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CONJECTURES INVOLVING ARITHMETICAL SEQUENCES

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Abstract. We pose thirty conjectures on arithmetical sequences, most of which are about monotonicity of sequences of the form $(\sqrt[n]{a_n})_{n\geqslant 1}$ or the form $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n\geqslant 1}$, where $(a_n)_{n\geqslant 1}$ is a number-theoretic or combinatorial sequence of positive integers. This material might stimulate further research.

1. Introduction

A sequence $(a_n)_{n\geqslant 0}$ of natural numbers is said to be *log-concave* (resp. *log-convex*) if $a_{n+1}^2 \geqslant a_n a_{n+2}$ (resp. $a_{n+1}^2 \leqslant a_n a_{n+2}$) for all $n = 0, 1, 2, \ldots$. The log-concavity or log-convexity of combinatorial sequences has been studied extensively by many authors (see, e.g., [5, 7, 8, 15, 17]).

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ let p_n denote the n-th prime. In 1982, Faride Firoozbakht conjectured that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for all } n \in \mathbb{Z}^+,$$

i.e., the sequence $(\sqrt[n]{p_n})_{n\geqslant 1}$ is strictly decreasing (cf. [20, p. 185]). This was verified for n up to 3.495×10^{16} by Mark Wolf [34].

Mandl's inequality (cf. [9, 21, 13]) asserts that $S_n < np_n/2$ for all $n \geqslant 9$, where S_n is the sum of the first n primes. Recently the author [31] proved that the sequence $(\sqrt[n]{S_n})_{n\geqslant 2}$ is strictly decreasing and moreover the sequence $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 5}$ is strictly increasing. Motivated by this, here we pose many conjectures on sequences $(\sqrt[n]{a_n})_{n\geqslant 1}$ and $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n\geqslant 1}$ for many number-theoretic or combinatorial sequences $(a_n)_{n\geqslant 1}$ of positive integers. Clearly, if $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n\geqslant N}$ is strictly increasing (decreasing) with limit 1, then the sequence $(\sqrt[n]{a_n})_{n\geqslant N}$ is strictly decreasing (resp., increasing).

Sections 2 and 3 are devoted to our conjectures involving number-theoretic sequences and combinatorial sequences respectively.

Key words and phrases. Primes, Artin's primitive root conjecture, Schinzel's hypothesis H, combinatorial sequences, monotonicity.

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2. Conjectures on number-theoretic sequences

2.1. Conjectures on sequences involving primes.

Conjecture 2.1. (2012-09-12) *For any* $\alpha > 0$ *we have*

$$\frac{1}{n}\sum_{k=1}^{n} p_k^{\alpha} < \frac{p_n^{\alpha}}{\alpha+1} \quad \text{for all } \ n \geqslant 2\lceil\alpha\rceil^2 + \lceil\alpha\rceil + 6.$$

Remark 2.1. We have verified the conjecture for $\alpha = 2, 3, \ldots, 700$ and $n \leqslant 10^6$. Our numerical computation suggests that for $\alpha = 2, 3, \ldots, 10$ we may replace $\lceil\alpha\rceil^2 + \lceil\alpha\rceil + 6$ in the inequality by 9, 15, 31, 47, 62, 92, 92, 122, 122 respectively. Note that Mandl's inequality (corresponding to the case $\alpha = 1$) can be restated as $\sum_{k=1}^{n} p_k < \frac{n-1}{2} p_{n+1}$ for $n \geqslant 8$, which provides a lower bound for p_{n+1} in terms of $p_1, \ldots, p_n$.

Our next conjecture is a refinement of Firoozbakht's conjecture.

Conjecture 2.2. (2002-09-11) *For any integer* $n > 4$, *we have the inequality*

$$\frac{\sqrt[n+1]{p_{n+1}}}{\sqrt[n]{p_n}} < 1 - \frac{\log\log n}{2n^2}.$$

Remark 2.2. The author has verified the conjecture for all $n \leqslant 3500000$ and all those n with $p_n < 4 \times 10^{18}$ and $p_{n+1} - p_n \neq p_{k+1} - p_k$ for all $1 \leqslant k < n$. Note that if $n = 49749629143526$ then $p_n = 1693182318746371$, $p_{n+1} - p_n = 1132$ and $(1 - \sqrt[n+1]{p_{n+1}}/\sqrt[n]{p_n})n^2/\log\log n \approx 0.5229$.

A well-known theorem of Dirichlet (cf. [14, pp. 249-268]) states that for any relatively prime positive integers a and q the arithmetic progression $a, a+q, a+2q, \ldots$ contains infinitely many primes; we use $p_n(a,q)$ to denote the n-th prime in this progression.

The following conjecture extends the Firoozbakht conjecture to primes in arithmetic progressions.

Conjecture 2.3. (2012-08-11) *Let* $q \geqslant a \geqslant 1$ *be positive integers with* a *odd,* q *even and* $\gcd(a,q) = 1$. *Then there is a positive integer* $n_0(a,q)$ *such that the sequence* $(\sqrt[n]{p_n(a,q)})_{n \geqslant n_0(a,q)}$ *is strictly decreasing. Moreover, we may take* $n_0(a,q) = 2$ *for* $q \leqslant 45$.

Remark 2.3. Note that $\sqrt[4]{p_4(13,46)} < \sqrt[5]{p_5(13,46)}$. Also, $\sqrt[3]{p_3(3,328)} < \sqrt[4]{p_4(3,328)}$ and $\sqrt[6]{p_6(23,346)} < \sqrt[7]{p_7(23,346)}$.

A famous conjecture of E. Artin asserts that if $a \in \mathbb{Z}$ is neither -1 nor a square then there are infinitely many primes p having a as a primitive root modulo p. This is still open, the reader may consult the survey [18] for known progress on this conjecture.

Conjecture 2.4. (2012-08-17) *Let $a \in \mathbb{Z}$ be not a perfect power (i.e., there are no integers $m > 1$ and x with $x^m = a$).*

(i) *Assume that $a > 0$. Then there are infinitely many primes p having a as the smallest positive primitive root modulo p. Moreover, if $p_1(a), \ldots, p_n(a)$ are the first n such primes, then the next such prime $p_{n+1}(a)$ is smaller than $p_n(a)^{1+1/n}$, i.e., $\sqrt[n]{p_n(a)} > \sqrt[n+1]{p_{n+1}(a)}$.*

(ii) *Suppose that $a < 0$. Then there are infinitely many primes p having a as the largest negative primitive root modulo p. Moreover, if $p_1(a), \ldots, p_n(a)$ are the first n such primes, then the next such prime $p_{n+1}(a)$ is smaller than $p_n(a)^{1+1/n}$ (i.e., $\sqrt[n]{p_n(a)} > \sqrt[n+1]{p_{n+1}(a)}$) with the only exception $a = -2$ and $n = 13$.*

(iii) *The sequence $(\sqrt[n+1]{P_{n+1}(a)}/\sqrt[n]{P_n(a)})_{n\geqslant 3}$ is strictly increasing with limit 1, where $P_n(a) = \sum_{k=1}^n p_k(a)$.*

Remark 2.4. Let us look at two examples. The first 5 primes having 24 as the smallest positive primitive root are $p_1(24) = 533821$, $p_2(24) = 567631$, $p_3(24) = 672181$, $p_4(24) = 843781$ and $p_5(24) = 1035301$, and we can easily verify that

$$p_1(24) > \sqrt{p_2(24)} > \sqrt[3]{p_3(24)} > \sqrt[4]{p_4(24)} > \sqrt[5]{p_5(24)}.$$

The first prime having -12 as the largest negative primitive root is $p_1(-12) = 7841$, and the second prime having -12 as the largest negative primitive root is $p_2(-12) = 16061$; it is clear that $p_1(-12) > \sqrt{p_2(-12)}$.

Recall that the Proth numbers have the form $k \times 2^n + 1$ with k odd and $0 < k < 2^n$. In 1878 F. Proth proved that a Proth number p is a prime if (and only if) $a^{(p-1)/2} \equiv -1 \pmod{p}$ for some integer a (cf. Ex. 4.10 of [6, p. 220]). A Proth prime is a Proth number which is also a prime number; the Fermat primes are a special kind of Proth primes.

Conjecture 2.5. (2012-09-07) (i) *The number of Proth primes not exceeding a large integer x is asymptotically equivalent to $c\sqrt{x}/\log x$ for a suitable constant $c \in (3, 4)$.*

(ii) *If $\mathrm{Pr}(1), \ldots, \mathrm{Pr}(n)$ are the first n Proth primes, then the next Proth prime $\mathrm{Pr}(n+1)$ is smaller than $\mathrm{Pr}(n)^{1+1/n}$ (i.e., $\sqrt[n]{\mathrm{Pr}(n)} > \sqrt[n+1]{\mathrm{Pr}(n+1)}$) unless $n = 2, 4, 5$. If we set $\mathrm{PR}(n) = \sum_{k=1}^n \mathrm{Pr}(k)$, then $\mathrm{PR}(n) < n\mathrm{Pr}(n)/3$ for all $n > 50$, and the sequence $(\sqrt[n+1]{\mathrm{PR}(n+1)}/\sqrt[n]{\mathrm{PR}(n)})_{n\geqslant 34}$ is strictly increasing with limit 1.*

Remark 2.5. We have verified that $\sqrt[n]{\mathrm{Pr}(n)} > \sqrt[n+1]{\mathrm{Pr}(n+1)}$ for all $n = 6, \ldots, 4000$, $\mathrm{PR}(n) < n\mathrm{Pr}(n)/3$ for all $n = 51, \ldots, 3500$, and

$$\sqrt[n+1]{\mathrm{PR}(n+1)}/\sqrt[n]{\mathrm{PR}(n)} < \sqrt[n+2]{\mathrm{PR}(n+2)}/\sqrt[n+1]{\mathrm{PR}(n+1)}$$

for all $n = 34, \ldots, 3200$.

In the remaining part of this section, we usually list certain primes of special types in ascending order as $q_1, q_2, q_3, \ldots$, and write $Q(n)$ for $\sum_{k=1}^n q_k$. Note that the inequality $\sqrt[n]{Q(n)}/\sqrt[n-1]{Q(n-1)} < \sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)}$ yields a lower bound for q_{n+1}.

Conjecture 2.6. (i) (2012-08-18) *Let $q_1, q_2, q_3, \ldots$ be the list (in ascending order) of those primes of the form x^2+1 with $x \in \mathbb{Z}$. Then we have $q_{n+1} < q_n^{1+1/n}$ unless $n = 1, 2, 4, 351$. Also, the sequence $(\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)})_{n\geqslant 13}$ is strictly increasing with limit 1.*

(ii) (2012-09-07) *Let $q_1, q_2, q_3, \ldots$ be the list (in ascending order) of those primes of the form x^2+x+1 with $x \in \mathbb{Z}$. Then we have $q_{n+1} < q_n^{1+1/n}$ unless $n = 3, 6$. Also, the sequence $(\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)})_{n\geqslant 20}$ is strictly increasing with limit 1.*

Remark 2.6. If we use the notation in part (i) of Conj. 2.6, then $q_{351} = 3536^2 + 1 = 12503297$, $q_{352} = 3624^2 + 1 = 13133377$, and $\sqrt[351]{q_{351}} < \sqrt[352]{q_{352}}$.

Schinzel's Hypothesis H (cf. [6, pp. 17-18]) states that if $f_1(x), \ldots, f_k(x)$ are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product $f_1(q)\cdots f_k(q)$ for all $q \in \mathbb{Z}$, then there are infinitely many $n \in \mathbb{Z}^+$ such that $f_1(n), \ldots, f_k(n)$ are all primes.

Here is a general conjecture related to Hypothesis H.

Conjecture 2.7. (2012-09-08) *Let $f_1(x), \ldots, f_k(x)$ be irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing $\prod_{j=1}^k f_j(q)$ for all $q \in \mathbb{Z}$. Let $q_1, q_2, \ldots$ be the list (in ascending order) of those $q \in \mathbb{Z}^+$ such that $f_1(q), \ldots, f_k(q)$ are all primes. Then, for all sufficiently large positive integers n, we have*

$$\frac{2}{n-1}Q(n) < q_{n+1} < q_n^{1+1/n}.$$

Also, for some $N \in \mathbb{Z}^+$ the sequence $(\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)})_{n\geqslant N}$ is strictly increasing with limit 1.

Remark 2.7. Obviously $2Q(n) < (n-1)q_{n+1}$ if and only if $Q(n+1) < (n+1)q_{n+1}/2$.

For convenience, under the condition of Conj. 2.7, below we set

$$E(f_1(x), \ldots, f_k(x)) = \{n \in \mathbb{Z}^+ : \ \sqrt[n]{q_n} > \sqrt[n+1]{q_{n+1}} \ \text{ fails}\}$$

and let $N_0(f_1(x), \ldots, f_k(x))$ stand for the least positive integer n_0 such that $2Q(n) < (n-1)q_{n+1}$ for all $n \geqslant n_0$, and let $N(f_1(x), \ldots, f_k(x))$ denote the smallest positive integer N such that $(\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)})_{n\geqslant N}$ is strictly increasing with limit 1.

If p and $p+2$ are both primes, then $\{p, p+2\}$ is said to be a pair of twin primes. The famous twin prime conjecture states that there are infinitely many twin primes.

Conjecture 2.8. (2012-08-18) *We have*

$$E(x, x+2) = \emptyset, \quad N_0(x, x+2) = 4, \quad \textit{and } N(x, x+2) = 9.$$

Remark 2.8. Let $q_1, q_2, \ldots$ be the list of those primes p with $p+2$ also prime. We have verified that $\sqrt[n]{q_n} > \sqrt[n+1]{q_{n+1}}$ for all $n = 1, \ldots, 500000$, $q_{n+1} > 2Q(n)/(n-1)$ for all $n = 4, \ldots, 2000000$, and $\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)} < \sqrt[n+2]{Q(n+2)}/\sqrt[n+1]{Q(n+1)}$ for all $n = 9, \ldots, 500000$. See also Conjecture 2.10 of the author [31].

Conjecture 2.9. (2012-08-20) *We have*

$$E(x, x+2, x+6) = E(x, x+4, x+6) = \emptyset,$$
$$N_0(x, x+2, x+6) = 3, \ N_0(x, x+4, x+6) = 6,$$
$$N(x, x+2, x+6) = N(x, x+4, x+6) = 13.$$

Remark 2.9. Recall that a prime triplet is a set of three primes of the form $\{p, p+2, p+6\}$ or $\{p, p+4, p+6\}$. It is conjectured that there are infinitely many prime triplets.

A prime p is called a Sophie Germain prime if $2p+1$ is also a prime. It is conjectured that there are infinitely many Sophie Germain primes, but this has not been proved yet.

Conjecture 2.10. (2012-08-18) *We have*

$$E(x, 2x+1) = \{3, 4\}, \ N_0(x, 2x+1) = 3, \ \textit{and } N(x, 2x+1) = 13.$$

Also,

$$E(x, 2x-1) = \{2, 3, 6\}, \ N_0(x, 2x-1) = 3, \ \textit{and } N(x, 2x-1) = 9.$$

Remark 2.10. When $q_1, q_2, \ldots$ gives the list of Sophie Germain primes in ascending order, we have verified that $\sqrt[n]{q_n} > \sqrt[n+1]{q_{n+1}}$ for all $n = 5, \ldots, 200000$, and $\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)} < \sqrt[n+2]{Q(n+2)}/\sqrt[n+1]{Q(n+1)}$ for every $n = 13, \ldots, 200000$.

One may wonder whether $E(x, x+d)$ or $E(x, 2x+d)$ with small $d \in \mathbb{Z}^+$ may contain relatively large elements. We have checked this for $d \leqslant 100$.

Here are few extremal examples suggested by our computation:

$$E(x,x+60)=\{187,3976,58956\},\ E(x,x+66)=\{58616\},$$
$$E(x,2x+11)=\{1,39593\},\ E(x,2x+81)=\{104260\}.$$

Conjecture 2.11. (2012-09-07) *We have*

$$E(x,x^2+x+1)=\{3,4,12,14\},$$
$$N_0(x,x^2+x+1)=3,\ N(x,x^2+x+1)=17.$$

Also,

$$E(x^4+1)=\{1,2,4\},\ N_0(x^4+1)=4,\ \text{and } N(x^4+1)=10.$$

Remark 2.11. Note that those primes p with p^2+p+1 prime are sparser than twin primes and Sophie Germain primes.

2.2. Conjectures on other number-theoretic sequences.

A positive integer n is called *squarefree* if $p^2 \nmid n$ for any prime p. Here is the list of all squarefree positive integers not exceeding 30 in ascending order:

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 11,\ 13,\ 14,\ 15,\ 17,\ 19,\ 21,\ 22,\ 23,\ 26,\ 29,\ 30.$$

Conjecture 2.12. (2012-08-14) *Let $s_1,s_2,s_3,\ldots$ be the list of squarefree positive integer in ascending order. Then the sequence $(\sqrt[n]{s_n})_{n\geqslant 7}$ is strictly decreasing, and the sequence $(\sqrt[n+1]{S(n+1)}/\sqrt[n]{S(n)})_{n\geqslant 7}$ is strictly increasing, where $S(n)=\sum_{k=1}^n s_k$.*

Remark 2.12. We have verified that $\sqrt[n]{s_n} > \sqrt[n+1]{s_{n+1}}$ for all $n=7,\ldots,500000$. Note that $\lim_{n\to\infty}\sqrt[n]{S(n)}=1$ since $S(n)$ does not exceed the sum of the first n primes.

Conjecture 2.13. (2012-08-25) *Let a_n be the n-th positive integer that can be written as a sum of two squares. Then the sequence $(\sqrt[n]{a_n})_{n\geqslant 6}$ is strictly decreasing, and the sequence $(\sqrt[n+1]{A(n+1)}/\sqrt[n]{A(n)})_{n\geqslant 6}$ is strictly increasing, where $A(n)=\sum_{k=1}^n a_k$.*

Remark 2.13. Similar things happen if we replace sums of squares in Conj. 2.13 by integers of the form x^2+dy^2 with $x,y\in\mathbb{Z}$, where d is any positive integer.

Recall that a partition of a positive integer n is a way of writing n as a sum of positive integers with the order of addends ignored. Also, a *strict partition* of $n\in\mathbb{Z}^+$ is a way of writing n as a sum of *distinct* positive integers with the order of addends ignored. For $n=1,2,3,\ldots$ we denote by $p(n)$

and $p_*(n)$ the number of partitions of n and the number of strict partitions of n respectively. It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{and} \quad p_*(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \qquad \text{as } n \to +\infty$$

(cf. [12] and [1, p. 826]) and hence

$$\lim_{n\to\infty} \sqrt[n]{p(n)} = \lim_{n\to\infty} \sqrt[n]{p_*(n)} = 1.$$

Conjecture 2.14. (2012-08-02) *Both* $(\sqrt[n]{p(n)})_{n\geqslant 6}$ *and* $(\sqrt[n]{p_*(n)})_{n\geqslant 9}$ *are strictly decreasing. Furthermore, the sequences* $(\sqrt[n+1]{p(n+1)}/\sqrt[n]{p(n)})_{n\geqslant 26}$ *and* $(\sqrt[n+1]{p_*(n+1)}/\sqrt[n]{p_*(n)})_{n\geqslant 45}$ *are strictly increasing.*

Remark 2.14. The author has verified the conjecture for n up to 10^5. [31] contains a stronger version of this conjecture.

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are rational numbers given by

$$B_0 = 1, \quad \text{and} \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad \text{for } n \in \mathbb{Z}^+.$$

It is well known that $B_{2n+1} = 0$ for all $n \in \mathbb{Z}^+$ and

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi).$$

(See, e.g., [14, pp. 228-232].) The Euler numbers $E_0, E_1, E_2, \ldots$ are integers defined by

$$E_0 = 1, \quad \text{and} \quad \sum_{\substack{k=0\\ 2\mid k}}^{n} \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+.$$

It is well known that $E_{2n+1} = 0$ for all $n = 0, 1, 2, \ldots$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \quad \left(|x| < \frac{\pi}{2}\right).$$

Conjecture 2.15. (2012-08-02) $(\sqrt[n]{(-1)^{n-1}B_{2n}})_{n\geqslant 1}$ *and* $\sqrt[n]{(-1)^n E_{2n}})_{n\geqslant 1}$ *are strictly increasing, where* $B_0, B_1, \ldots$ *are Bernoulli numbers and* $E_0, E_1, \ldots$ *are Euler numbers. Moreover, the sequences*

$$\left(\sqrt[n+1]{(-1)^n B_{2n+2}}/\sqrt[n]{(-1)^{n-1}B_{2n}}\right)_{n\geqslant 2}$$

and

$$\left(\sqrt[n+1]{(-1)^{n+1} E_{2n+2}}/\sqrt[n]{(-1)^n E_{2n}}\right)_{n\geqslant 1}$$

are strictly decreasing.

Remark 2.15. It is known that both $(-1)^{n-1}B_{2n}$ and $(-1)^nE_{2n}$ are positive for all $n = 1, 2, 3, \ldots$.

For $m, n \in \mathbb{Z}^+$ the n-th harmonic number $H_n^{(m)}$ of order m is defined as $\sum_{k=1}^n 1/k^m$.

Conjecture 2.16. (2012-08-12) *For any positive integer m, the sequence*

$$\left(\sqrt[n+1]{H_{n+1}^{(m)}}\Big/\sqrt[n]{H_n^{(m)}}\right)_{n\geqslant 3}$$

is strictly increasing.

Remark 2.16. It is easy to show that $\left(\sqrt[n]{H_n^{(m)}}\right)_{n\geqslant 2}$ is strictly decreasing for any $m \in \mathbb{Z}^+$. Some fundamental congruences on harmonic numbers can be found in [29].

Conjecture 2.17. (2012-09-01) *Let $q > 1$ be a prime power and let $\mathbb{F}_q$ be the finite field of order q. Let $M_n(q)$ denote the number of monic irreducible polynomials of degree at most n over $\mathbb{F}_q$.*

(i) *We have $M_q(n+1)/M_q(n) < M_q(n+2)/M_q(n+1)$ unless $q < 5$ and $n \in \{2, 4, 6, 8, 10, 12\}$.*

(ii) *If $n > 2$, then $\sqrt[n]{M_q(n)} < \sqrt[n+1]{M_q(n+1)}$ unless $q < 7$ and $n \in \{3, 5\}$.*

(iii) *When $n > 3$, we have*

$$\sqrt[n+1]{M_q(n+1)}\Big/\sqrt[n]{M_q(n)} > \sqrt[n+2]{M_q(n+2)}\Big/\sqrt[n+1]{M_q(n+1)}$$

unless $(q < 8$ & $n \in \{5, 7, 9, 11, 13\})$ or $(9 < q < 14$ & $n = 4)$.

Remark 2.17. It is known that the number of monic irreducible polynomials of degree n over the finite field $\mathbb{F}_q$ equals $\frac{1}{n}\sum_{d\mid n}\mu(d)q^{n/d}$, where μ is the Möbius function (cf. [14, p. 84]).

3. Conjectures on combinatorial sequences

The Fibonacci sequence $(F_n)_{n\geqslant 0}$ is given by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots).$$

the reader may consult [24, p. 46] for combinatorial interpretations of Fibonacci numbers.

Conjecture 3.1. (2012-08-11) *The sequence $(\sqrt[n]{F_n})_{n\geqslant 2}$ is strictly increasing, and moreover the sequence $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n\geqslant 4}$ is strictly decreasing. Also, for any integers $A > 1$ and $B \neq 0$ with $A^2 > 4B$ and ($A > 2$ or*

$B \geqslant -9$), *the sequence* $(\sqrt[n+1]{u_{n+1}}/\sqrt[n]{u_n})_{n\geqslant 4}$ *is strictly decreasing with limit* 1, *where*

$$u_0 = 0,\ u_1 = 1,\ \textit{and}\ u_{n+1} = Au_n - Bu_{n-1}\ (n = 1, 2, 3, \ldots).$$

Remark 3.1. By [25, Lemma 4], if $A > 1$ and $B \neq 0$ are integers with $A^2 > 4B$ then the sequence $(u_n)_{n\geqslant 0}$ defined in Conjecture 3.1 is strictly increasing.

For $n = 1, 2, 3, \ldots$ the n-th Bell number B_n denotes the number of partitions of $\{1, \ldots, n\}$ into disjoint nonempty subsets. It is known that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ (with $B_0 = 1$) and $B_n = e^{-1} \sum_{k=0}^{\infty} k^n/k!$ for all $n = 0, 1, 2, \ldots$ (cf. [22, A000110]).

Conjecture 3.2. (2012-08-11) *The sequence* $(\sqrt[n]{B_n})_{n\geqslant 1}$ *is strictly increasing, and moreover the sequence* $(\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n})_{n\geqslant 1}$ *is strictly decreasing with limit* 1, *where* B_n *is the* n*-th Bell number.*

Remark 3.2. In 1994 K. Engel [10] proved the log-convexity of $(B_n)_{n\geqslant 1}$. [32] contains a curious congruence property of the Bell numbers.

For $n \in \mathbb{Z}^+$ the n-th derangement number D_n denotes the number of permutations σ of $\{1, \ldots, n\}$ with $\sigma(i) = i$ for no $i = 1, \ldots, n$. It has the following explicit expression (cf. [24, p. 67]):

$$D_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

Conjecture 3.3. (2012-08-11) *The sequence* $(\sqrt[n]{D_n})_{n\geqslant 2}$ *is strictly increasing, and the sequence* $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n\geqslant 3}$ *is strictly decreasing.*

Remark 3.3. As $D_n = nD_{n-1} + (-1)^n$ for $n \in \mathbb{Z}^+$, it is easy to see that $(D_{n+1}/D_n)_{n\geqslant 1}$ is strictly increasing.

During his study of irreducible root systems of a special type related to Weyl groups, T. A. Springer [23] introduced the Springer numbers $S_0, S_1, \ldots$ defined by

$$\frac{1}{\cos x - \sin x} = \sum_{n=0}^{\infty} S_n \frac{x^n}{n!}.$$

The reader may consult [22, A001586] for various combinatorial interpretations of Springer numbers.

Conjecture 3.4. (2012-08-05) *The sequence* $(S_{n+1}/S_n)_{n\geqslant 0}$ *is strictly increasing, and the sequence* $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 1}$ *is strictly decreasing with limit* 1, *where* S_n *is the* n*-th Springer number.*

Remark 3.4. It is known (cf. [22, A001586]) that S_n coincides with the numerator of $|E_n(1/4)|$, where $E_n(x)$ is the Euler polynomial of degree n.

Conjecture 3.5. (2012-08-18) *For the tangent numbers* $T(1), T(2), \ldots$ *given by*

$$\tan x = \sum_{n=1}^{\infty} T(n)\frac{x^{2n-1}}{(2n-1)!},$$

the sequences $(T(n+1)/T(n))_{n\geqslant 1}$ *and* $(\sqrt[n]{T(n)})_{n\geqslant 1}$ *are strictly increasing, and the sequence* $(\sqrt[n+1]{T(n+1)}/\sqrt[n]{T(n)})_{n\geqslant 2}$ *is strictly decreasing.*

Remark 3.5. The tangent numbers are all integral, see [22, A000182] for the sequence $(T(n))_{n\geqslant 1}$. It is known that $T(n) = (-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}/(2n)$ for all $n \in \mathbb{Z}^+$, where B_{2n} is the $2n$-th Bernoulli number.

The n-th central trinomial coefficient T_n is the coefficient of x^n in the expansion of $(x^2+x+1)^n$. Here is an explicit expression:

$$T_n = \sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$ (cf. [22, A002426]). It is known that $(n+1)T_{n+1} = (2n+1)T_n + 3nT_{n-1}$ for all $n \in \mathbb{Z}^+$.

Conjecture 3.6. (2012-08-11) *The sequence* $(\sqrt[n]{T_n})_{n\geqslant 1}$ *is strictly increasing, and the sequence* $(\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n})_{n\geqslant 1}$ *is strictly decreasing.*

Remark 3.6. Via the Laplace-Heine formula (cf. [33, p. 194]) for Legendre polynomials, $T_n \sim 3^{n+1/2}/(2\sqrt{n\pi})$ as $n \to +\infty$. In 2011, the author [28] found many series for $1/\pi$ involving generalized central trinomial coefficients.

The n-th Motzkin number

$$M_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}\frac{1}{k+1}$$

is the number of lattice paths from $(0,0)$ to $(n,0)$ which never dip below the line $y=0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$ (cf. [22, A001006]). It is known that $(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}$ for all $n \in \mathbb{Z}^+$.

Conjecture 3.7. (2012-08-11) *The sequence* $(\sqrt[n]{M_n})_{n\geqslant 1}$ *is strictly increasing, and moreover the sequence* $(\sqrt[n+1]{M_{n+1}}/\sqrt[n]{M_n})_{n\geqslant 1}$ *is strictly decreasing.*

Remark 3.7. The log-convexity of the sequence $(M_n)_{n\geqslant 1}$ was first established by M. Aigner [2] in 1998.

For $r = 2, 3, 4, \ldots$ define

$$f_n^{(r)} := \sum_{k=0}^{n} \binom{n}{k}^r \quad (n = 0, 1, 2, \ldots).$$

Note that $f_n^{(2)} = \binom{2n}{n}$, and those $f_n = f_n^{(3)}$ are called Franel numbers (cf. [22, A000172]).

Conjecture 3.8. (2012-08-11) *For each $r = 2, 3, 4, \ldots$ there is a positive integer $N(r)$ such that the sequence $\big(\sqrt[n+1]{f_{n+1}^{(r)}}/\sqrt[n]{f_n^{(r)}}\big)_{n\geqslant N(r)}$ is strictly decreasing with limit 1. Moreover, we may take*

$$N(2) = \cdots = N(6) = 1, \quad N(7) = N(8) = N(9) = 3, \; N(10) = N(11) = 5,$$
$$N(12) = N(13) = 7, \; N(14) = N(15) = N(16) = 9, \; N(17) = N(18) = 11.$$

Remark 3.8. It is known that $(f_n^{(r)})_{n\geqslant 1}$ is log-convex for $r = 2, 3, 4$ (cf. [7]). [27] contains some fundamental congruences for Franel numbers.

Conjecture 3.9. (2012-08-15) *Set $g_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$ for $n = 0, 1, 2, \ldots$. Then $(\sqrt[n]{g_n})_{n\geqslant 1}$ is strictly increasing and the sequence $(\sqrt[n+1]{g_{n+1}}/\sqrt[n]{g_n})_{n\geqslant 1}$ is strictly decreasing.*

Remark 3.9. It is known that $g_n = \sum_{k=0}^{n} \binom{n}{k} f_k$, where $f_k = \sum_{j=0}^{k} \binom{k}{j}^3$ is the k-th Franel number. Both $(f_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 0}$ are related to the theory of modular forms, see D. Zagier [35].

For $r = 1, 2, 3, \ldots$ define

$$A_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r \quad (n = 0, 1, 2, \ldots).$$

Those $A_n^{(1)}$ and $A_n = A_n^{(2)}$ are called central Delannoy numbers and Apéry numbers respectively. The Apéry numbers play a key role in Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (cf. [3, 19]).

Conjecture 3.10. (2012-08-11) *For each $r = 1, 2, 3, \ldots$ there is a positive integer $M(r)$ such that the sequence $\big(\sqrt[n+1]{A_{n+1}^{(r)}}/\sqrt[n]{A_n^{(r)}}\big)_{n\geqslant M(r)}$ is strictly decreasing with limit 1. Moreover, we may take*

$$M(1) = \cdots = M(16) = 1, \quad M(17) = M(18) = M(19) = 9, \; M(20) = 12.$$

Remark 3.10. The log-convexity of $(A_n)_{n\geqslant 0}$ was proved by T. Došlić [7]. The reader may consult [30] for some congruences involving Apéry numbers and Apéry polynomials.

The n-th Schröder number

$$S_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}\frac{1}{k+1}$$

is the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$ that never rise above the line $y=x$ (cf. [22, A006318] and [24, p. 185]).

Conjecture 3.11. (2012-08-11) *The sequence* $(\sqrt[n]{S_n})_{n\geqslant 1}$ *is strictly increasing, and moreover the sequence* $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 1}$ *is strictly decreasing, where* S_n *stands for the* n*-th Schröder number.*

Remark 3.11. The reader may consult [26] for some congruences involving central Delannoy numbers and Schröder numbers.

Conjecture 3.12. (2012-08-13) *For the Domb numbers*

$$D(n)=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\ (n=0,1,2,\ldots),$$

the sequences $(D(n+1)/D(n))_{n\geqslant 0}$ *and* $(\sqrt[n]{D(n)})_{n\geqslant 1}$ *are strictly increasing. Moreover, the sequence* $(\sqrt[n+1]{D(n+1)}/\sqrt[n]{D(n)})_{n\geqslant 1}$ *is strictly decreasing.*

Remark 3.12. For combinatorial interpretations of the Domb number $D(n)$, the reader may consult [22, A002895]. [4] contains some series for $1/\pi$ involving Domb numbers.

The Catalan-Larcombe-French numbers $P_0,P_1,P_2,\ldots$ (cf. [16]) are given by

$$P_n=\sum_{k=0}^n\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}=2^n\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}^2 4^{n-2k},$$

they arose from the theory of elliptic integrals (see [11]). It is known that $(n+1)P_{n+1}=(24n(n+1)+8)P_n-128n^2P_{n-1}$ for all $n\in\mathbb{Z}^+$. The sequence $(P_n)_{n\geqslant 0}$ is also related to the theory of modular forms, see D. Zagier [35].

Conjecture 3.13. (2012-08-14) *The sequences* $(P_{n+1}/P_n)_{n\geqslant 0}$ *and* $(\sqrt[n]{P_n})_{n\geqslant 1}$ *are strictly increasing. Moreover, the sequence* $(\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n})_{n\geqslant 1}$ *is strictly decreasing.*

Remark 3.13. We also have the following conjecture related to Euler numbers:

$$\sum_{k=0}^{p-1}\frac{P_k}{8^k}\equiv 1+2\left(\frac{-1}{p}\right)p^2E_{p-3}\pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}\frac{P_k}{16^k}\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3}\pmod{p^3}$$

for any odd prime p, where $(\frac{\cdot}{p})$ is the Legendre symbol.

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References

[1] M. Abramowitz and I. A. Stegun (eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing, New York, Dover, 1972.

[2] M. Aigner, *Motzkin numbers*, European J. Combin. **19** (1998), 663–675.

[3] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Astérisque **61** (1979), 11–13.

[4] H. H. Chan, S. H. Chan and Z.-G. Liu, *Domb's numbers and Ramanujan-Sato type series for* $1/\pi$, Adv. in Math. **186** (2004), 396–410.

[5] W. Y. C. Chen, *Recent developments on log-concavity and q-log-concavity of combinatorial polynomials*, a talk given at the 22nd Inter. Confer. on Formal Power Series and Algebraic Combin. (San Francisco, 2010).

[6] R. Crandall and C. Pomerance, Prime Numbers: A Computational Perspective, 2nd Edition, Springer, New York, 2005.

[7] T. Došlić, *Log-balanced combinatorial sequences*, Int. J. Math. Math. Sci. **4** (2005), 507–522.

[8] T. Došlić, *Log-convexity of combinatorial sequences from their convexity*, J. Math. Inequal. **3** (2009), 437–442.

[9] P. Dusart, *Sharper bounds for* ψ,θ,π,p_k, Rapport de Recherche, 1998.

[10] K. Engel, *On the average rank of an element in a filter of the partition lattice*, J. Comb. Theory Ser. A **65** (1994), 67–78.

[11] P. Larcombe and D. French, *On the 'other' Catalan numbers: a historical formulation re-examined*, Congr. Numer. **143** (2000), 33–64.

[12] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatorial analysis*, Proc. London Math. Soc. **17** (1918), 75–115.

[13] M. Hassani, *A remark on the Mandl's inequality*, Octogon Math. Magazine **15** (2007), 567–572.

[14] K. Ireland and M. Rosen, A Classical Introduction to Modern Number Theory, 2nd Edition, Springer, New York, 1990.

[15] J. E. Janoski, *A collection of problems in combinatorics*, PhD Thesis, Clemson Univ., May 2012.

[16] F. Jarvis and H. A. Verrill, *Supercongruences for the Catalan-Larcombe-French numbers*, Ramanujan J. **22** (2010), 171–186.

[17] L. L. Liu and Y. Wang, *On the log-convexity of combinatorial sequences*, Adv. in Appl. Math. **39** (2007), 453–476 .

[18] R. Murty, *Artin's conjecture for primitive roots*, Math. Intelligencer **10** (1988), 59–67.

[19] A. van der Poorten, *A proof that Euler missed...Apéry's proof of the irrationality of* $\zeta(3)$, Math. Intelligencer **1** (1978/79), 195–203.

[20] P. Ribenboim, The Little Book of Bigger Primes, 2nd Edition, Springer, New York, 2004.

[21] J. B. Rosser and L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$*, Math. Comp. **29** (1975), 243–269.

[22] N. J. A. Sloane, Sequences A000110, A000172, A000182, A001006, A001586, A002426, A002895, A006318 in OEIS (On-Line Encyclopedia of Integer Sequences), `http://www.oeis.org`.

[23] T. A. Springer, *Remarks on a combinatorial problem*, Nieuw Arch. Wisk. **19** (1971), 30–36.

[24] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Univ. Press, Cambridge, 1997.

[25] Z. W. Sun, *Reduction of unknowns in diophantine representations*, Sci. China Ser. A **35** (1992), 257–269.

[26] Z. W. Sun, *On Delannoy numbers and Schröder numbers*, J. Number Theory **131** (2011), 2387–2397.

[27] Z. W. Sun, *Congruences for Franel numbers*, preprint, `arXiv:1112.1034`.

[28] Z. W. Sun, *List of conjectural series for powers of π and other constants*, preprint, `arXiv:1102.5649`.

[29] Z. W. Sun, *Arithmetic theory of harmonic numbers*, Proc. Amer. Math. Soc. **140** (2012), 415–428.

[30] Z. W. Sun, *On sums of Apéry polynomials and related congruences*, J. Number Theory **132** (2012), 2673–2699.

[31] Z. W. Sun, *On a sequence involving sums of primes*, Bull. Aust. Math. Soc., to appear.

[32] Z. W. Sun and D. Zagier, *On a curious property of Bell numbers*, Bull. Aust. Math. Soc. **84** (2011), 153–158.

[33] G. Szegö, Orthogonal Polynomials, 4th Edition, Amer. Math. Soc., Providence, RI, 1975.

[34] M. Wolf, Personal communications, August 15, 2012.

[35] D. Zagier, *Integral solutions of Apéry-like recurrence equations*, in: Groups and Symmetries: from Neolithic Scots to John McKay, CRM Proc. Lecture Notes 47, Amer. Math. Soc., Providence, RI, 2009, pp. 349–366.

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