on $\Sigma^0$. We can complete these sets to sets of $n$ elements linearly independent on $\Sigma^0$.

Then we can write

(5) $$\Phi(P_1,\ldots,P_r)=F^0+\sum_{k=1}^{n} f_k A_k.$$

Similarly, if

$$R_j=P_j^0+\sum_{k=1}^{m} p_{jk} B_k \qquad (j=1,2,\ldots,r),$$

we can write

(6) $$\Phi(R_1,\ldots,R_r)=G^0+\sum_{k=1}^{n} g_k B_k,$$

where $G^0 \epsilon \Sigma^0$ and the $g_k$ are real numbers.

To have the theorem, it suffices to prove that $F^0=G^0$, $f_k=g_k$ for $1\leqslant k\leqslant m$ and $f_k=g_k=0$ for $m<k\leqslant n$.

There exists a transformation $\Theta A$ such that

$$\Theta A_k=B_k \qquad (k=1,2,\ldots,n).$$

Then $\Theta P_j=R_j$ $(j=1,\ldots,r)$. By (5) we have

(7) $$\Theta\Phi(P_1,\ldots,P_r)=F^0+\sum_{n=1}^{n} f_k B_k.$$

But, in view of (4), the left members of (7) and (6) are equal, and, consequently, so must be their right members:

(8) $$F^0+\sum_{k=1}^{n} f_k B_k=G^0+\sum_{k=1}^{n} g_k B_k.$$

Hence

$$\sum_{k=1}^{n}(f_k-g_k)B_k=G^0-F^0,$$

which implies $f_k=g_k$ $(k=1,\ldots,n)$ and $F^0=G^0$, since the elements $B_1,\ldots,B_n$ are linearly independent on $\Sigma^0$. Thus, in the case of $m=n$, the theorem is proved.

If $m<n$, there exists a transformation $\Theta_1 A$ such that

$$\Theta_1 A_k=\begin{cases} B_k & \text{for } 1\leqslant k\leqslant m,\\ -B_k & \text{for } m<k\leqslant n.\end{cases}$$

By (5) we have

(9) $$\Theta_1\Phi(P_1,\ldots,P_r)=F^0+\sum_{k=1}^{m} f_k B_k-\sum_{k=m+1}^{n} f_k B_k.$$

Again, by (4), the left members of (9) and (6) are equal and so must be their right members:

(10) $$F^0+\sum_{k=1}^{m} f_k B_k-\sum_{k=m+1}^{n} f_k B_k=G^0+\sum_{k=1}^{n} g_k B_k.$$



Subtracting (10) from (8), we get

$$2\sum_{k=m+1}^{n} f_k B_k=0,$$

which implies $f_k=0$ for $m<k\leqslant n$. This completes the proof.

## III. Algebraic foundations of the Dimensional Analysis

**1.** The foundation of the Dimensional Analysis can be conveniently embedded into the theory of linear space. However, with regard to the physical interpretation, we shall use the multiplicative form of linear space.

Let $\Pi$ denote the space in question. The following axioms for $\Pi$ correspond to the axioms of $\Sigma$, given in Chapter II ($A$, $B$, $C$, $X$ are elements of $\Pi$ and $a,b$ are real numbers):

1. $AB=BA$;
2. $(AB)C=A(BC)$;
3. the solution $X$ of $AX=B$ exists for any pair $A$, $B$ of elements of $\Pi$;
4. $A^{a+b}=A^a A^b$;
5. $(AB)^a=A^a B^a$;
6. $(A^a)^b=A^{ab}$;
7. $A^1=A$.

Assume that the positive numbers $a$ belong to $\Pi$ and that their powers $a^b$ are calculated as usually. Thus the positive numbers can be considered as a subspace $\Pi^0$ of $\Pi$ (satisfying the same axioms as $\Pi$). It is easy to show that $1A=A$ and $A^0=1$ for any element $A\epsilon\Pi$.

Any element of $\Pi$ which does not belong to $\Pi^0$, *i. e.* which is not a number, will be called a *dimensional quantity*.

The elements $A_1,\ldots,A_m$ of $\Pi$ will be called *dimensionally independent* when the equality

$$A_1^{a_1}\ldots A_m^{a_m}=a,$$

where $a$ is a number, implies $a_1=\ldots=a_m=0$ (and $a=1$).

We shall suppose, in what follows, that there exist, in $\Pi$, $n$ elements dimensionally independent, but not $n+1$ such elements.

Any set $X_1,\ldots,X_n$ of dimensionally independent elements of $\Pi$ will be called a *system of units*. Obviously, such a system cannot contain numbers.

If $X_1, \ldots, X_n$ is a system of units, then any element $A$ of $\Pi$ can be uniquely represented in the form

$$A = \alpha X_1^{a_1} \ldots X_n^{a_n}, \tag{1}$$

where $a_1, \ldots, a_n$ are real numbers, and $\alpha > 0$.

If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are two systems of units, there exist $n$ positive numbers $\xi_1, \ldots, \xi_n$ and a non-singular square matrix $(t_{ij})$ of $n^2$ real numbers $t_{ij}$ such that

$$X_i = \xi_i \prod_{j=1}^{n} Y_j^{t_{ij}} \qquad (i = 1, 2, \ldots, n). \tag{2}$$

These properties of systems of units correspond to the well known properties of linearly independent elements in the theory of linear space.

Substituting (2) in (1), we have

$$A = \alpha \prod_{i=1}^{n} \prod_{j=1}^{n} \xi_i^{a_i} Y_j^{t_{ij} a_j}. \tag{3}$$

This formula enables us to pass from a given system of units to any other.

**2.** A transformation $\Delta A$, mapping $\Pi$ into itself, will be called a *dimensional transformation* when

$$\Delta(AB) = (\Delta A)(\Delta B),$$

$$\Delta(A^a) = (\Delta A)^a,$$

$$\Delta A = 1 \quad \text{implies} \quad A = 1,$$

$$\Delta \alpha = \alpha \quad \text{for } \alpha \in \Pi^0.$$

The first two of these axioms correspond to the axioms of linear transformation in the theory of linear space. The third axiom ensures that the transformation $\Delta$ shall be one-to-one. The last axiom means that $\Delta$ is identical on $\Pi^0$.

**3.** We shall further consider the functions

$$\Phi(Q_1, \ldots, Q_s),$$

where $Q_1, \ldots, Q_s$ and the value of $\Phi$ belong to $\Pi$.

Such a function will be called *dimensionally invariant* if the identity

$$\Phi(\Delta Q_1, \ldots, \Delta Q_s) = \Delta \Phi(Q_1, \ldots, Q_s) \tag{4}$$

holds for any dimensional transformation $\Delta$ and any set $Q_1, \ldots, Q_s$ of elements of $\Pi$.



It will be called *dimensionally homogeneous* if, given any system of elements $Q_1, \ldots, Q_s$ of $\Pi$ and any system $\varrho_1, \ldots, \varrho_s$ of positive numbers, there exists a number $\varrho$ such that

$$\Phi(\varrho_1 Q_1, \ldots, \varrho_s Q_s) = \varrho \Phi(Q_1, \ldots, Q_s). \tag{5}$$

**4.** THEOREM II. *Let $\Phi(A_1, \ldots, A_m; P_1, \ldots, P_r)$ be a dimensionally invariant and homogeneous function. If $A_1, \ldots, A_m$ $(m \leqslant n)$ are dimensionally independent and*

$$P_j = \pi_j \prod_{k=1}^{m} A_k^{p_{jk}} \qquad (j = 1, 2, \ldots, r),$$

*where $\pi_j$ and $p_{jk}$ are real numbers, $\pi_j > 0$, then*

$$\Phi(A_1, \ldots, A_m; P_1, \ldots, P_r) = \varphi \prod_{k=1}^{m} A_k^{f_k}, \tag{6}$$

*where the coefficient $\varphi$ (positive number) does not depend on $A_1, \ldots, A_m$, and the exponents $f_k$ (real numbers) depend neither on $A_1, \ldots, A_m$ nor on $\pi_1, \ldots, \pi_r$.*

To obtain the proof of this theorem, it suffices to write it in the additive form. Then the theorem follows directly from Theorems I and II of the preceding Chapter.

## IV. Remarks on Theorem II

**1.** Usually, the conditions under which Theorem II holds are not clearly specified. Thus it will be of interest to show that this theorem fails when either of the two assumptions on $\Phi$, its dimensional invariance or its homogeneity, is omitted. In fact:

1° Let $n = 1$ and let $X_0$ be any *fixed* dimensional unit. Then, each element of $\Pi$ is of the form

$$A = \alpha X_0^a,$$

where $\alpha$ and $a$ are real numbers, $\alpha > 0$. The function $\Phi(A) = a$ is obviously dimensionally homogeneous but not dimensionally invariant, for, if $\Delta A = \alpha X_0^{2a}$, we have

$$\Delta \Phi(A) = a \qquad \text{and} \qquad \Phi(\Delta A) = 2a.$$

The Theorem II fails, for the coefficient $\varphi$ (equal to $a$) depends on $A$.

2° The function $\Phi(\pi A) = A^{\pi}$ is dimensionally invariant but it is not homogeneous, since

$$\Phi(\varrho \pi A) = A^{\varrho \pi}.$$

The Theorem II fails, for the exponents depend on $\pi$.

**2.** On the other hand, it is customary to write $\varphi(\pi_1, \ldots, \pi_r)$ instead of $\varphi$ in the formula (6). The aim is to make it evident that $\varphi$ depends on

$\pi_1, \ldots, \pi_r$ but not on $A_1, \ldots, A_m$. But then it is not clear enough what relationship there is between $\varphi$ and $p_{jk}$. The answer is that $\varphi$ really depends on $p_{jk}$.

In fact, it is a question of mere verification to show that, in (6), $\varphi$ can be a completely arbitrary positive-valued function of $\pi_j$ and $p_{jk}$. Similarly, the exponents $f_k$ can be completely arbitrary functions of $p_{jk}$. That means that the conversion of Theorem II holds without any restrictions.

Thus, the use of the symbol $\varphi(\pi_1, \ldots, \pi_r)$ is correct only under the assumption that all numbers $p_{jk}$ are fixed in the problem in question. In that case, the exponents $f_k$ are to be treated, of course, as constant numbers.

**3.** Lastly, it is worth while to notice that, in the particular case where the function $\Phi$ of Theorem II depends on $A_1, \ldots, A_m$ only ($A_1, \ldots, A_m$ are dimensionally independent), we have

$$\Phi(A_1, \ldots, A_m) = \varphi \prod_{k=1}^{m} A_k^{f_k},$$

and the numbers $\varphi$ and $f_k$ are constant. This is a direct corollary of Theorem II. However, in this case, the assumption that $\Phi$ is a dimensionally homogeneous function is superfluous. To see this, it suffices again to write the theorem in the additive form and apply Theorem II of Chapter II.

## V. The physical interpretation

**1.** As examples of dimensional quantities appearing in physics we can give the length of a segment, the mass of a body, its velocity, a time interval, the temperature of a gas, its pressure, the electrical charge of a particle. These quantities are to be considered as elements of $\Pi$.

In physics, one considers products and powers of these quantities and their products by numbers. The laws of these operations are expressed by axioms 1-7 of Chapter III.

**2.** In considering any physical phenomenon we must introduce some system of units. If a quantity $A$ is represented in the system $X_1, \ldots, X_n$ by formula (1) of Chapter III, then formula (3) allows us to represent it in another system $Y_1, \ldots, Y_n$. That formula can also be interpreted as a transformation of a system of units. However, instead of considering transformations of systems of units it is convenient to consider transformations of the quantity $A$ itself, which is mathematically the same. Namely, the transformation of $A$, corresponding to (3) is described, in the system of units $X_1, \ldots, X_n$, by the formula obtained from (3) by replacing $Y_j$ by $X_j$ in it. Such a transformation is characterized by the four axioms of Chapter III; thus it is a dimensional transformation $\Delta$.



**3.** The formula (4) of Chapter III means that, if a transformation $\Delta$ is applied to arguments of a function $\Phi$, then the same result can be obtained by applying the same transformation $\Delta$ to the value of the function $\Phi$ itself. Coming back to transformations of systems of units we reason as follows. The function $\Phi$ depends on quantities $Q_j$. Those quantities are given in two systems of units $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Thus the value of $\Phi$ can be calculated in either of the two systems by means of appropriate coefficients and exponents. The condition imposed by (4) is that in both cases the value of $\Phi$ should be the same quantity (but expressed in different systems of units). This property is often formulated as follows: the form of the function does not depend on the chosen system of units [2], [8].

**4.** Lastly, the formula (5) means that if each argument of $\Phi$ is multiplied by a numerical factor, then the value of $\Phi$ should also be multiplied by a numerical factor. The physical meaning of this will be explained by introducing the notion of dimension.

## VI. The dimension

**1.** It is usual to say that *two elements $A$ and $B$ of $\Pi$ have the same dimension* if there is a number $a$ such that $B = aA$. Then we write in symbols, introduced by Maxwell,

$$[A] = [B].$$

For example 3 m and 5 cm have the same dimension called *length*, for $3\,\mathrm{m} = 60 \cdot 5\,\mathrm{cm}$; in symbols

$$[3\,\mathrm{m}] = [5\,\mathrm{cm}] = [L] \text{ (length)}.$$

The space $\Pi$ can be divided into disjoint classes such that all elements of the same class have the same dimension. Thus, it is natural, from the formal point of view, to identify the dimension of a given quantity with the class to which this quantity belongs. It is convenient to introduce, for practical reasons, the notion of product and of power of dimensions; the definitions are as follows:

$$[A][B] = [AB], \quad [A]^a = [A^a].$$

Particularly, if $a$ is a number, it follows from the first formula that $[a][B] = [B]$ and hence $[a] = 1$. Thus, the dimension of any number is 1.

It is easy to verify that the dimensions satisfy the axioms 1-7 from Chapter III. Thus, the set of all dimensions can be treated as a space of the same type as the space $\Pi$.

**2.** Now, it is easily seen from formula (5) that if the arguments of $\Phi$ retain their dimension, so does the value of $\Phi$. This is the physical interpretation of the formula (5) and constitutes the starting point for *model testing*.

In physics, a phenomenon is described by a function $\Phi$. We say that a phenomenon, called *model*, is similar to a phenomenon, called *prototype*, when the following conditions are satisfied: the function $\Phi$ for the model and for the prototype is the same, the arguments and values of $\Phi$ for the model and these for the prototype have the same dimensions and differ by such numerical factors that the numbers $\pi_1,\ldots,\pi_r$ of Theorem $\Pi$ retain their values. This is the *principle of similitude*. The numbers $\pi_1,\ldots,\pi_r$ are called *invariants of similitude*.

**3.** Let the arguments $A_1,\ldots,A_m$ have in a system of units $X_1,\ldots,X_n$ the dimensions

$$[A_j]=\prod_{k=1}^{n}[X_k^{a_{jk}}] \qquad (j=1,2,\ldots,m).$$

The quantities $A_1,\ldots,A_m$ are dimensionally independent if, and only if, the determinant $|a_{jk}|$ does not vanish.

Let a quantity $F$ have, in the same system of units $X_1,\ldots,X_n$, the dimension

$$[F]=\prod_{k=1}^{n}[X_k^{a_k}].$$

In applications, it is often important to know whether $F$ can be expressed as a function $\Phi$ of $A_1,\ldots,A_m$ (satisfying (4) and (5)). The answer is that it is possible if, and only if, the equations

$$\sum_{j=1}^{n}a_{jk}f_j=a_k \qquad (k=1,2,\ldots,n)$$

have exactly one solution.

This is an easy corollary of Theorem $\Pi$.

**4.** So far, the only operations on dimensional quantities have been products, powers and multiplication by numbers. However, for practical purposes, it is convenient to introduce also addition and subtraction. The last two operations can be performed only on quantities of the same dimension. The equalities

$$\alpha A+\beta A=(\alpha+\beta)A,$$
$$\alpha A-\beta A=(\alpha-\beta)A,$$



are, formally, to be considered as definition of those operations. In order to make subtraction always possible it is natural to introduce also the elements $aA$ with non-positive coefficients. Thus we put stress on the fact that the quantities $aA$ with non-positive coefficient $a$ do not belong to the original space $\Pi$.

Addition and subtraction are defined, so to say, within the dimensions under consideration; they consist in performing appropriate operations on numerical coefficients. In the same way, it is possible to introduce other operations, like differentiation, integration, etc. *E. g.* we can introduce the limit of an infinite sequence, writing

$$\lim_{n\to\infty}(a_nA)=(\lim_{n\to\infty}a_n)A;$$

all elements of the sequence, as well as their limit, have, of course, the same dimension.

### VII. Examples and paradoxes

We shall consider some examples, the first two of them being known as paradoxical [2], [7], [10], [11], [12].

**1.** A rigid ball of diameter $D$ and heat capacity $C$ moves with the velocity $V$ in a gas, of which the heat conductivity is $H$. The temperature in the centre of the ball differs from the temperature of the gas (at a great distance from the ball) by $T$. How does the rate $U$ of the change of the heat depend on the quantities $D, C, V, H, T$?

To obtain the solution we must determine the form of the function

$$U=\Phi(D,H,T,V,C).$$

Adopt the system of the *four* units: cm, g, sec, degree. Then, as we learn in physics, the unit of heat, cal, has the dimension

$$[\mathrm{cal}]=[\mathrm{cm^2\,g\,sec^{-2}}]$$

and

$$[D]=[\mathrm{cm}],\qquad [C]=[\mathrm{cm^{-3}\,cal\,deg^{-1}}]=[\mathrm{cm^{-1}\,g\,sec^{-2}\,deg^{-1}}],$$
$$[V]=[\mathrm{cm\,sec^{-1}}],\qquad [H]=[\mathrm{cm^{-1}\,sec^{-1}\,cal\,deg^{-1}}]=[\mathrm{cm\,g\,sec^{-3}\,deg^{-1}}],$$
$$[T]=[\mathrm{deg}],\qquad [U]=[\mathrm{sec^{-1}\,cal}]=[\mathrm{cm^2\,g\,sec^{-3}}].$$

The four quantities $D, V, H, T$ are dimensionally independent, for

$$\begin{vmatrix}1&0&0&0\\1&0&-1&0\\1&1&-3&-1\\0&0&0&1\end{vmatrix}=-1\neq0.$$

Then, the quantity $C$ can be expressed by $D, V, H, T$:

$$C=\pi_1 D^{-1} V^{-1} H.$$

By Theorem II, we have

$$U=\varphi_1(\pi_1) D^d V^v H^h T^t,$$

where $\varphi_1$ is a numerical factor which depends on $\pi_1$ only, and the real exponents $d, v, h, t$ can be determined from the equality

$$[\mathrm{cm}^2\,\mathrm{g}\,\mathrm{sec}^{-3}]=[(\mathrm{cm})^d(\mathrm{cm}\,\mathrm{sec}^{-1})^v(\mathrm{cm}\,\mathrm{g}\,\mathrm{sec}^{-3}\,\mathrm{deg}^{-1})^h(\mathrm{deg})^t].$$

We obtain $d=1$, $v=0$, $h=1$, $t=1$. Thus

(1) $$U=\varphi_1(\pi_1) DHT,$$

where $\pi_1=DCV/H$.

Now, we are going to solve the same problem using a complementary assertion of statistical physics which says that

(1.2) $$[\mathrm{deg}]=[\mathrm{cal}].$$

Thus, the system gets reduced to *three* units: cm, g, sec. By Theorem II, we have

(2) $$U=\varphi_2(\pi_1,\pi_2) DHT,$$

where $\pi_1=DCV/H$ and $\pi_2=D^3C$.

The only difference between the formulae (1) and (2) is that the numerical coefficients depend on the single variable $\pi_1$ in the first case, and on two variables $\pi_1$, $\pi_2$ in the second case. Thus, formula (2) gives us less information on the phenomenon described than formula (1) does. This seems to be a paradox, for, to establish formula (2), we have used complementarily the hypothesis (1.2), and therefore we should expect a more precise result.

But this paradox turns out to be only apparent if we notice that, in formula (1), the use of Theorem II needs another hypothesis, namely that the four dimensions, [cm], [g], [sec] and [deg], are independent. Thus, the number of hypotheses in both cases is the same.

The qualitative difference between the first hypothesis, namely that cm, g, sec, deg are dimensionally independent, and the second hypothesis (1.2) is the following:

The first hypothesis means that the equality

$$[\mathrm{cm}^{a_1}\,\mathrm{g}^{a_2}\,\mathrm{sec}^{a_3}\,\mathrm{deg}^{a_4}]=1$$

implies

$$a_1=a_2=a_3=a_4=0.$$



The second hypothesis means that the equality

$$[\mathrm{cm}^{a_1}\,\mathrm{g}^{a_2}\,\mathrm{sec}^{a_3}\,\mathrm{deg}^{a}]=[\mathrm{cm}^{a_1}\,\mathrm{g}^{a_2}\,\mathrm{sec}^{a_3}(\mathrm{cm}^2\,\mathrm{g}\,\mathrm{sec}^{-2})^{a_4}]=1$$

implies

$$a_1+2a_4=a_2+a_4=a_3-2a_4=0.$$

We see that the first hypothesis implies the second one, thus the first of them is *stronger*.

Therefore the class of the functions considered by the second hypothesis is larger than the class of the functions considered by the first hypothesis. Each function, which exists in the space with four units, cm, g, sec, deg, exists also in the space with three units, cm, g. sec, provided we assume (1.2). The physical meaning of this remark is that the hypothesis (1.2) enlarges the physical theory of the same phenomenon.

The formulae (1) and (2) describe the same phenomenon in two different theories. Which of the two theories is true is a question which concerns not the Dimensional Analysis but experience. However, we see that the Dimensional Analysis allows us, after appropriate experiments, to argue about the correctness of physical theories.

Lastly, let us remark that formula (2) can better be adapted to the result of the experiment than formula (1). In fact, if formula (1) is verified by an experiment, then formula (2) is also verified by the same experiment since formula (2) is then formally valid, for we can always assume that $\varphi_2(\pi_1,\pi_2)$ is constant in regard to $\pi_2$. On the other hand, if formula (1) is not verified by experiment, then it is still possible that formula (2) will be verified by experiment.

This example shows how the result furnished by the Dimensional Analysis can and must be completed by experiments or other theories. In the example from Chapter I the determining of the numerical coefficient $\alpha$ belongs to experiment, if, *nota bene*, the formula is verified by experiment.

**2.** How does the period $T$ of vibrations of a mathematical pendulum depend on its length $L$, mass $M$ and on the gravity acceleration $G$?

To determine by Dimensional Analysis the form of the function

$$T=\Phi(L, M, G)$$

we adopt the system of units cm, g, sec. Then

$$[T]=[\mathrm{sec}], \qquad [L]=[\mathrm{cm}], \qquad [M]=[\mathrm{g}], \qquad [G]=[\mathrm{cm}\,\mathrm{sec}^{-2}].$$

By Theorem II, we have

$$T=\text{number}\sqrt{\frac{L}{G}}.$$

It should seem that the assumption of the period $T$ depending on the gravity acceleration $G$ is artificial. It is, namely, a well known empirical fact, that in any fixed place of the earth the period $T$ depends on the length $L$ only. But if we assume that

$$T=\Phi(L),$$

then from the Corollary of Theorem II, Chapter VI, 3, it follows that such a function does not exist. This seems to be a paradox (cf. also [6]).

In my opinion, this is a good example of how the Dimensional Analysis can sometimes improve our empirical observations.

**3.** How does the size $P$ on a sample depend on the size $N$ of a lot of merchandise, on the cost $K$ of the control and on the standard deviation $S$ of value in the lot?

To determine, by the Dimensional Analysis, the form of the function

$$P=\Phi(N, K, S)$$

we adopt the following system of units: the size of the lot of merchandise (this unit we call PIECE), the size of the sample (this unit we call piece) and the value of the merchandise (this unit we call £). We construct, [5], the theory of quality control by sampling, where it is proved that

$$[P]=[\text{piece}],\quad [N]=[\text{PIECE}],\quad [K]=[£\,\text{piece}^{-1}],\quad S=[£\,\text{PIECE}^{-1/2}].$$

By Theorem II, we have

$$P=\beta\frac{S\sqrt{N}}{K}, \tag{3}$$

where $\beta$ is a numerical factor.

It is interesting to notice that formula (3) holds for various economical conditions. If a definite economical condition is given, the factor $\beta$ can be determined. Formula (3) is known in practice as empirical.

**4.** Let us consider the differential equation

$$FX^2\frac{\partial^2 F}{\partial X\,\partial Y}=K\left\{\left(X\frac{\partial F}{\partial X}\right)^2-\left(Y\frac{\partial F}{\partial Y}\right)^2\right\}, \tag{4}$$

where $F$ is an unknown function and $K$ is a parameter.

We shall consider the variables $F, X, Y, K$ as dimensional quantities. We adopt the quantities $X, Y$ as the system of units. Then, from (4), it follows that

$$\xi=\frac{KY}{X}$$



is a number. Theorem II leads to the equality

$$F(X,Y;K)=\varphi(\xi)X^aY^b,$$

where $a, b$ are real numbers and $\varphi(\xi)$ satisfies the ordinary differential equation

$$\xi^2\frac{d^2\varphi}{d\xi^2}-\{2(a+b)\xi^2+(a-b-1)\xi\}\frac{d\varphi}{d\xi}+\{(a^2-b^2)\xi-ab\}\varphi=0.$$

The general solution of this equation is a four-parameter family of the particular solutions of (4). For example, if $a=b=0$, we get

$$F=a\int_{\xi_0}^{KY/X}\exp\left(-\frac{1}{2}\xi^2\right)d\xi,$$

where $a$ and $\xi_0$ are arbitrary numerical constants.

For some linear partial equations, this method can give a general solution, [12].

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