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Some Impredicative Definitions in the Axiomatic Set-Theory.

By

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Let (S) denote the Zermelo-Fraenkel set-theory based on the following axioms

$$(A_1)\quad (x_1,x_2)\,[(x_3)\,(x_3 \in x_1 \equiv x_3 \in x_2) \supset x_1 = x_2],$$

$$(A_2)\quad (x_1,x_2)\,(\exists x_3)\,(x_4)\,[x_4 \in x_3 \equiv (x_4 = x_1 \vee x_4 = x_2)],$$

$$(A_3)\quad (x_1)\,(\exists x_2)\,(x_3)\,[x_3 \in x_2 \equiv (x_4)\,(x_4 \in x_3 \supset x_4 \in x_1)],$$

$$(A_4)\quad (x_1)\,(\exists x_2)\,(x_3)\,[x_3 \in x_2 \equiv (\exists x_4)\,(x_3 \in x_4 \cdot x_4 \in x_1)],$$

$$(A_5)\quad (\exists x_1)\,(\exists x_2)\,(x_2 \in x_1 \cdot (x_2)\,\{x_2 \in x_1 \supset (\exists x_3)\,[x_2 \neq x_3 \cdot x_3 \in x_1 \cdot (x_4)\,(x_4 \in x_2 \supset x_4 \in x_3)]\}),$$

$$(A_6)\quad (x_k)\,(x_{k_1},\ldots,x_{k_p})\,\{(x_l)\,[x_l \in x_k \supset (\exists x_m)\,(x_n)\,(\Phi \equiv x_n = x_m)] \supset \supset (\exists x_q)\,(x_n)\,[x_n \in x_q \equiv (\exists x_l)\,(x_l \in x_k \cdot \Phi)]\},$$

$$(A_7)\quad (x_{k_1},\ldots,x_{k_p})\,\{(\exists x_k)\Phi \supset (\exists x_k)\,[\Phi \cdot (x_l)\,(x_l \in x_k \supset \sim \Phi')]\}\ [1]).$$

(A_6) and (A_7) are axiom schemata. The letter Φ in (A_6) replaces any expression (with free variables $x_l, x_n, x_{k_1}, \ldots, x_{k_p}$, and x_k [2]) built up according to the following rules: If i and j are integers, then $x_i \in x_j$ and $x_i = x_j$ are formulas; if Θ is a formula and j an integer, then $(\exists x_j)\Theta$ is a formula; if Θ and Z are formulas, then so is $\Theta | Z$ [3]). We assume that x_q is not free in Φ.

The letter Φ in (A_7) replaces a formula with free variables $x_k, x_{k_1}, \ldots, x_{k_p}$ and Φ' replaces the formula resulting from Φ by substitution of the letter x_l for x_k on every place where x_k is free in Φ. It is supposed that x_l is not bound in Φ.

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[1]) (A_1) is the axiom of extensionality, (A_2) — the pair-axiom, (A_3) — the powerset axiom, (A_4) — the sum-set axiom, (A_5) — the axiom of infinity, (A_6) — the axiom of replacement, and (A_7) — the restrictive axiom (the „Axiom der Fundierung" of Zermelo).

[2]) x_k must not necessarily be a free variable of Φ.

[3]) Other logical connectives can be defined by the stroke | in the well-known manner.

We assume in (S) the well-known rules of proof, namely the *modus ponens*, the rule of substitution and the rules of omission and of introduction of quantifiers. Furthermore we assume special rules which enable us to prove every tautological formula including the identity-symbol:

R_1. *If Φ, Ψ, and Θ are formulas, then the formulas*

$$(\Phi\supset\Psi)\supset[(\Psi\supset\Theta)\supset(\Psi\supset\Theta)],\quad \Phi\supset(\sim\Phi\supset\Psi),\quad (\sim\Phi\supset\Phi)\supset\Phi$$

are provable.

R_2. *The formula $x_k=x_k$ is provable.*

R_3. *If Φ is a formula, x_l is not a bound variable of Φ and Φ' differs from Φ only by containing free occurrences of x_l on one or several places where Φ contains free occurrences of x_k, then the formula*

$$x_k=x_l\supset(\Phi\equiv\Phi')$$

is provable.

Let (S') be the Bernays-Gödel system of set-theory. We shall not describe the details of this system because it is sufficiently well known from the literature [4]). We remark only that every expression meaningful in (S) is also meaningful in (S') and every axiom of (S) is provable in (S').

It has been proved by Novak [5]) that if (S) is consistent, then (S') is also consistent [6]). Since this proof is formalizable in (S'), it follows that the consistency of (S) cannot be proved in (S'). On the other hand (S') arises from (S) by addition of variables of the next higher type and therefore the so-called definition of truth for (S) is formalizable in (S') [7]). Since the „whole theory of truth" makes it possible to prove the consistency of a system for which the notion of satisfaction has been defined [8]), we infer that certain properties of the notion of truth for (S) cannot be established in (S').

[4]) See, e. g., Bernays [1] or Gödel [2].

[5]) See Novak [3] and Rosser-Wang [5].

[6]) It can even be shown that every formula provable in (S') and expressible in (S) must be provable already in (S). We give here a simple proof based on results established by Novak [3]. Suppose that Φ is expressible in (S), provable in (S') but not provable in (S). Let (S_1) be the system got from (S) by addition of $\sim\Phi$ as a new axiom. Then (S_1) is consistent and the corresponding system (S_1') obtained from (S_1) by the method described by Novak must be consistent too. On the other hand (S_1') is at least as strong as (S') and therefore Φ is provable in (S_1') which is a contradiction because $\sim\Phi$ is evidently provable in (S_1').

A more elaborate proof is given in Rosser-Wang [5].

[7]) See section 1 below.

[8]) See Tarski [6], pp. 359, 392.

An exact analysis of this situation leads to the following three theorems the proofs of which will be sketched in this paper:

Theorem I. *There is an expression $V(x_1)$ of (S') with exactly one free variable x_1 such that if Φ is an arbitrary expression of (S) without free variables and n the Gödel number of Φ, then the equivalence*

$$\Phi\equiv V(n)$$

is provable in (S') [9]). *The formula $V(x_1)$ has the form $(\exists X)\,A(X,x_1)$ where $A(X,x_1)$ is a formula without bound class variables. If Φ is a theorem of (S), then $V(n)$ is provable in (S'), but the general theorem*

$$(x_1)\,[x_1 \text{ is the Gödel number of a theorem of } (S)\supset V(x_1)]$$

is not provable in (S') provided that (S') is consistent.

Theorem II. *There is an expression $\Theta(x_1)$ of the form $(\exists X)\,B(X,x_1)$ where $B(X,x_1)$ does not contain bound class variables such that the formulas*

$$\Theta(1)\quad and\quad (n)\,[\Theta(n)\supset\Theta(n+1)]$$

are both provable in (S') but $(n)\,\Theta(n)$ [10]) *is not provable in (S') provided that (S') is consistent.*

Theorem III. *There is an expression $\Phi(x_1)$ of the form $(\exists X)\,C(X,x_1)$ where $C(X,x_1)$ does not contain bound class variables such that the formula*

$$(\exists X)\,(x_1)\,[x_1\,\epsilon\,X\equiv\Phi(x_1)]$$

is not provable in (S') provided that (S') is consistent [11]).

1. We begin with the proof of theorem I. In order to construct the formula $V(x_1)$ with the properties required in the theorem we shall formalize in (S') the definition of satisfaction and of truth for (S). Let us therefore recall briefly these definitions.

[9]) $V(n)$ is the expression resulting from $V(x_1)$ by substitution of the n-th numeral (suitably defined in (S')) for the variable x_1.

[10]) The variable n ranges over the set of integers defined in (S').

[11]) This theorem shows that the system NQ considered by Wang [7] is essentially stronger than the system of Bernays-Gödel. This follows also from the fact that the consistency of the Bernays-Gödel system is provable in NQ.

For every formula Φ of (S) there exists a sequence of formulas

(1) $$\Phi_1, \Phi_2, \ldots, \Phi_n = \Phi$$

such that for every $i \leqslant n$ Φ_i is either a formula of the form $x_k = x_l$ or of the form $x_k \in x_l$ or one of the following two cases is satisfied:

(2) *there are integers* j, h *less than* i *such that* $\Phi_i = \Phi_j | \Phi_h$,

(3) *there are integers* j *and* m *such that* $j < i$ *and* $\Phi_i = (\exists x_m)\Phi_j$.

A sequence (1) satisfying these conditions will be called a *construction-sequence* or briefly a *C-sequence* for Φ.

We denote by s_i the set of integers q such that x_q is free in Φ_i.

A *finite sequence of sets* is defined as a finite set f of ordered pairs $\langle u,v \rangle$ such that if $\langle u,v \rangle \in f$ and $\langle u,v_1 \rangle \in f$, then $v = v_1$. The u's of the pairs $\langle u,v \rangle$ belonging to f form the *domain* $D(f)$ of f. If $\langle u,v \rangle \in f$, we write $v = f(u)$. If $s \subset D(f)$, then $f|s$ is the set of pairs $\langle u,v \rangle$ such that $u \in s$ and $\langle u,v \rangle \in f$.

A *finite sequence of classes* is defined as a class F of finite sequences of sets with a common domain D which is at the same time called the *domain of* F[12]. If $x \in D$, then the class of all y's such that there is a sequence f with the properties $\langle x,y \rangle \in f \in F$ is called the x-th term of F and denoted by F_x. Note that elements of F_x can be arbitrary sets, in particular arbitrary finite sequences of sets.

To each C-sequence (1) we let correspond a finite sequence of classes F with the domain D consisting of all integers $\leqslant n$. If $i \leqslant n$ and Φ_i is the formula $x_k = x_j$ or $x_k \in x_j$, then F_i is the class of all sequences f such that $D(f) = \{k,j\}$ and $f(k) = f(j)$ or $f(k) \in f(j)$.

If Φ_i satisfies the condition (2), then F_i is the class of all sequences f such that $D(f) = s_i$ and either $f|s_j \operatorname{non} \in F_j$ or $f|s_h \operatorname{non} \in F_h$.

Finally if Φ_i satisfies the condition (3) and x_m is free in Φ_i, then F_i is the class of all sequences f such that $D(f) = s_i$ and there exist an integer m and a set a for which $f + \{\langle m,a \rangle\} \in F_j$. If x_m is not free in Φ_j, then we put $F_i = F_j$.

A sequence of classes F which satisfies the above conditions is called an *S-sequence* for Φ corresponding to the C-sequence (1).

This definition says of course nothing about the existence of S-sequences.

We say that a sequence f *satisfies* Φ if there is an S-sequence F for Φ such that $f \in F_n$.

[12]) See Robinson [4].

If Φ has no free variables, then the only sequence which can possibly satisfy Φ is the void sequence 0. If $0 \in F_n$, we say that Φ is *true*, otherwise that Φ is *false*[13]).

It is clear that these definitions can be expressed in (S'). The only difficulty lies in the presence of the meta-mathematical notions „formula“, „quantifier“ etc. It is however possible to eliminate all these notions in favour of the purely arithmetical ones, using the well-known technique of the Gödel numbers and identifying the „linguistic“ concepts with their arithmetical counterparts.

The definition of satisfaction thus formalized in (S') takes on the form of a formula $Stsf(x_1,x_2) \equiv (\exists X) M(X,x_1,x_2)$ where x_1 runs over the set of the Gödel numbers of formulas and x_2 over the class of finite sequences of sets. The definition of truth takes on the form of a formula $V(x_1)$ of the form $(\exists X) A(X,x_1)$:

$$V(x_1) \equiv (\exists x_2)\,(x_3)\,[\sim(x_3 \in x_2) \cdot Stsf\,(x_1,x_2)]$$
$$\equiv (\exists X)\,\{(\exists x_2)\,(x_3)\,[\sim(x_3 \in x_2) \cdot M(X,x_1,x_2)]\}.$$

We shall now prove a series of lemmas which will lead to the proof of theorem I. All these lemmas are concerned with properties of formulas of (S) and since we wish to state and to prove them in (S') we must explain in a few words in what way such theorems can be expressed in (S').

There are two different ways to express in (S') meta-mathematical theorems about (S). One of them uses the method of Gödel and identifies expressions with their Gödel numbers. Instead to say that every expression Φ of this or other class K possesses a property P we say that every integer which is the Gödel number of an expression from the class K possesses the property P' obtained from P by substitution of the arithmetical notions for the corresponding meta-mathematical ones.

Whether this method is applicable or not depends on the nature of the class K and of the property P and notably on the possibility to define K and P by means formalizable in (S').

A theorem about formulas expressed in this way in the symbolism of (S') becomes a single theorem of (S').

Another possible method is to express theorems about formulas of (S) as theorem schemata of (S'). The general theorem *every* Φ *has the property* P is then expressed in the form of an in-

[13]) See Tarski [6], pp. 313-314.

finite sequence of theorems of (S') each formula Φ contributing one theorem to the sequence. This method is sometimes advantageous because the arithmetical counterpart P' of P is not always definable in (S') and even if it is definable in (S'), the general theorem described in the preceeding paragraph does not need to be provable in (S') although each formula of the sequence representing the theorem schema is provable in (S'). We shall see later that both situations can actually occur (cf. lemmas Σ_4 and Σ_5).

In order to facilitate our exposition we shall always identify formulas of (S) with their Gödel numbers and shall avoid as far as possible the use of logical symbols. These simplifications, convenient though they are, obliterate sometimes completely the difference between single theorems and theorem schemata of (S'). We shall therefore denote theorems by the letter „T" and theorem schemata by the letter „Σ". We remark that proofs of all theorems and of all particular instances of the schemata are based exclusively on the axioms of (S').

T_1. *For every formula of the form* $x_k = x_j$ *or* $x_k \in x_j$ *there exists an* S*-sequence.*

Proof. It is sufficient to take for this sequence a one term sequence F such that F_1 is the class of all sequences $\{\langle k,a\rangle, \langle j,b\rangle\}$ where $a=b$ or $a \in b$.

T_2. *If* Φ *and* Ψ *are two formulas for which there exist* S*-sequences, then there exists an* S*-sequence for the formula* $\Phi|\Psi$.

Proof. Let F and G be the S-sequences for the formulas Φ and Ψ and let F_m and G_n be the last terms of these sequences. Finally let s_1 be the set of integers i for which x_i is free in Φ and s_2 the set of integers i for which x_i is free in Ψ. From the axioms of (S') follows easily the existence of the class Z of all sequences f such that $D(f) = s_1 + s_2$ and either $f|s_1$ does not belong to F_m or $f|s_2$ does not belong to G_n. We obtain now an S-sequence H for the formula $\Phi|\Psi$ putting $H_i = F_i$ for $i \leqslant m$, $H_{m+j} = G_j$ for $j \leqslant n$, and $H_{m+n+1} = Z$.

T_3. *If there exists an* S*-sequence for an expression* Φ, *then there exists also an* S*-sequence for the expression* $(\exists x_m)\Phi$.

Proof is similar to that of T_2.

Let now n be one of the integers $1,2,3,\ldots$ Applying T_1, T_2, and T_3 n times we obtain the following theorem schema:

Σ_4. *If* Φ *is a formula of* (S), *then for every* C*-sequence* (1) *ending with* Φ *there exists a corresponding* S*-sequence.*

As indicated, Σ_4 is a theorem schema; the existence of the S-sequences stated in the schema is provable for each formula Φ separately. The general theorem

$$(\Phi)(\exists F)\quad [F \text{ is an } S\text{-sequence for } \Phi]$$

is expressible in (S') but we see no way to prove it from the axioms of (S').

Σ_5. *If* Φ *is an expression of* (S) *with free variables* $x_{k_1},\ldots,x_{k_p}$, (1) *a* C*-sequence for* Φ, *and* F *a corresponding* S*-sequence, then*

$$f = \{\langle k_1, x_{k_1}\rangle, \ldots, \langle k_p, x_{k_p}\rangle\} \supset (f \in F_n \equiv \Phi). \tag{4}$$

Proof. We proceed by induction with respect to n, the length of the C-sequence (1). If $n=1$, then Φ has either the form $x_{k_1} = x_{k_2}$ or the form $x_{k_1} \in x_{k_2}$ and F is a one termed sequence such that F_1 is the class of all sequences $\{\langle k_1, a\rangle, \langle k_2, b\rangle\}$, where $a=b$ or $a \in b$. Hence (4) becomes in this case one of the tautological formulas

$$f = \{\langle k_1, x_{k_1}\rangle, \langle k_2, x_{k_2}\rangle\} \supset (f \in F_1 \equiv x_{k_1} = x_{k_2}),$$
$$f = \{\langle k_1, x_{k_1}\rangle, \langle k_2, x_{k_2}\rangle\} \supset (f \in F_1 \equiv x_{k_1} \in x_{k_2}).$$

Suppose that (4) is provable for formulas with C-sequences shorter than n and let Φ be a formula with a C-sequence (1) of the length n. We have to consider the two cases (2) and (3).

In case (2) we have $\Phi = \Phi_j|\Phi_h$ with $j<n$ and $h<n$. Let s_j be the common domain of sequences from F_j and s_h the common domain of sequences from F_h. By definition of S-sequences we obtain

$$f \in F_n \equiv (f|s_j \text{ non} \in F_j \vee f|s_h \text{ non} \in F_h). \tag{5}$$

The inductive assumption gives the equivalences

$$f|s_j \in F_j \equiv \Phi_j, \qquad f|s_h \in F_h \equiv \Phi_h$$

and we obtain from them and from (5)

$$f \in F_n \equiv (\sim\Phi_j \vee \sim\Phi_h) \equiv \Phi_j|\Phi_h \equiv \Phi.$$

In case (3) we have $\Phi = (\exists x_m)\Phi_j$ with $j<n$. We can suppose that x_m is free in Φ_j since otherwise the theorem is trivial. From the inductive assumption we obtain the equivalence

$$f + \{\langle m, x_m\rangle\} \in F_j \equiv \Phi_j$$

and the definition of satisfaction gives another equivalence

$$f \in F_n \equiv (\exists x_m)\,[f + \{\langle m, x_m\rangle\} \in F_j].$$

Both equivalences together entail the equivalence

$$f \in F_n \equiv (\exists x_m)\Phi_f \equiv \Phi.$$

Theorem Σ_5 is thus proved.

The difference between the schemata Σ_4 and Σ_5 lies in the fact that Σ_5 is not expressible in (S') as a single theorem. If we try to express Σ_5 as a single statement of (S'), we must replace the Φ on the right side of the equivalence by its Gödel number and the theorem evidently looses sense because on both sides of an equivalence must stay formulas and not numbers.

2. Theorem Σ_5 shows that the definition of truth which we adopted for the system (S) satisfies the conditions imposed on that notion by Tarski [6], p. 305. Furthermore this fact can be proved in (S') for each particular formula of (S). We shall now analyse the question why the consistency of (S) cannot be proved in (S') although a „good" definition of truth is formalizable in (S'). We shall show that the real source of this illusionary paradox lies in the fact that although every particular instance of the schema

(Σ) *if Φ is provable in (S), then Φ is true*

can be proved in (S'),yet the general theorem

(T) *if Φ is provable in (S), then Φ is true*

cannot be deduced from the axioms of (S').

T_6. *The axioms (A_1)—(A_5) are true.*

This is merely a restatement of the fact that the axioms (A_1)—(A_5) are at the same time axioms of (S'). For the sake of completeness we indicate the method of proof for the axiom (A_1).

The void sequence satisfies the formula (A_1) if and only if every two termed sequence $f=\{\langle 1,a\rangle, \langle 2,b\rangle\}$ satisfies the formula

(6) $$(x_3)(x_3 \in x_1 \equiv x_3 \in x_2) \supset x_1 = x_2.$$

Applying the definition of satisfaction we infer that f satisfies the formula (6) if and only if it either does not satisfy the formula $(x_3)(x_3 \in x_1 \equiv x_3 \in x_2)$ or does satisfy the formula $x_1 = x_2$. Applying again the definition of satisfaction we transform this condition into an equivalent one as follows: either there is a set c such that the conditions

the sequence $\{\langle 1,a\rangle, \langle 3,c\rangle\}$ satisfies the formula $x_3 \in x_1$,
the sequence $\{\langle 2,b\rangle, \langle 3,c\rangle\}$ satisfies the formula $x_3 \in x_2$

are not equivalent or f satisfies the formula $x_1 = x_2$.

Applying still once more the definition of satisfaction we reduce the above conditions to the following: either there exists a set c such that $\sim(c \in a \equiv c \in b)$ or $a = b$.

It follows now directly from the axiom (A_1) which is valid in (S') that these conditions are satisfied.

Σ_7. *For every formula Φ of (S) the formulas (A_6) and (A_7) corresponding to the formula Φ are true.*

It will be sufficient to indicate the method of proof for the axiom schema (A_6). Let Φ be a formula of (S) with the free variables $x_l, x_n, x_{k_1}, \ldots, x_{k_p}, x_k$. Applying the definition of satisfaction we prove easily that the assertion of Σ_7 is equivalent to the following statement: *If*

(7) *x_k is a set and for every x_l in x_k there exists exactly one set x_n such that Φ,*

then there exists a set x_q such that $x_n \in x_q$ if and only if there is an x_l in x_k such that Φ.

In order to prove this statement let us assume (7). By theorem Σ_5 there exists a class X such that

$$\{\langle k,x_k\rangle, \langle l,x_l\rangle, \langle n,x_n\rangle, \langle k_1,x_{k_1}\rangle, \ldots, \langle k_p,x_{k_p}\rangle\} \in X \equiv \Phi.$$

Let U be the class of pairs $\langle x_l, x_n\rangle$ such that

$$\{\langle k,x_k\rangle, \langle l,x_l\rangle, \langle n,x_n\rangle, \langle k_1,x_{k_1}\rangle, \ldots, \langle k_p,x_{k_p}\rangle\} \in X.$$

The existence of U (which depends of course on „parameters" $x_k, x_{k_1}, \ldots, x_{k_p}$) follows from the class theorem which is valid in (S'). It follows from (7) that for every x_l in x_k there exists exactly one x_n such that $\langle x_l, x_n\rangle \in U$. We apply now to U and x_k the axiom of replacement and obtain a set x_q with the desired properties.

The above argument would remain valid with the letter „Φ" replaced everywhere by *„the sequence $f = \{\langle k,x_k\rangle, \langle l,x_l\rangle, \langle n,x_n\rangle, \langle k_1,x_{k_1}\rangle, \ldots, \langle k_p,x_{k_p}\rangle\}$ satisfies Φ"* if we only knew that there exists a class X of sequences which satisfy Φ. Hence applying theorems T_2 and T_3 we obtain the following theorem:

T_8. *If Φ and Ψ are formulas of (S) such that the instances of the axiom schemata (A_6) and (A_7) corresponding to these formulas are true, then the formulas $\Phi|\Psi$ and $(\exists x_m)\Phi$ have the same property.*

As is well known the rules of proof lead from true formulas again to true formulas. To express conveniently this theorem we introduce the notion of valid formulas.

A formula Φ with free variables $x_{k_1},\dots,x_{k_p}$ is called *valid* if every sequence f with the domain $\{k_1,\dots,k_p\}$ satisfies Φ.

T_9. *If Φ and Ψ are two valid formulas, then all formulas resulting from them by the rules of proof are also valid.*

This theorem follows immediately from the fact that the rules of proof admitted in (S) are also valid in (S'). The method of proving this will be exemplified sufficiently well on the following example.

One of the rules states that if the formula $\Phi\supset\Psi$ is already proved and the variable x_m is not free in Ψ, then the formula $(\exists x_m)\Phi\supset\Psi$ can also be considered as proved. Now let us assume that the formula $\Phi\supset\Psi$ is valid but the formula $(\exists x_m)\Phi\supset\Psi$ is not.

Les s_1 be the set of integers i such that x_i is free in Φ and s_2 the set of integers j such that x_j is free in Ψ. From the definition of satisfaction follows the existence of a sequence f such that $f|s_1$ satisfies $(\exists x_m)\Phi$ but $f|s_2$ does not satisfy Ψ. If x_m is not free in Φ we have already a contradiction since $f|s_1$ satisfies Φ but $f|s_2$ does not satisfy Ψ, hence f does not satisfy the formula $\Phi\supset\Psi$ against our assumption that this formula is valid. If x_m is free in Φ then there exists an a such that the sequence $f|s_1+\{\langle m,a\rangle\}$ satisfies Φ and hence the sequence $f+\{\langle m,a\rangle\}$ does not satisfy the implication $\Phi\supset\Psi$ contrary to the assumption that this implication is valid.

We arrange now all the formulas falling under the schemata (A_6) and (A_7) into an infinite sequence

$$B_1, B_2, B_3, \dots \tag{8}$$

in such a way that the formulas corresponding to the composite expressions $\Phi|\Psi$ and $(\exists x_m)\Phi$ occur later in the sequence than the formulas corresponding to the simple expressions Φ and Ψ.

We shall say that Φ is a *theorem of at most n-th order* of (S) if Φ is provable from the axioms (A_1)—(A_5) and at most n first terms of the sequence (8) by at most n applications of the rules of proof.

From theorems T_8 and T_9 we obtain

T_{10}. *If every theorem of the n-th order is true, then every theorem of the $n+1$-st order is true.*

Using T_6 and applying successively the theorem T_{10}, we see that the following schema contains exclusively formulas provable in (S'):

Σ_{11}. *Every theorem of the n-th order* $(n=1,2,\dots)$ *is true.*

It follows from this schema that if m is the Gödel number of an arbitrary theorem of (S), then $V(m)$ is provable in (S'). The general theorem however

$$(n)\,(\Phi)\,[(\Phi \textit{ is a theorem of the n-th order})\supset(\Phi \textit{ is true})]$$

though expressible in (S') is not provable in (S') provided that (S') is consistent. If this theorem were provable in (S'), then the theorem

$$(\Phi)\,[(\Phi \textit{ is a theorem of } (S))\supset(\Phi \textit{ is true})]$$

would also be provable in (S') and since the theorem

$$(\Phi \textit{ is true})\supset\sim(\sim\Phi \textit{ is true})$$

is provable in (S'), we would infer that the consistency of (S) is provable in (S'). This however entails the inconsistency of (S) and hence the inconsistency of (S').

In view of Σ_5 and Σ_{11} the last remarks complete the proof of the theorem I. At the same time we have explained why the consistency of (S) is unprovable in (S') in spite of the fact that a satisfactory definition of „truth" for the system (S) is formalizable in (S').

3. We shall now deduce from the previous results the theorems II and III mentioned in the introduction.

Let $\Theta(x)$ be the formula which we obtain writing in the symbols of (S') the following statement:

x is an integer and every theorem of the x-th order is true.

It follows from T_6 and T_{10} that the formulas

$$\Theta(1) \quad\text{and}\quad (n)\,[\Theta(n)\supset\Theta(n+1)]$$

are provable in (S') whereas the discussion given at the end of section **2** shows that if (S') is consistent, then the general statement $(n)\,\Theta(n)$ is not provable in (S').

As to the form of the formula $\Theta(x)$ it is immediate that it can be written as

(9) $$(m)\,[m<\varphi(x)\supset(\exists X)\,A(X,m)]$$ [14]

where $\varphi(x)$ is a number-theoretic function such that $\varphi(x)$ exceeds the Gödel numbers of all theorems of the order x at most. Indeed $\Theta(x)$ says that for every theorem of an order $\leqslant x$ there exists an S-sequence such that the void sequence belongs to its last term.

We can now transform the expression (9) into an equivalent one which says that there is a finite sequence Y with the domain consisting of integers less than $\varphi(x)$ and such that if $m\leqslant\varphi(x)$, then the m-th term Y_m of Y satisfies the condition $A(Y_m,m)$.

In this way we give to $\Theta(x)$ the form $(\exists X)\,B(X,x)$ required in theorem II.

Finally we prove theorem III. Let us take as $\Phi(x)$ the formula

$$x \textit{ is an integer and } \sim\Theta(x).$$

Suppose that the formula

$$(\exists X)\,(x)\,[x\,\epsilon\,X\equiv\Phi(x)]$$

is provable in (S') and consider the class X such that $x\,\epsilon\,X\equiv\Phi(x)$. Using the restrictive axiom [15] we infer that

(10) $$X=0\vee(\exists x)\,[(x\,\epsilon\,X)\cdot(\textit{no element of } x \textit{ is in } X)].$$

Since $\Theta(1)$ is provable in (S'), we obtain $x\,\epsilon\,X\supset x\neq 1$. Hence (10) entails [16] that

$$X=0\vee(\exists x)\,[(x\,\epsilon\,X\cdot(x-1 \text{ non } \epsilon\,X)]$$

and therefore we obtain

$$X=0\vee(\exists x)\,[\Theta(x-1)\cdot\sim\Theta(x)].$$

Since $\Theta(x-1)\supset\Theta(x)$ is provable in (S') we can simplify this formula to $X=0$. But if this formula were provable in (S'), then the formula $(n)\,\Theta(n)$ would also be provable in (S') and hence (S') would be inconsistent. Theorem III is thus proved.

[14]) The variable m ranges over the set of Gödel numbers of theorems of (S).

[15]) See Bernays [1], axiom VII or Gödel [2], axiom D.

[16]) We recall that in the Bernays theory of integers the *less-than* relation is identical with ϵ. Cf. Bernays [1], pp. 8-9.

4. We conclude with the following general remark which should clarify the intention of the paper.

People working with Tarski's theory of truth generally believe that if a satisfactory definition of truth for a system (or „language") (S) can be set up in another system („meta-language") (S'), then the consistency of (S) is provable in (S'). By a satisfactory definition of truth is meant a definition which satisfies the „convention $\mathfrak{W}$" given on p. 305 of Tarski [6].

It follows from theorem I proved above that one should be careful making such general statements. If the meta-language of (S) is very weak (though stronger than (S) itself), then as our theorem I shows the statement in question can even be false.

In order to be sure that the consistency of (S) is provable in (S') by the methods used in the theory of „truth", one has to require that the general theorem

T. Each formula provable in (S) is true,

be provable in (S'). This is certainly the case if the following two theorems

T′. Each axiom of (S) is true,

T″. If Φ arises from true formulas by means of a rule of proof admitted in (S), then Φ is true

are provable in (S') and if the induction principle

if $\Theta(1)$ and $(n)\,[\Theta(n)\supset\Theta(n+1)]$ are provable in (S'), then so is $(n)\,\Theta(n)$

holds in (S') for arbitrary formulas.

It is entirely conceivable (although I did not succeed to find a suitable example) that for certain systems (S) and (S') both T' and T'' are provable in (S') and yet the consistency of (S) is not provable in (S') because of the lack of a sufficiently strong induction principle in (S').

On the other hand the consistency of (S) is evidently provable in each ω-complete system (S') in which a satisfactory definition of truth for (S) is formalizable. This remark is however of little practical value, since according to the well-known fundamental theorem of Gödel no finitary system containing arithmetic of integers is ω-complete.

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Remarks on some topological spaces of high power.

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The subject of this paper [1]) is the study of those topological spaces (called *ω_μ-additive spaces*) [2]) which satisfy the following axioms:

I. For every α-sequence [3]) of sets $\{X_\xi\}$, $\alpha<\omega_\mu$,

$$\overline{\sum_{0\leqslant\xi<\alpha} X_\xi} = \sum_{0\leqslant\xi<\alpha} \overline{X}_\xi.$$

II. $\overline{X}=X$ for every finite set X.

III. $\overline{X}=\overline{\overline{X}}$.

If $\mu=0$, axioms I-III coincide with the well-known axioms of Kuratowski [4]). If $\mu>0$, axiom I is stronger than the first axiom of Kuratowski.

It will be shown that in the case $\mu>0$ it is convenient to modify some topological notions and definitions. The idea of the modification is that the words: „an enumerable sequence", „a finite set", „an enumerable set" should be replaced by „an ω_μ-sequence", „a set of a potency $<\aleph_\mu$", „a set of the power $\aleph_\mu$" respectively. After this modification many topological theorems on separable metric spaces holds also for ω_μ-additive spaces whose power, in general, is $\geqslant\aleph_\mu$.

It is not the purpose of this paper to specify all topological theorems which can be generalized in the above-mentioned way. Only the direction of the generalization will be shown and some singularities which appear in connection with the notion of compactness and of completeness will be discussed. The final section contains an application to the theory of Boolean algebras.

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[1]) Presented at the Mathematical Congress in Wrocław on December 14, 1946.

[2]) ω_μ always denotes an initial ordinal (i. e. ω_μ is the least ordinal such that the set of all ordinals $\xi<\omega_\mu$ is of the power $\aleph_\mu$).

[3]) For brevity's sake we say „an α-sequence" instead of „a (transfinite) sequence of the type α".

[4]) Kuratowski [1], p. 20.