[12] L. J. Mordell, *On the conjecture for the rational points on a cubic surface*, J. London Math. Soc. 40 (1965), pp. 149–158.
[13] E. S. Selmer, *The Diophantine equation* $ax^3+by^3+cz^3=0$, Acta Math. 85 (1951), pp. 203–362.
[14] T. A. Springer, *Some properties of cubic forms over fields with a discrete valuation*, Nederl. Akad. Wetensch. Proc., Ser. A, 58 (1955), pp. 512–516.
[15] H. P. F. Swinnerton-Dyer, *Two special cubic surfaces*, Mathematika 9 (1962), pp. 54–56.
[16] G. Terjanian, *Un contre-exemple à une conjecture d'Artin*, C. R. Acad. Sci. Paris, Ser. A–B, 262 (1966), p. 612.

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Frieze patterns

by

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In memory of Harold Davenport,
my companion of college days

1. Introduction. The idea of a frieze pattern is most quickly conveyed by means of an example, such as the following pattern of order 7:

0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 …
… 1 2 2 3 1 2 4 1 2 2 3
1 3 5 2 1 7 3 1 3 5 …
… 2 1 7 3 1 3 5 2 1 7 3
1 2 4 1 2 2 3 1 2 4 …
… 1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0 …

Apart from the borders of zeros and ones, the essential property is that every four adjacent numbers forming a square

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

satisfy the "unimodular" equation $ad-bc=1$. Moreover, we insist that all the numbers (except the borders of zeros) shall be positive. The surprising conclusion is that every such pattern is periodic. More precisely, it is symmetrical by a glide: the product of a horizontal translation and a horizontal reflection.

After giving some historical background, we shall prove this periodicity and deduce some cyclic sequences based on continued fractions. Finally, we shall give a necessary and sufficient condition for a frieze pattern to consist of integers.

2. Frieze patterns of order 5. The story begins in 1602, when Nathaniel Torporley (1564–1632) began to investigate the five "parts"

a, A, b, B, c of a right-angled spherical triangle (right-angled at C). According to De Morgan [7], Torporley anticipated by a dozen years the famous rules of Napier ([12], p. 32) which Gauss embodied in his *pentagramma mirificum* ([8], p. 484). Gauss used the identity

$$(1+\gamma)(1+\beta-\delta\varepsilon)-(1+\beta)(1+\gamma-\varepsilon\alpha) = \varepsilon\{(1+\alpha-\gamma\delta)-(1+\delta-\alpha\beta)\}$$

to prove that any three of the four relations

$$1+\alpha = \gamma\delta, \quad 1+\beta = \delta\varepsilon, \quad 1+\gamma = \varepsilon\alpha, \quad 1+\delta = \alpha\beta$$

implies the remaining one and also $1+\varepsilon = \beta\gamma$. This remark establishes the periodicity of the frieze pattern

$$\begin{array}{cccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& & \alpha & & \beta & & \gamma & & \delta & & \varepsilon & & \alpha & & \beta & & \gamma \\
& & & \delta & & \varepsilon & & \alpha & & \beta & & \gamma & & \delta & & \varepsilon & & \alpha \ldots \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& & & & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

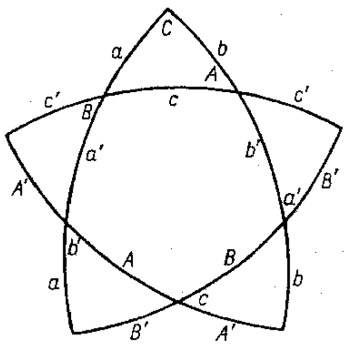

Fig. 1

The *pentagramma mirificum* (Figure 1) is a spherical pentagram formed by five successively orthogonal great-circle arcs. The "core" of the pentagram is a pentagon whose vertices are (obviously) poles of these five arcs; it is thus a *self-polar pentagon*. The whole figure can be derived from the right-angled triangle ABC (appearing at the top) by extending the sides and drawing also the polar great circles of the vertices A and B. Using a prime to indicate the complement

$$x' = \tfrac{1}{2}\pi - x,$$

we easily see that the sides of the self-polar pentagon are

$$A, B. b', c, a'$$

while the remaining arcs and angles are as indicated in Figure 1. Clearly, any equation connecting the five "parts" (thus amended) remains valid when they are cyclically permuted. This is Napier's observation (except that he used the alternated cycle a', B, c, A, b').

Gauss defined

$$\alpha = \tan^2 A, \quad \beta = \tan^2 B, \quad \gamma = \tan^2 b',$$
$$\delta = \tan^2 c, \quad \varepsilon = \tan^2 a'$$

and used one of the classical relations (such as $\cos c = \cot A \cot B$, [5], p. 234) to derive

$$\sec^2 A = \gamma\delta, \quad \sec^2 B = \delta\varepsilon, \quad \sec^2 b' = \varepsilon\alpha,$$
$$\sec^2 c = \alpha\beta, \quad \sec^2 a' = \beta\gamma.$$

He mentioned the "schöne Gleichung"

$$3+\alpha+\beta+\gamma+\delta+\varepsilon = \alpha\beta\gamma\delta\varepsilon = \sqrt{(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\varepsilon)}$$

and the related formula

$$(1+i\sqrt{\alpha})(1+i\sqrt{\beta})(1+i\sqrt{\gamma})(1+i\sqrt{\delta})(1+i\sqrt{\varepsilon}) = \alpha\beta\gamma\delta\varepsilon e^{iS},$$

where $S = 2\pi-(A+B+b'+c+a')$.

Incidentally, the obvious identity

$$\frac{1}{\alpha\beta} = \frac{\varepsilon}{\beta\gamma}\,\frac{\gamma}{\varepsilon\alpha}$$

yields the spherical analogue of the theorem of Pythagoras:

$$\cos c = \sin a' \sin b' = \cos a \cos b.$$

Lobachevsky ([9], p. 18; [10], p. 36; [5], p. 281) showed that any right-angled triangle ABC in the hyperbolic plane corresponds (in a special manner) to a spherical triangle with hypotenuse $\Pi(a)$ and catheti B and $\Pi(c)$ opposite to its angles A' and $\Pi(b)$. Recalling that $\cot \Pi(a) = \sinh a$, so that $\tan \Pi(a) = \operatorname{csch} a$, we obtain the following hyperbolic interpretation for the frieze pattern of Gauss:

$$\cot^2 A = \alpha, \quad \operatorname{csch}^2 b = \beta, \quad \sinh^2 c = \gamma, \quad \operatorname{csch}^2 a = \delta, \quad \cot^2 B = \varepsilon,$$
$$\csc^2 A = \gamma\delta, \quad \coth^2 b = \delta\varepsilon, \quad \cosh^2 c = \varepsilon\alpha, \quad \coth^2 a = \alpha\beta, \quad \csc^2 B = \beta\gamma.$$

The revised notation

$$u_1 = \alpha, \quad u_2 = \delta, \quad u_3 = \beta, \quad u_4 = \varepsilon, \quad u_5 = \gamma$$

yields the frieze pattern

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& & u_1 & & u_3 & & u_5 & & u_2 & & u_4 & & u_1 & & u_3 & & u_5 \\
& & & u_2 & & u_4 & & u_1 & & u_3 & & u_5 & & u_2 & & u_4 & & u_1 & \ldots \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& & & & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

and the relations

$$u_1u_3 = 1+u_2, \quad u_2u_4 = 1+u_3, \quad u_3u_5 = 1+u_4, \quad \ldots,$$

which are thus seen to imply $u_{r+5} = u_r$ for all r (provided $u_r \neq 0$). This 5-cycle seems to have been transmitted in the form of mathematical gossip for a long time. One published account is by Lyness [11]. The following straightforward proof has kindly been provided by Israel Halperin. Given

$$u_1u_3 = 1+u_2, \quad u_2u_4 = 1+u_3, \quad u_3u_5 = 1+u_4, \quad u_4u_6 = 1+u_5$$

and $u_3u_4 \neq 0$, we have

$$\begin{aligned} u_1u_3u_4 &= (1+u_2)u_4 = u_4+1+u_3 \\ &= u_3+1+u_4 = u_3(1+u_5) = u_3u_4u_6; \end{aligned}$$

therefore $u_6 = u_1$.

Any reader who is in a hurry may now turn to § 5, where a notation is proposed which will enable us to establish, in § 6, the periodicity of the pattern of order n.

3. Continued fractions. A pretty variant of the 5-cycle $u_{r-1}u_{r+1} = 1+u_r$ can be obtained in terms of $c_r = 1+u_r$:

$$c_{r+1} = 1-\frac{c_r}{1-c_{r-1}}$$

or, in the notation of continued fractions,

$$c_{r+1} = 1-c_r/1-c_{r-1}.$$

This suggests a possible generalization. Let $m-1$ positive numbers $c_1, \ldots, c_{m-1}$ be given, and let the sequence $\{c_r\}$ be continued so that

(3.1) $$c_{m+r} = 1-c_{m+r-1}/1-\ldots c_{r+2}/1-c_{r+1}$$

for $r = 0, 1, \ldots$ The sequence clearly has period 2 when $m = 2$, and § 2 shows that it has period 5 when $m = 3$. Is it still periodic when $m > 3$?

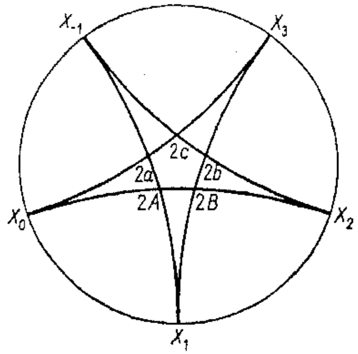

Fig. 2

Let X_{-1}, X_0, X_1 be three distinct points on a circle (as in Figure 2, where $m = 3$), or on the real projective line. Let $m-1$ further points $X_2, \ldots, X_m$ be chosen to form cross ratios

(3.2) $$\{X_{r-2}X_{r+1}, X_{r-1}X_r\} = c_r \quad (r = 1, \ldots, m-1).$$

Applying a well-known property of cross ratios ([5], p. 77) to the five points $X_{-1}, X_{r-2}, X_{r-1}, X_r, X_{r+1}$, we find

$$\{X_{r-2}X_{r+1}, X_{r-1}X_r\} = \{X_{r-2}X_{-1}, X_{r-1}X_r\}\{X_{-1}X_{r+1}, X_{r-1}X_r\},$$

so that

$$\begin{aligned} c_r/\{X_{r-2}X_{-1}, X_{r-1}X_r\} &= \{X_{-1}X_{r+1}, X_{r-1}X_r\} \\ &= 1-\{X_{-1}X_{r-1}, X_{r+1}X_r\} = 1-\{X_{r-1}X_{-1}, X_rX_{r+1}\}. \end{aligned}$$

Setting $r = m-1, m-2, \ldots, 2$, in turn, and then using (3.1) with $r = 0$, we have

$$\begin{aligned} \{X_{m-2}X_{-1}, X_{m-1}X_m\} &= 1-c_{m-1}/\{X_{m-3}X_{-1}, X_{m-2}X_{m-1}\} \\ &= 1-c_{m-1}/1-c_{m-2}/1-\ldots c_2/\{X_0X_{-1}, X_1X_2\} \\ &= 1-c_{m-1}/1-\ldots c_2/1-c_1 \\ &= c_m. \end{aligned}$$

Thus, if we regard X_{m+1} as an alternative name for X_{-1}, (3.2) will hold not only for $r < m$ but also for $r = m$. By the same procedure, specializing X_0 instead of X_{-1}, we conclude that, if X_{m+2} is another name for X_0, (3.2) will hold also for $r = m+1$; and so on. Altogether, we have a cycle of $m+2$ points X_r and a cycle of $m+2$ numbers c_r: the period is $m+2$ for all $m > 2$.

4. Schläfli's orthoscheme. Figure 2 shows arcs $X_{-1}X_1, X_0X_2, \ldots$, orthogonal to the given cycle. If the point-pairs X_rX_t and X_sX_u separate each other, one of the angles of intersection of the arcs X_rX_t and X_sX_u is twice the angle whose cosine is $\sqrt{\{X_rX_u, X_sX_t\}}$. This may be seen by inverting the figure in a circle with centre X_u, so that this point becomes the point at infinity, the remaining three become collinear, the arc X_rX_t

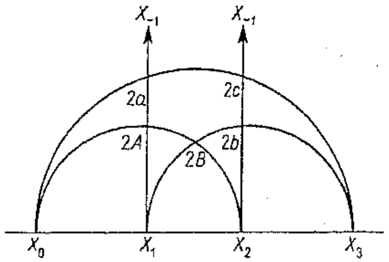

Fig. 3

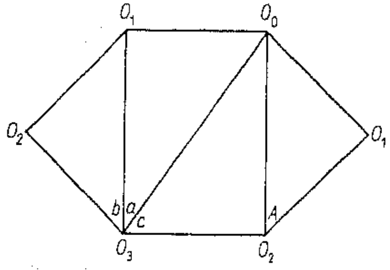

Fig. 4

becomes a semicircle, X_sX_u becomes a "vertical" straight line, and $\{X_rX_u, X_sX_t\}$ becomes the simple ratio X_rX_s/X_rX_t ([2], p. 123; [5], p. 276).

Figure 3 shows the result of inverting Figure 2 in a circle with centre X_{-1}. Now $X_0, X_1, \ldots, X_m$ occur in their natural order along a line, so that, if $r < s < t$, $X_rX_s + X_sX_t = X_rX_t$.

In a Euclidean m-space, choose points $O_0, O_1, \ldots, O_m$ so that the m line-segments $O_0O_1, \ldots, O_{m-1}O_m$ are mutually perpendicular and of lengths

$$O_0O_1 = \sqrt{X_0X_1}, \quad \ldots, \quad O_{m-1}O_m = \sqrt{X_{m-1}X_m}.$$

Then (by Pythagoras) we have $(O_rO_s)^2 = X_rX_s$ for all r, s, the triangle $O_rO_sO_t$ is right-angled at O_s whenever $r < s < t$, and the simplex $O_0O_1 \ldots O_m$ is the special kind for which Schläfli devised the name *orthoscheme* ([13], p. 243; [14], pp. 169, 240).

Figure 4 is an unfolded *net* of a 3-dimensional orthoscheme $O_0O_1O_2O_3$: a tetrahedron whose faces consist of four right-angled triangles. If its size in such that $O_3O_2 \geqslant 1$, the faces at O_3 determine, on the unit sphere with its centre at that vertex, a right-angled spherical triangle ABC with A on O_3O_2, B on O_3O_0, C on O_3O_1. In terms of the "parts" of this spherical triangle, the face-angles of the tetrahedron are A, a, b, c, as indicated in Figure 4. The corresponding angles $2A, 2a, 2b, 2c$ in Figures 3 and 2 are derived from the relations

$$\cos^2 A = \left(\frac{O_1O_2}{O_0O_2}\right)^2 = \frac{X_1X_2}{X_0X_2} = \{X_{-1}X_2, X_0X_1\},$$

$$\sin^2 a = \left(\frac{O_0O_1}{O_0O_3}\right)^2 = \frac{X_0X_1}{X_0X_3} = \{X_3X_1, X_{-1}X_0\},$$

$$\sin^2 b = \left(\frac{O_1O_2}{O_1O_3}\right)^2 = \frac{X_1X_2}{X_1X_3} = \{X_1X_{-1}, X_2X_3\},$$

$$\cos^2 c = \left(\frac{O_2O_3}{O_0O_3}\right)^2 = \frac{X_2X_3}{X_0X_3} = \{X_2X_0, X_3X_{-1}\}.$$

The remaining angle B (which appears in the tetrahedron as the dihedral angle at the edge O_0O_3) can then be inferred in Figure 2 from the fact that a and b are interchanged when the points $X_{-1}X_0X_1X_2X_3$ are replaced by $X_3X_2X_1X_0X_{-1}$, and then the $2B$ can be inserted in the corresponding position in Figure 3. Recalling that $X_4 = X_{-1}$, $X_5 = X_0$, and $X_6 = X_1$, we deduce that, in the notation of (3.2),

$$c_1 = \cos^2 A, \quad c_2 = \cos^2 B, \quad c_3 = \sin^2 b, \quad c_4 = \cos^2 c, \quad c_5 = \sin^2 a.$$

We have thus returned to Napier's cycle from a new point of view. In fact,

$$A, \quad B, \quad \tfrac{1}{2}\pi - b, \quad c, \quad \tfrac{1}{2}\pi - a$$

can be described as the angles between adjacent pairs in a cycle of five planes such that any two non-adjacent planes are perpendicular. The first four of these planes are the face-planes of the orthoscheme (opposite to the vertices O_0, O_1, O_2, O_3), and the fifth is perpendicular to the "long" edge O_0O_3.

It appears from the work of Wythoff [15] that all the angular properties of a four-dimensional orthoscheme can be derived by the analogous procedure with $m = 4$. The m-dimensional case was described by Schläfli ([13], pp. 256–260; [14], pp. 174, 249) in terms of the determinantal equation

$$(4.1) \qquad \begin{vmatrix} 1 & \sqrt{c_{r+1}} & 0 & \ldots & 0 & 0 \\ \sqrt{c_{r+1}} & 1 & \sqrt{c_{r+2}} & \ldots & 0 & 0 \\ 0 & \sqrt{c_{r+2}} & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \sqrt{c_{r+m}} & 1 \end{vmatrix} = 0,$$

which is easily seen to be equivalent to (3.1).

Figure 5 illustrates the case when $m = 5$. The angles $2\theta_r$ of the innermost "heptagon" are given by

$$\cos^2\theta_r = c_r \quad (r = 1, \ldots, 7).$$

In Euclidean 5-space, θ_r appears as the angle between adjacent hyperplanes in a cycle of seven, so arranged that any two non-adjacent hyperplanes are perpendicular. When any one of these hyperplanes is omitted, the remaining six bound an orthoscheme whose acute dihedral angles are five of the seven θ's. By drawing further arcs, such as X_1X_4, we exhibit the doubles of all the remaining angular properties of this orthoscheme ([2], pp. 125–134).

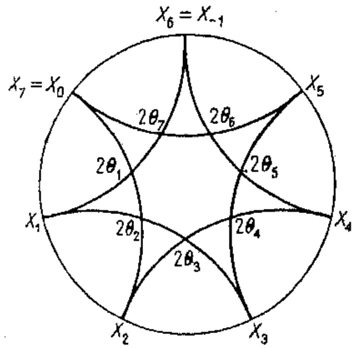

Fig. 5

5. The two-digit symbols (r, s). To investigate the general frieze pattern, we shall find it convenient to represent its elements by symbols (r, s) as follows:

$(0,0)$ $(1,1)$ $(2,2)$ $(3,3)$ $(4,4)$

$(0,1)$ $(1,2)$ $(2,3)$ $(3,4)$ ...

$(-1,1)$ $(0,2)$ $(1,3)$ $(2,4)$ $(3,5)$

. . . .

$(-1,n-3)$ $(0,n-2)$ $(1,n-1)$ $(2,n)$

$(-1,n-2)$ $(0,n-1)$ $(1,n)$ ...

$(-1,n-1)$ $(0,n)$ $(1,n+1)$

([3], p. 159 = [6], p. 204).

The borders of zeros and ones are given by the specifications

(5.1) $$(r, r) = 0, \qquad (r, r+n) = 0,$$

(5.2) $$(r, r+1) = 1, \qquad (r+1, r+n) = 1.$$

We have also $(r, s) > 0$ for $r < s < r+n$, and the "unimodular rule"

(5.3) $$(r-1, s)(r, s+1) - (r, s)(r-1, s+1) = 1,$$

which implies

(5.4) $$(r, r-1) = -1, \qquad (r-1, r+n) = -1.$$

Clearly, the whole pattern is determined by the elements in one diagonal, such as

$$(0,0), \quad (0,1), \quad (0,2), \quad \ldots, \quad (0, n-1), \quad (0, n)$$

(beginning with 0 and 1, and ending with 1 and 0). For instance, $(1, 2) = 1$, and setting $r = 1$ in (5.3) we obtain

$$(1,3) = \frac{(1,2)(0,3)+1}{(0,2)}, \qquad (1,4) = \frac{(1,3)(0,4)+1}{(0,3)},$$

and so on. Accordingly, we adopt the auxiliary notation

(5.5) $$f_s = (-1, s), \qquad g_s = (0, s),$$

so that

$$f_{-2} = g_{-1} = f_n = g_{n+1} = -1,$$
$$f_{-1} = g_0 = f_{n-1} = g_n = 0,$$
$$f_0 = g_1 = f_{n-2} = g_{n-1} = 1,$$
$$g_2 = \frac{g_1 f_2 + 1}{f_1}, \quad g_3 = \frac{g_2 f_3 + 1}{f_2}, \quad \ldots, \quad g_{n-2} = \frac{g_{n-3} f_{n-2} + 1}{f_{n-3}}.$$

We easily verify that, for the relevant values of r and s,

(5.6) $$(r, s) = f_r g_s - f_s g_r.$$

In fact, this definition of (r, s) (for all integers r and s) implies

(5.7) $$(r, s)(t, u) + (r, t)(u, s) + (r, u)(s, t) = 0,$$

(5.8) $$(s, r) = -(r, s),$$

and

$$(r-1, s)(r, s+1) - (r, s)(r-1, s+1) - (r-1, r)(s, s+1) = 0.$$

The last relation agrees with (5.3), since $(r-1, r) = (s, s+1) = 1$.

This procedure was inspired by D. S. Mitrinović's proof ([1], p. 189) that the general solution of (5.7) is

$$(r, s) = f(r) g(s) - f(s) g(r)$$

for arbitrary functions f and g.

6. The periodicity of the frieze pattern. Using (5.4), (5.2), (5.7), (5.1), in turn, we have

(6.1) $$(r, s) + (r, s+n) = -(r, s)(s-1, s+n) + (r, s+n)(s-1, s) = -(r, s-1)(s, s+n) = 0.$$

In particular,

(6.2) $$f_{s+n} = -f_s, \quad g_{s+n} = -g_s.$$

Thus the values of f_s for the pattern in § 1 are

0, 1, 1, 1, 1, 2, 1, 0, −1, −1, −1, −1, −2, −1, 0, 1, ...

From (5.8) and (6.1) it follows that

(6.3) $$(r, s) = (s, r+n) = (r+n, s+n).$$

These equations show that the frieze pattern is symmetrical by a glide, and therefore also by the "square" of this glide, which is a horizontal translation. Hence, if we define

(6.4) $$a_r = (r-1, r+1),$$

we will have

(6.5) $$a_{r+kn} = a_r$$

for all integers k. However, the n numbers $a_0, \ldots, a_{n-1}$ are not independent. It will soon be seen that, instead of deriving the whole pattern from $f_1, \ldots, f_{n-3}$, we can equally well derive it from $a_0, \ldots, a_{n-4}$.

As Böhm remarks ([1], p. 189), since

$$(r, s) = (s-2, s)(r, s-1) - (r, s-2)(s-1, s)$$
$$= a_{s-1}(r, s-1) - (r, s-2),$$

it follows by induction that

(6.6) $$(r, s) = \begin{vmatrix} a_{r+1} & 1 & 0 & \ldots & 0 & 0 \\ 1 & a_{r+2} & 1 & \ldots & 0 & 0 \\ 0 & 1 & a_{r+3} & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 & a_{s-1} \end{vmatrix}$$

([2], p. 134; [4], p. 160). In particular, (5.2) (with $r = -2$) shows that a_{n-3} can be derived from $a_0, \ldots, a_{n-4}$ by the linear equation

(6.7) $$\begin{vmatrix} a_0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & a_1 & 1 & \ldots & 0 & 0 \\ 0 & 1 & a_2 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 & a_{n-3} \end{vmatrix} = 1 \quad (n > 3).$$

Accordingly, the periodicity (6.5) can be expressed as follows:

Let $n-3$ positive numbers $a_0, \ldots, a_{n-4}$ be given, and let the sequence $\{a_r\}$ be continued so that

$$\begin{vmatrix} a_r & 1 & 0 & \ldots & 0 & 0 \\ 1 & a_{r+1} & 1 & \ldots & 0 & 0 \\ 0 & 1 & a_{r+2} & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 & a_{r+n-3} \end{vmatrix} = 1 \quad (r = 0, 1, \ldots).$$

Then the sequence is periodic: $a_r = a_{r+n}$.

7. Continued fractions, again. Another consequence of (5.2) and (5.7) is

$$(r-1, s) = (r, r+1)(r-1, s) = (r, s)(r-1, r+1) - (r-1, r)(r+1, s)$$
$$= (r, s)a_r - (r+1, s),$$

whence

$$\frac{(r-1, s)}{(r, s)} = a_r - \frac{(r+1, s)}{(r, s)} = a_r - 1\Big/\frac{(r, s)}{(r+1, s)}.$$

Since this result remains valid when r is replaced by $r+1$, it follows that, if $r < s < r+n$,

(7.1) $$\frac{(r-1, s)}{(r, s)} = a_r - 1/a_{r+1} - 1/\ldots - 1/a_{s-1}.$$

In particular, by (5.5), if $0 < s < n$,

(7.2) $$\frac{f_s}{g_s} = a_0 - 1/a_1 - 1/\ldots - 1/a_{s-1}.$$

Setting $s = n-1$, we obtain

(7.3) $$a_0 - 1/a_1 - 1/\ldots - 1/a_{n-2} = 0.$$

Similarly, since $(r, s+1) = (r, s)a_s - (r, s-1)$, we have, for $r < s < r+n$,

$$\frac{(r, s+1)}{(r, s)} = a_s - 1/a_{s-1} - 1/\ldots - 1/a_{r+1}.$$

In particular, if $0 \leqslant s < n-1$,

(7.4) $$\frac{f_{s+1}}{f_s} = a_s - 1/a_{s-1} - 1/\ldots - 1/a_0.$$

Setting $s = n-2$, we obtain

(7.5) $$a_{n-2} - 1/a_{n-3} - 1/\ldots - 1/a_0 = 0.$$

The periodicity may now be expressed as follows:

Let $n-2$ positive numbers satisfying (6.7) *be given, and let the sequence be continued so that*

$$a_{r+n-2} = 1/a_{r+n-3} - 1/\ldots - 1/a_r \quad (r = 0, 1, \ldots).$$

Then the sequence is periodic: $a_r = a_{r+n}$.

Notice that we have obtained two kinds of cyclic sequence: $\{c_r\}$, of period $m+2$, and $\{a_r\}$, of period n. It is natural to ask whether they are related by any more significant equation than $n = m+2$. Comparing (4.1) with

$$\begin{vmatrix} a_r & 1 & 0 & \ldots & 0 & 0 \\ 1 & a_{r+1} & 1 & \ldots & 0 & 0 \\ 0 & 1 & a_{r+2} & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & a_{r+n-2} \end{vmatrix} = 0$$

(see (5.1) and (6.6)), we find the precise connection

$$c_r = (a_{r-1} a_r)^{-1}.$$

8. Frieze patterns of integers. If all the numbers (r, s) are integers (as in the example chosen in § 1), the unimodular rule shows that each (r, s) is prime to its four neighbours

$$(r \pm 1, s) \quad \text{and} \quad (r, s \pm 1).$$

In particular, for any six of the numbers arranged thus:

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & & f \\ & & e & \end{array}$$

c and d are coprime. As in the familiar theory of Farey series, we have $ad - bc = 1 = cf - de$,

$$(8.1) \qquad (a+e)d = (b+f)c,$$

c divides $a+e$, and d divides $b+f$. In other words, *three consecutive entries on a diagonal are such that the middle one divides the sum of the other two.*

Conversely, if, in a frieze pattern, a, b, c, d, e are integers such that c divides $a+e$, then (8.1) shows that $b+f$ is an integer, and therefore f is an integer. Returning to the notation of (5.5), we infer that, if $f_0, \ldots, f_{n-2}$ is a sequence of integers beginning and ending with 1, and if

$$(8.2) \qquad f_s \text{ divides } f_{s-1} + f_{s+1},$$

then all the g_s are integers and therefore, by (5.6), all the (r, s) are integers. In other words, a frieze pattern consists of integers if and only if the generating sequence $f_0, \ldots, f_{n-2}$ (beginning and ending with 1) consists of integers satisfying (8.2).

For instance, $f_0, \ldots, f_{n-2}$ may be $1, \ldots, 1$, or the sequence of numerators or denominators of a Farey series, or a suitable subsequence such as the numerators of the first or second half of a Farey series. Several suitable sequences can be juxtaposed to make a new one; for instance,

$$1, 2, 5, 3, 1 \quad \text{and} \quad 1, 2, 3, 4, 1$$

can be combined to form

$$1, 2, 5, 3, 1, 2, 3, 4, 1,$$

and of course each 1 may be replaced by a string of any number of 1's. After choosing f_{s-1} and f_s, we can take f_{s+1} to be $mf_s - f_{s-1}$ for any integer $m > f_{s-1}/f_s$. The only difficulty lies in making sure that, for a pattern of order n, $f_{n-2} = 1$.

The following example (based on the above sequence) illustrates the fact that a pattern of integers does not necessarily include a diagonal consisting entirely of 1's and 2's:

$$\begin{array}{l}
\quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \ldots \\
\quad 2 \quad 3 \quad 1 \quad 2 \quad 5 \quad 2 \quad 2 \quad 1 \quad 4 \quad 2 \quad 2 \quad 3 \\
\qquad 5 \quad 2 \quad 1 \quad 9 \quad 9 \quad 3 \quad 1 \quad 3 \quad 7 \quad 3 \quad 5 \quad \ldots \\
\qquad\quad 3 \quad 1 \quad 4 \quad 16 \quad 13 \quad 1 \quad 2 \quad 5 \quad 10 \quad 7 \quad 3 \\
\qquad\qquad 1 \quad 3 \quad 7 \quad 23 \quad 4 \quad 1 \quad 3 \quad 7 \quad 23 \quad 4 \quad \ldots \\
\qquad\quad 1 \quad 2 \quad 5 \quad 10 \quad 7 \quad 3 \quad 1 \quad 4 \quad 16 \quad 13 \quad 1 \\
\qquad\qquad 1 \quad 3 \quad 7 \quad 3 \quad 5 \quad 2 \quad 1 \quad 9 \quad 9 \quad 3 \quad \ldots \\
\qquad\qquad\quad 1 \quad 4 \quad 2 \quad 2 \quad 3 \quad 1 \quad 2 \quad 5 \quad 2 \quad 2 \\
\qquad\qquad\qquad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \ldots \\
\qquad\qquad\quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0
\end{array}$$

References

[1] J. Böhm, *Zu Coxeters Integrationsmethode in gekrümmten Räumen*, Math. Ann. 27 (1964), pp. 179–214.

[2] H. S. M. Coxeter, *On Schläfli's generalization of Napier's Pentagramma Mirificum*, Bull. Calcutta Math. Soc. 28 (1936), pp. 123–144.

[3] — *Regular honeycombs in hyperbolic space*, Proc. Internat. Congr. Math., Amsterdam 1954, Vol. III, pp. 155–169.

[4] — *Regular Polytopes*, New York 1963.

[5] — *Non-Euclidean Geometry*, Toronto 1966.

[6] — *Twelve Geometric Essays*, Carbondale, Ill., 1968.

[7] A. De Morgan, *On the invention of the circular parts*, Phil. Mag. (3) 22 (1843), 350–353.

[8] C. F. Gauss, *Werke*, Vol. III, Göttingen 1876.
[9] N. I. Lobachevsky, *Zwei geometrische Abhandlungen*, Leipzig 1898.
[10] — *Geometrical Researches on the Theory of Parallels*, Chicago 1914.
[11] R. C. Lyness, *Notes* 1581, 1847, 2952, Math. Gazette 26 (1942), p. 62; 29 (1945), p. 231, 45 (1961), pp. 207-209.
[12] J. Napier, *Logarithmorum Canonis Descriptio*, London 1620.
[13] L. Schläfli, *Gesammelte Mathematische Abhandlungen, I*, Basel 1950.
[14] — *Gesammelte Mathematische Abhandlungen, II*, Basel 1953.
[15] W. A. Wythoff, *The rule of Neper in the four dimensional space*, K. Akad. van Wetensch. te Amsterdam, Proc. Sect. of Sci., 9 (1907), pp. 529–534.

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A problem in comparative prime number theory

by

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In memory of Harold Davenport

1. Introduction. Let $\pi(y,k,a)$ denote the number of primes $\leqslant y$ that are congruent to a (mod k). In a series of papers, Knapowski and Turán [3] considered among other questions the problem of whether $\pi(y,k,a)-\pi(y,k,b)$ changes sign infinitely often. The first result of this nature is due to Littlewood who showed that $\pi(y,4,1)-\pi(y,4,3)$ changes sign infinitely often. Knapowski and Turán were able to handle many other cases under the assumption that no L-series with a character mod k has a real zero strictly between 0 and 1 (an assumption that has been checked for $k\leqslant 24$ and is quite possibly true for all k). For example, under this assumption they were able to show that $\pi(y,k,1)-\pi(y,k,a)$ changes sign infinitely often. However the general problem is still open.

The first unknown case is that of $\pi(y,5,4)-\pi(y,5,2)$. We prove below (Theorem 2 and 6) that there are positive constants c_1 and c_2 such that

$$\liminf_{y\to\infty}\frac{\pi(y,5,4)-\pi(y,5,2)}{\sqrt{y}/\log y}\leqslant -c_1,$$

$$\limsup_{y\to\infty}\frac{\pi(y,5,4)-\pi(y,5,2)}{\sqrt{y}/\log y}\geqslant c_2.$$

The first inequality is actually easy; the real difficulty is in the second. The "correct" values of c_1 and c_2 are undoubtedly $+\infty$, but this remains unestablished.

More generally, one can consider sign changes of $\varphi(k)\pi(y,k,a)-\varphi(K)\pi(y,K,A)$. We prove below a general result (Theorem 1) that applies to this situation. Unfortunately most cases will require a numerical calculation to reach the desired conclusion. This will be discussed in Sections 5 and 6.

2. Notation and other preliminaries. Throughout, k, K, a, A will be positive integers and if $k=K$, we will assume that $a\not\equiv A \pmod k$.