PLANAR DIAGRAMS
AND
TENSOR ALGEBRA

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INTRODUCTION

The algebraic view of this paper is that it provides topological precision for a diagrammatic technique to calculate with morphisms between tensor products. The geometric view is that it shows tensor categories and autonomous tensor categories to be the algebraic structures underlying fairly naturally occurring types of planar diagrams to within appropriate deformation.

CHAPTER 1

TENSOR CATEGORIES AND PLANAR DIAGRAMS

§1. Tensor categories

Let us recall the concept [, , , ,] of tensor category $\mathbb{V} = (\mathbb{V}, \otimes, I, a, l, r)$, also called "monoidal category". It consists of a category $\mathbb{V}$, a functor $\otimes : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ (called the tensor product), an object $I \in \mathbb{V}$ (called the unit object) and natural isomorphisms

$$a = a_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

$$l = l_A : I \otimes A \longrightarrow A, \qquad r = r_A : A \otimes I \longrightarrow A$$

(called the constraints of associativity, left unit, right unit, respectively) such that the following diagrams commute:

a) the pentagon for associativity

$$\begin{tikzcd}
 & (A\otimes B)\otimes(C\otimes D) \arrow[dr, "a"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ur, "a"] \arrow[d, "a\otimes 1"'] & & A\otimes(B\otimes(C\otimes D)) \\
(A\otimes(B\otimes C))\otimes D \arrow[r, "a"'] & A\otimes((B\otimes C)\otimes D) \arrow[ur, "1\otimes a"'] &
\end{tikzcd} \quad ;$$

b) the triangle for the unit

$$\begin{tikzcd}(A\otimes I)\otimes B \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & A\otimes(I\otimes B) \arrow[dl, "1\otimes l"] \\ & A\otimes B & \end{tikzcd}$$

Commutativity of the two triangles

$$\begin{tikzcd}(I\otimes A)\otimes B \arrow[rr, "a"] \arrow[dr, "l\otimes 1"'] & & I\otimes(A\otimes B) \arrow[dl, "l"] \\ & A\otimes B & \end{tikzcd} \qquad \begin{tikzcd}(A\otimes B)\otimes I \arrow[rr, "a"] \arrow[dr, "r"'] & & A\otimes(B\otimes I) \arrow[dl, "1\otimes r"] \\ & A\otimes B & \end{tikzcd}$$

and the equality $r = l : I\otimes I \to I$ are (not-so-obvious) consequences of these axioms [ ]. The coherence theorem of Mac Lane [ ] states that all diagrams built up from a, l, r by tensoring, substituting and composing, commute. It follows that all the objects obtained by computing the tensor product of a sequence $A_1\otimes\ldots\otimes A_n$, by bracketing it differently and by cancelling units, are coherently identified with each other. More precisely, the different ways of computing the tensor product $A_1\otimes\ldots\otimes A_n$ produce a clique; that is, a non-empty family $(C_i \mid i\in I)$ of objects together with a family $(u_{ji}: C_i \longrightarrow C_j \mid (i,j) \in I \times I)$ of maps such that $u_{ii} = 1$ and $u_{ki} = u_{kj} u_{ji}$ (so that $u_{ij} = u_{ji}^{-1}$). The cliques in $\mathbb{V}$ are the objects of a category $\overline{\mathbb{V}}$ in which a map $f : (C_i \mid i \in I) \longrightarrow (D_k \mid k \in K)$ is a family of maps $f_{ki} : C_i \longrightarrow D_k$ such that

$$\begin{tikzcd} C_i \arrow[r, "f_{ki}"] \arrow[d, "u_{ji}"'] & D_k \arrow[d, "u_{mk}"] \\ C_j \arrow[r, "f_{mj}"'] & D_m \end{tikzcd}$$

commutes for every $(i,j) \in I^2$, $(k,m) \in K^2$. It is sometimes convenient to think of the n-fold tensor product as a functor

$$\mathbb{V}^n \longrightarrow \overline{\mathbb{V}}.$$

The functor $\mathbb{V} \longrightarrow \overline{\mathbb{V}}$ which associates to each $A \in \mathbb{V}$ the singleton clique $(A) \in \overline{\mathbb{V}}$ is full and faithful. Since any clique is isomorphic to the singleton clique of any one of its members, this functor is an equivalence. This equivalence between $\mathbb{V}$ and $\overline{\mathbb{V}}$ shows that the ambiguity which exists in computing the n-fold tensor product is not a real one.

Furthermore, any tensor category is equivalent to a strict one [ ]; that is, one in which each constraint is an identity arrow. In principle, most results obtained with the hypothesis that a tensor category is strict can be reformulated and proved without this condition. Since, in this paper, we wish to focus on aspects other than associativity of tensor product, we shall avoid putting brackets on n-fold tensor products when clarity is gained and rigor preserved.

Example. Let k be a commutative ring. A bialgebra A over k is an associative algebra with unit equipped with a pair of algebra homomorphisms

$$\Delta : A \to A \otimes A, \qquad \varepsilon : A \to k,$$

(called the diagonal and the counit) such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\otimes A \arrow[d, "\Delta\otimes 1"] \\ A\otimes A \arrow[r, "1\otimes\Delta"'] & A\otimes A\otimes A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "1"'] \arrow[d, "\Delta"] \arrow[dr, "1"] & \\ A \arrow[r, "\varepsilon\otimes 1"'] & A\otimes A & A \arrow[l, "1\otimes\varepsilon"] \end{tikzcd}$$

commute. Let $\mathrm{Mod}(A)$ denote the category of left A-modules. For any $V, W \in \mathrm{Mod}(A)$, the tensor product $V \otimes W$ is an $(A \otimes A)$-module which becomes an A-module if we restrict the action of $A \otimes A$ along the diagonal $\Delta : A \to A \otimes A$. This defines a tensor product on $\mathrm{Mod}(A)$ yielding a tensor category for which the unit object is k equipped with the A-module structure given by $\varepsilon : A \to k$. //

In a tensor category there are two operations for constructing new arrows from old ones: composition $f \circ g$ and tensor product $f \otimes g$. Using ordinary algebraic notation, we are immediately faced with expressions like

$$(B \otimes c \otimes d) \circ (B \otimes B \otimes b \otimes C) \circ (a \otimes B \otimes C) = w_1$$

and

$$(B \otimes C \otimes d) \circ (B \otimes c \otimes D \otimes C) \circ (a \otimes b \otimes C) = w_2 .$$

In this form it is sometimes unclear when two words like w_1, w_2 are equivalent; that is, when they can be proved to be equal from the tensor category axioms alone. The graphical notation which we shall develop in this chapter will make it easier to detect such equalities and hence will provide a convenient technique for computation in a tensor category.

To motivate the precise topological details of the graphical notation, we point out that the two words w_1, w_2 given above will be respectively represented by the two diagrams Fig. 1, Fig. 2.

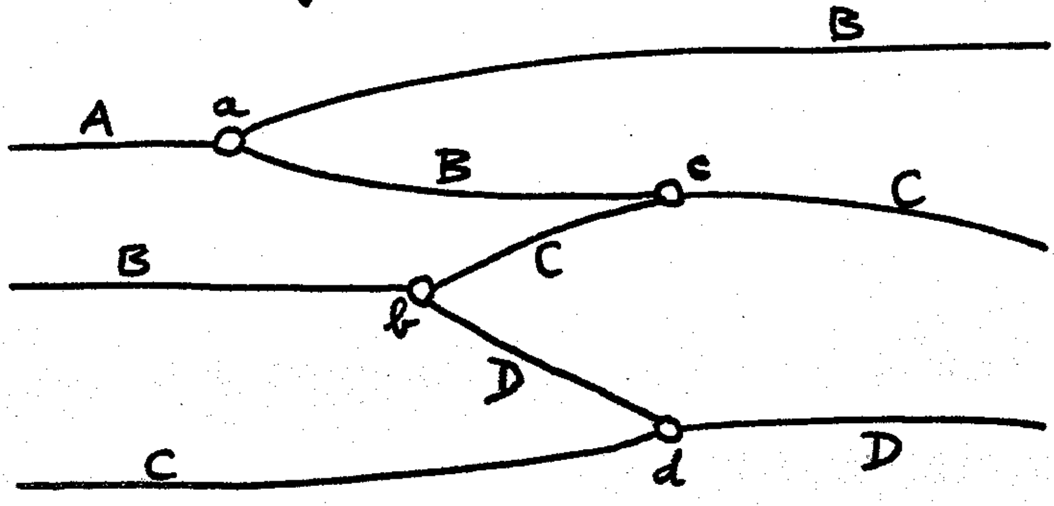


Fig. 1

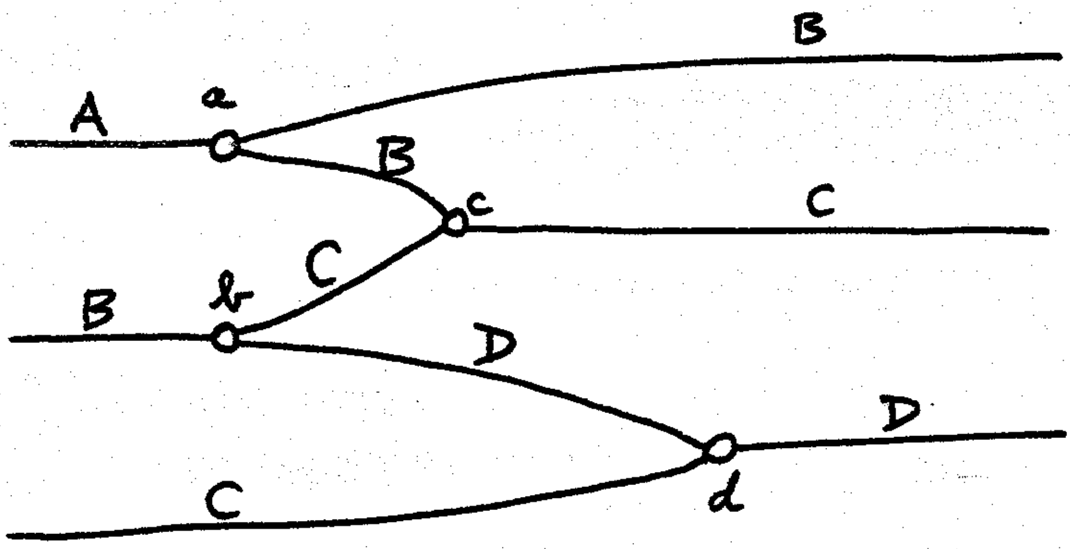


Fig. 2

In these diagrams the inner nodes represent the maps

$a: A \longrightarrow B \otimes B$, $\quad b: B \longrightarrow C \otimes D$, $\quad c: B \otimes C \longrightarrow C$,

$d : D \otimes C \longrightarrow D$,

whereas the edges represent objects. It is obvious that Fig. 1 and Fig. 2 are deformations of one another; this will enable us to deduce the equality $w_1 = w_2$.

§2. Graphs

This section introduces the basic concepts of graph theory needed for this paper. Its purpose is also to fix the notation and terminology.

Let us recall the concept of (*topological*) *graph* $\Gamma = (\Gamma, \Gamma_0)$, also called "1-dimensional cell complex". It consists of a Hausdorff space Γ and a closed subset $\Gamma_0 \subseteq \Gamma$ such that

(i) Γ_0 is discrete,

(ii) $\Gamma - \Gamma_0$ is the topological sum of its connected components each of which is homeomorphic to an open interval,

(iii) the closure in Γ of each component of $\Gamma - \Gamma_0$ is compact.

An element of Γ_0 is called a 0-*cell*; a component of $\Gamma - \Gamma_0$ is called an *open* 1-*cell*. Each open

1-cell e can be compactified to a <u>closed</u> 1-<u>cell</u> $\hat{e}$ by adjoining two end-points. The resulting space $\hat{e}$ is homeomorphic to a closed interval and the inclusion $e \hookrightarrow \Gamma$ can be extended by continuity to a map $\hat{e} \to \Gamma$, called the <u>structural map</u>, which sends the boundary $\partial\hat{e}$ into Γ_0.

We recall that a <u>(combinatorial) graph</u> G consists of a pair (G_1, G_0) of sets, and two functions

$$\begin{aligned} G_1 &\longrightarrow G_0 \times G_0 \\ \gamma &\longmapsto (\gamma(0), \gamma(1)) \end{aligned} \qquad\qquad \begin{aligned} G_1 &\longrightarrow G_1 \\ \gamma &\longmapsto \gamma^0 \end{aligned}$$

satisfying the following conditions for each $\gamma \in G_1$:

$$(\gamma^0)^0 = \gamma, \qquad \gamma^0 \neq \gamma, \qquad \gamma^0(0) = \gamma(1).$$

An element $x \in G_0$ is called a <u>vertex</u> or a <u>node</u> of G; an element $\gamma \in G_1$ is called an (<u>oriented</u>) <u>edge</u> of G and γ^0 is called the <u>opposite edge</u>. The vertex $\gamma(0)$ is called the <u>source</u> or the <u>origin</u> of γ and the vertex $\gamma(1)$ is called the <u>target</u> of γ. An <u>orientation</u> of G is a subset G_1^+ of G_1 such that

$$(G_1^+)^0 \cap G_1 = \emptyset \quad \text{and} \quad (G_1^+)^0 \cup G_1^+ = G_1.$$

The pair (G_1^+, G_0) together with the function

$$\begin{aligned} G_1^+ &\longrightarrow G_0 \times G_0 \\ \gamma &\longmapsto (\gamma(0), \gamma(1)) \end{aligned}$$

constitute an <u>oriented graph</u>.

Each topological graph Γ defines a combinatorial graph (Γ_1, Γ_0) by taking the vertices to be the 0-cells and the edges to be the <u>oriented</u> open 1-cells of Γ. To each orientation of a cell e corresponds a linear order on $\hat{e}$, or equivalently, a linear order on $\partial\hat{e}$: the source of an oriented cell e is the image of the first element of $\partial\hat{e}$ by the structural map $\hat{e} \to \Gamma$; the target is the image of the last element; the opposite edge is obtained by reversing orientation.

A continuous real-valued function f defined on an open interval is said to be <u>regular</u> at a point r in its domain if there exists $\varepsilon > 0$ such that $f(x) - f(r)$ is non-zero yet changes sign on the set $\{x : 0 < |x - r| < \varepsilon\}$.

Definition 1. Let $a < b$ be real numbers. A planar graph (between slices a and b) is a finite topological graph Γ embedded in $[a,b] \times \mathbb{R}$ such that every $x \in \Gamma \cap (\{a,b\} \times \mathbb{R})$ is in Γ_0 and belongs to the closure of exactly one 1-cell of Γ.//

A node $x \in \Gamma_0$ is called inner when $x \in (a,b) \times \mathbb{R}$; otherwise it is called outer. We shall say that a vertical line $\{u\} \times \mathbb{R}$ crosses an open 1-cell e transversally when, for all $r \in e \cap (\{u\} \times \mathbb{R})$, the restriction of the first projection

$$p_1 : [a,b] \times \mathbb{R} \longrightarrow [a,b]$$

is regular at r.

A number $u \in [a,b]$ is called a regular slice if the vertical line $\{u\} \times \mathbb{R}$ contains no inner nodes and transversally crosses the open 1-cells of Γ. If $c < d$ are regular slices of Γ, we write $\Gamma[c,d]$ for the graph $\Gamma \cap ([c,d] \times \mathbb{R})$ whose set of nodes is the union of $\Gamma_0 \cap ([c,d] \times \mathbb{R})$ and $\Gamma \cap (\{c,d\} \times \mathbb{R})$. The graph $\Gamma[c,d]$ is a planar graph between slices c and d; it is called a strip of Γ.

For any number $u \in [a,b]$, write $u \cap \Gamma$ for the intersection of the vertical line $\{u\} \times \mathbb{R}$ and Γ. As a subset of the line $\{u\} \times \mathbb{R}$ ($\simeq \mathbb{R}$), the set $u \cap \Gamma$ inherits a linear order. Suppose Γ is the disjoint union of two subgraphs Γ^1 and Γ^2. We shall say that the pair (Γ^1, Γ^2) is a <u>vertical decompostion</u> of Γ, and write $\Gamma = \Gamma^1 \otimes \Gamma^2$, when the ordered set $u \cap \Gamma^1$ is a final segment of the ordered set $u \cap \Gamma$ for all $u \in [a,b]$. This extends in the obvious way to n-fold decompositions
$\Gamma = \Gamma^1 \otimes \cdots \otimes \Gamma^n$.

We now make precise the kind of deformation we shall consider. Let Γ be a topological graph and T be a (path) connected space.

<u>Definition 2.</u> <u>A deformation of planar graphs</u> (between slices a and b) is a continuous function

$$h : \Gamma \times T \longrightarrow [a,b] \times \mathbb{R}$$

such that

(i) for all $t \in T$, the function

$$h(-,t) : \Gamma \longrightarrow [a,b] \times \mathbb{R}$$

is an embedding whose image is a planar graph $\Gamma(t)$ between slices a and b,

(ii) for all $x \in \Gamma_0$, if $h(x,t)$ is inner for one value of t then it is inner for all values of $t \in T$. //

In view of the isomorphism $\Gamma \simeq \Gamma(t)$ of (i) above, it is possible to define many structures on Γ by transporting the structures defined on the $\Gamma(t)$ for some $t \in T$. Some of these structures on Γ are independent of the choice of $t \in T$. For example, our definition of deformation forces the inner and outer nodes to keep their natures independent of $t \in T$. Another example of invariant structure is the circular order that can be defined on the set of edges $\{\gamma \in \Gamma_1 : \gamma(0) = x\}$ for any inner node $x \in \Gamma_0$; the definition of the order uses the standard orientation of $\mathbb{R}^2$.

§3. Recumbent diagrams

Definition 3. A planar graph Γ between slices a and b is recumbent when the first projection

$$p_1 : [a,b] \times \mathbb{R} \longrightarrow [a,b]$$

is injective on each 1-cell of Γ.

For a recumbent graph Γ, there is an orientation defined by the set

$$\Gamma_1^+ = \{ \gamma \in \Gamma_1 : p_1 \gamma(0) < p_1 \gamma(1) \}.$$

For each inner node $x \in \Gamma_0$, we distinguish the two sets of edges:

$$in(x) = \{ \gamma \in \Gamma_1^+ : \gamma(1) = x \},$$

$$out(x) = \{ \gamma \in \Gamma_1^+ : \gamma(0) = x \},$$

called the <u>input</u> and <u>output</u> of x, respectively. These sets are linearly ordered as follows. If $u \in [a,b]$ is chosen smaller than but close enough to $p_1(x)$, each edge $\gamma \in in(x)$ intersects the line $\{u\} \times \mathbb{R}$ in one point which is different for different edges. This defines a bijection between $in(x)$ and a subset of $\{u\} \times \mathbb{R}$ ($\simeq \mathbb{R}$) and so induces a linear order on $in(x)$. The order on $out(x)$ is defined similarly by intersecting with $\{u\} \times \mathbb{R}$ slicing at u larger than but close enough to $p_1(x)$.

<u>Definition 4.</u> A <u>valuation</u> $v : \Gamma \longrightarrow \mathbb{V}$ of a recumbent graph Γ in a tensor category $\mathbb{V}$ is a pair

of functions

$$v_0 : \Gamma_1^+ \longrightarrow ob(\mathbb{V}), \qquad v_1 : \Gamma_0' \longrightarrow mor(\mathbb{V})$$

(where Γ_0' is the set of inner nodes of Γ) such that, for all $x \in \Gamma_0'$,

$$v_1(x) : v_0(\gamma_1) \otimes \cdots \otimes v_0(\gamma_m) \longrightarrow v_0(\delta_1) \otimes \cdots \otimes v_0(\delta_n)$$

where $\gamma_m < \cdots < \gamma_1$, $\delta_n < \cdots < \delta_1$ are the ordered lists of elements of $in(x)$, $out(x)$, respectively. The pair (Γ, v) is called a <u>(recumbent) diagram</u>, and is denoted merely by Γ when the context is clear. //

Each diagram (Γ, v) has a <u>value</u> $v(\Gamma) \in mor(\mathbb{V})$ associated with it in the following manner. First, for any point $p \in \Gamma$, we define a morphism $v(p)$ of $\mathbb{V}$ as follows:

$$v(p) = \begin{cases} v_1(p) & \text{if } p \text{ is an inner node,} \\ 1_A & \text{otherwise,} \end{cases}$$

where $A = v_0(e)$ and e is the unique cell whose closure contains p. Second, for any $u \in [a, b]$, define

$$v(u \cap \Gamma) = v(p_1) \otimes \cdots \otimes v(p_r)$$

where $p_r < \cdots < p_1$ are the elements of the ordered set $u \cap \Gamma$. Finally, we put

$$v(\Gamma) = v(u_1 \cap \Gamma) \circ \cdots \circ v(u_n \cap \Gamma)$$

where $u_n < \cdots < u_1$ is a non-empty subset of

$[a,b]$ including all the critical (=non-regular) slices of Γ. That $v(\Gamma)$ is well defined follows from the two observations:

(i) if $u < u'$ and the interval (u, u') contains no critical slices then the domain of $v(u' \cap \Gamma)$ is equal to the codomain of $v(u \cap \Gamma)$,

(ii) $v(u \cap \Gamma)$ is an identity morphism for any regular slice u.

Let $c < d$ be regular slices of a recumbent diagram $\Gamma = (\Gamma, v)$. The valuation v "restricts" in an obvious way to a valuation on the strip $\Gamma[c,d]$ and we also denote this by v. Similarly, if $\Gamma = \Gamma^1 \otimes \Gamma^2$, the valuation v restricts to valuations on Γ^1 and Γ^2 again denoted by v.

<u>Proposition 1</u>. *Let Γ be a recumbent diagram between slices a and b. If $a = a_0 < \dots < a_n = b$ are regular slices for Γ then*

$$v(\Gamma) = v(\Gamma[a_{n-1}, a_n]) \circ \dots \circ v(\Gamma[a_0, a_1]).$$

<u>Proof</u>. This is a direct consequence of the definitions. $\square$

Proposition 2. *If* $\Gamma = \Gamma^1 \otimes \cdots \otimes \Gamma^n$ *then*

$$v(\Gamma) = v(\Gamma^1) \otimes \cdots \otimes v(\Gamma^n).$$

Proof. It suffices to prove the case $n=2$. Let $u_1 < \cdots < u_r$ be a non-empty sequence in $[a,b]$ containing all the critical slices of Γ. Put $f_i = v(u_i \cap \Gamma^1)$, $g_i = v(u_i \cap \Gamma^2)$. By definition, $v(u_i \cap \Gamma) = f_i \otimes g_i$. Using the functoriality of the tensor product we obtain

$$\begin{aligned} v(\Gamma) &= (f_r \otimes g_r) \circ \cdots \circ (f_1 \otimes g_1) \\ &= (f_r \circ \cdots \circ f_1) \otimes (g_r \circ \cdots \circ g_1) \\ &= v(\Gamma^1) \otimes v(\Gamma^2). \quad \square \end{aligned}$$

Definition 5. A *deformation of recumbent graphs* is a deformation $h : \Gamma \times T \longrightarrow [a,b] \times \mathbb{R}$ of planar graphs (see Definition 2) such that the image $\Gamma(t)$ of $h(-,t)$ is recumbent for all $t \in T$. //

For any inner node $x \in \Gamma_0$ we can define $in(x)$, $out(x)$ as the sets of edges corresponding to $in(h(x,t))$, $out(h(x,t))$ via the isomorphism $\Gamma \cong \Gamma(t)$. It is easy to see that we have isomorphisms

$$\mathrm{in}(x) \simeq \mathrm{in}(h(x,t)), \quad \mathrm{out}(x) \simeq \mathrm{out}(h(x,t)),$$

and that these ordered sets do not depend on the choice of $t \in T$. Moreover, a valuation defined on one $\Gamma(t_0)$ for some $t_0 \in T$ can be transported along the isomorphisms $\Gamma(t_0) \simeq \Gamma \simeq \Gamma(t)$ to a valuation on $\Gamma(t)$ for all $t \in T$. In this way h becomes <u>a deformation of diagrams</u>.

<u>Theorem 3</u>. <u>If</u> $h : \Gamma \times T \longrightarrow [a,b] \times \mathbb{R}$ <u>is a deformation of recumbent diagrams then the value</u> $v(\Gamma(t))$ <u>is independent of</u> $t \in T$.

<u>Proof</u>. Since T is connected, it suffices to show that $v(\Gamma(t))$ is a locally constant function of t. Take $t_0 \in T$. We shall prove that $v(\Gamma(t)) = v(\Gamma(t_0))$ for t close enough to t_0. Let $a = a_0 < a_1 < \ldots < a_n = a$ be regular slices for $\Gamma(t_0)$ such that each interval $[a_i, a_{i+1}]$ contains at most one critical slice for $\Gamma(t_0)$. It t is close enough to t_0, the numbers a_i are all regular slices for $\Gamma(t)$ and we have (Proposition 1)

$$v(\Gamma(t)) = v(\Gamma(t)[a_n, a_{n-1}]) \circ \ldots \circ v(\Gamma(t)[a_1, a_0]).$$

It remains to prove that, for t close enough to t_0, we have

$$v(\Gamma(t)[c,d]) = v(\Gamma(t_0)[c,d])$$

where $c = a_i$, $d = a_{i+1}$. This is immediate when $[c,d]$ contains no critical slices for $\Gamma(t_0)$, since then $\Gamma(t)[c,d]$ has no inner nodes and $v(\Gamma(t)[c,d])$ is an identity arrow for t close enough to t_0. Hence we may suppose that $[c,d]$ contains exactly one critical slice k. Let $p_1 > \cdots > p_r$ be a listing of the ordered set $k \cap \Gamma(t_0)$. Let $p_1(t), \ldots, p_r(t)$ be the image of $p_1, \ldots, p_r$ under the composite isomorphism $\Gamma(t_0) \simeq \Gamma \simeq \Gamma(t)$. It t is close enough to t_0, each $p_i(t)$ belongs to exactly one connected component $\Gamma^i(t)$ of $\Gamma(t)[c,d]$ and we have

$$\Gamma(t)[c,d] = \Gamma^1(t) \otimes \ldots \otimes \Gamma^r(t).$$

By Proposition 2,

$$v(\Gamma(t)[c,d]) = v(\Gamma^1(t)) \otimes \ldots \otimes v(\Gamma^r(t)).$$

It remains to see that $v(\Gamma^i(t)) = v(\Gamma^i(t_0))$ when t is close enough to t_0. This follows from the equality

$$v(\Gamma^i(t)) = v(p_i(t)) = v(p_i(t_0)) = v(\Gamma^i(t_0)). \qquad \square$$

§4. Tensor functors and Tensor schemes.

Let $\mathbb{V}$, $\mathbb{W}$ be tensor categories. Let us recall the concept [ ] of tensor functor $F = (F, \varphi^2, \varphi^0) : \mathbb{V} \longrightarrow \mathbb{W}$, also called "strong monoidal functor". It consists of a functor $F : \mathbb{V} \longrightarrow \mathbb{W}$, a natural isomorphism

$$\varphi^2 = \varphi^2_{A,B} : FA \otimes FB \longrightarrow F(A \otimes B),$$

and an isomorphism $\varphi^0 : I \longrightarrow FI$, such that the following diagrams commute:

$$\begin{tikzcd}
FA \otimes FB \otimes FC \arrow[r, "\varphi^2 \otimes 1"] \arrow[d, "1 \otimes \varphi^2"'] & F(A \otimes B) \otimes FC \arrow[d, "\varphi^2"] \\
FA \otimes F(B \otimes C) \arrow[r, "\varphi^2"'] & F(A \otimes B \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
 & FA \arrow[dl, "F(\varphi^0 \otimes 1)"'] \arrow[dr, "\varphi^0 \otimes 1"] & \\
F(I \otimes A) \arrow[rr, "\varphi^2"'] & & FI \otimes FA
\end{tikzcd}
\qquad
\begin{tikzcd}
 & FA \arrow[dl, "F(1 \otimes \varphi^0)"'] \arrow[dr, "1 \otimes \varphi^0"] & \\
F(A \otimes I) \arrow[rr, "\varphi^2"'] & & FA \otimes FI.
\end{tikzcd}$$

We can define natural isomorphisms

$$\varphi^n : FA_1 \otimes \cdots \otimes FA_n \longrightarrow F(A_1 \otimes \cdots \otimes A_n)$$

inductively as follows: φ^0 is given, $\varphi^1 = 1_{FA}$, φ^2 is given, and φ^{n+1} is the composite

$$FA_1 \otimes \cdots \otimes FA_{n+1} \xrightarrow{1 \otimes \varphi^n} FA_1 \otimes F(A_2 \otimes \cdots \otimes A_{n+1}) \xrightarrow{\varphi^2} F(A_1 \otimes \cdots \otimes A_{n+1}).$$

It follows (by induction) that the following triangle commutes.

$$\begin{tikzcd}
FA_1 \otimes \cdots \otimes FA_m \otimes FB_1 \otimes \cdots \otimes FB_n \arrow[rr, "\varphi^{m+n}"] \arrow[dr, "\varphi^m \otimes \varphi^n"'] & & F(A_1 \otimes \cdots \otimes A_m \otimes B_1 \otimes \cdots \otimes B_n) \\
& F(A_1 \otimes \cdots \otimes A_m) \otimes F(B_1 \otimes \cdots \otimes B_n) \arrow[ur, "\varphi^2"'] &
\end{tikzcd}$$

Composition of tensor functors $\mathbb{U} \to \mathbb{V}$, $\mathbb{V} \to \mathbb{W}$ is defined in the obvious manner [,].

Suppose $F, G : \mathbb{V} \to \mathbb{W}$ are tensor functors. A <u>tensor transformation</u> $\alpha : F \to G$ is a natural transformation α such that the following diagrams commute.

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\varphi^2"] \arrow[d, "\alpha_A \otimes \alpha_B"'] & F(A \otimes B) \arrow[d, "\alpha_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "\varphi^2"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
& FI \arrow[d, "\alpha_I"] \\
I \arrow[ur, "\varphi^0"] \arrow[r, "\varphi^0"'] & GI
\end{tikzcd}$$

We write $\mathrm{Ten}(\mathbb{V}, \mathbb{W})$ for the category of tensor functors $\mathbb{V} \to \mathbb{W}$ and tensor transformations between them.

The following concept is in the literature under various names and various levels of generality [ ].

<u>Definition 6.</u> A <u>tensor scheme</u> $\mathbb{D}$ consists of two sets $\mathrm{ob}\,\mathbb{D}$ and $\mathrm{mor}\,\mathbb{D}$ together with a function which assigns to each element $d \in \mathrm{mor}\,\mathbb{D}$ a pair $(d(0), d(1))$ of words in the elements of $\mathrm{ob}\,\mathbb{D}$. Write

$$d : X_1 \ldots X_m \longrightarrow Y_1 \ldots Y_n$$

for $d \in \mathrm{mor}\,\mathbb{D}$ with $d(0) = X_1 \ldots X_m$, $d(1) = Y_1 \ldots Y_n$. //

For a tensor scheme $\mathbb{D}$ and a tensor category $\mathbb{V}$, there is a category $[\mathbb{D}, \mathbb{V}]$ described as follows. An object K is a pair of functions

$$K : \mathrm{ob}\,\mathbb{D} \longrightarrow \mathrm{ob}\,\mathbb{V}, \qquad K : \mathrm{mor}\,\mathbb{D} \longrightarrow \mathrm{mor}\,\mathbb{V}$$

such that, for all $d : X_1 \ldots X_m \longrightarrow Y_1 \ldots Y_n$, we have

$$Kd : KX_1 \otimes \ldots \otimes KX_m \longrightarrow KY_1 \otimes \ldots \otimes KY_n .$$

A morphism $\theta : K \longrightarrow L$ is a family of morphisms

$$\theta_X : KX \longrightarrow LX \ , \quad X \in \operatorname{ob} \mathbb{D},$$

in $\mathbb{V}$ such that, for all d (as before) the following diagram commutes.

$$\begin{tikzcd}
KX_1 \otimes \cdots \otimes KX_m \arrow[r, "Kd"] \arrow[d, "\theta_{X_1} \otimes \cdots \otimes \theta_{X_m}"'] & KY_1 \otimes \cdots \otimes KY_n \arrow[d, "\theta_{Y_1} \otimes \cdots \otimes \theta_{Y_n}"] \\
LX_1 \otimes \cdots \otimes LX_m \arrow[r, "Ld"'] & LY_1 \otimes \cdots \otimes LY_n
\end{tikzcd}$$

There is a "composition" functor

$$\mathrm{Ten}(\mathbb{V}, \mathbb{W}) \times [\mathbb{D}, \mathbb{V}] \longrightarrow [\mathbb{D}, \mathbb{W}]$$

$$(F, K) \longmapsto F \circ K, \qquad (\alpha, \theta) \longmapsto \alpha \circ \theta$$

where $(F \circ K)X = FKX$, $(F \circ K)d$ is the composite

$$FKX_1 \otimes \cdots \otimes FKX_m \xrightarrow{\varphi^m} F(KX_1 \otimes \cdots \otimes KX_m) \xrightarrow{FKd} F(KY_1 \otimes \cdots \otimes FY_n) \xrightarrow{(\varphi^n)^{-1}}$$

$$FKY_1 \otimes \cdots \otimes FKY_n,$$

and $(\alpha \circ \theta)_X = \alpha_{KX} \circ G\theta_X$.

<u>Definition 7.</u> <u>A free tensor category on a tensor scheme</u> $\mathbb{D}$ is a tensor category $\mathbb{F}$ together with an object N of $[\mathbb{D}, \mathbb{F}]$ such that, for all tensor categories $\mathbb{V}$, the functor

$$-\circ N : \mathrm{Ten}(\mathbb{F}, \mathbb{V}) \longrightarrow [\mathbb{D}, \mathbb{V}]$$

is an equivalence of categories.//

If $\mathbb{F}', N'$ constitute another free tensor category on $\mathbb{D}$ then there exists an equivalence $E : \mathbb{F} \to \mathbb{F}'$ of tensor categories with an isomorphism $E \circ N \cong N'$ in $[\mathbb{D}, \mathbb{F}]$. The existence of free tensor categories can be proved algebraically; we shall give a topological construction in terms of planar graphs.

A planar graph Γ is said to be <u>centred</u> when it is between slices -1 and $+1$ and is contained in $[-1,1] \times (-1,1)$. The <u>domain</u> m (resp. <u>codomain</u> n) of Γ is the number of outer nodes on the slice -1 (resp. $+1$). Write $\Gamma : m \to n$.

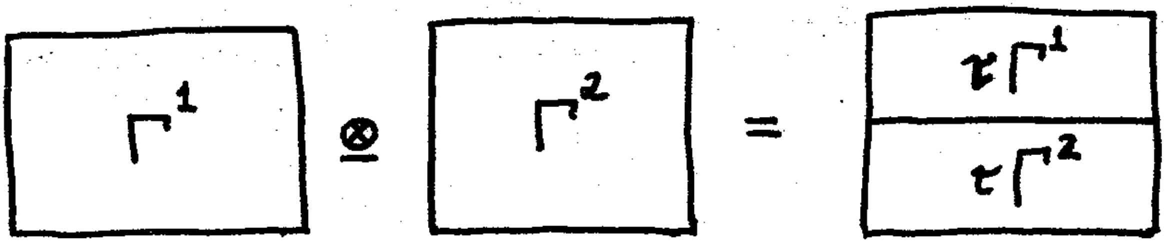
We shall require the functions $\gamma, \tau : \mathbb{R}^2 \to \mathbb{R}^2$ which take (x, y) to $(\frac{1}{3}x, y), (x, \frac{1}{2}y)$, respectively. For a subset $S \subset \mathbb{R}^2$, the notation $\gamma(S + e_1)$ for example will denote the set

$$\{ (\tfrac{1}{3}(x+1), y) \in \mathbb{R}^2 \mid (x, y) \in S \}.$$

(Here e_1, e_2 are the points $(1,0), (0,1) \in \mathbb{R}^2$.)

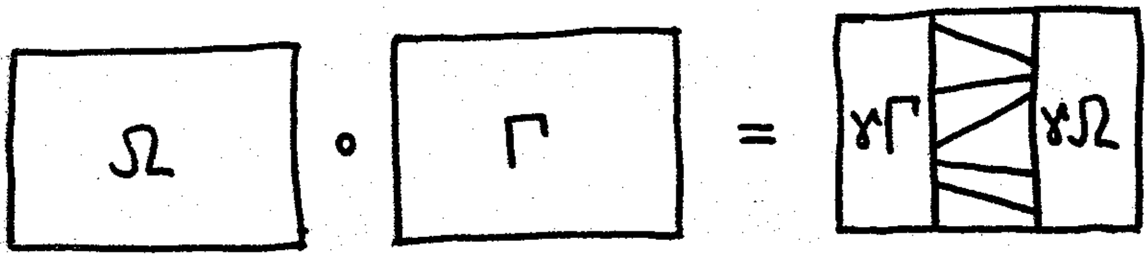
The <u>tensor product</u> $\Gamma^1 \otimes \Gamma^2$ of two centred planar graphs Γ^1, Γ^2 is defined by the equation

$$\Gamma^1 \otimes \Gamma^2 = \tau\big((\Gamma^1 + e_2) \cup (\Gamma^2 - e_2)\big).$$



The <u>composite</u> $\Omega\circ\Gamma : m \longrightarrow p$ of centred planar graphs $\Gamma : m \longrightarrow n$, $\Omega : n \longrightarrow p$ is defined as follows: let $a_1 > a_2 > \dots > a_n$ be the outer nodes of Γ on slice $+1$ and let $b_1 > b_2 > \dots > b_n$ be the outer nodes of Ω on slice -1, then

$$\Omega\circ\Gamma = \gamma\Big(\big((\Gamma - 2e_1)\cup[a_1, b_1]\cup\dots\cup[a_n, b_n]\cup(\Omega + 2e_1)\big)\Big).$$



<u>A deformation of centred planar graphs</u> is a deformation $h : \Gamma\times T \longrightarrow [-1,1]^2$ of planar graphs such that each $\Gamma(t)$ is centred. Notice that the domain and codomain of $\Gamma(t)$ are independent of t. Two centred planar graphs Γ^1, Γ^2 are said to be <u>isotopic</u>, denoted $\Gamma^1 \sim \Gamma^2$, when there exists such an h with $\Gamma^1 = \Gamma(t_1)$, $\Gamma^2 = \Gamma(t_2)$ for some $t_1, t_2 \in T$.

<u>Proposition 4</u>. <u>For centred planar graphs</u>:

(a) <u>if</u> $\Gamma^1 \sim \Gamma^2$ <u>and</u> $\Omega^1 \sim \Omega^2$ <u>then</u> $\Gamma^1 \underline{\otimes} \Omega^1 \sim \Gamma^2 \underline{\otimes} \Omega^2$ <u>and, provided the composites are defined,</u> $\Omega^1 \circ \Gamma^1 \sim \Omega^2 \circ \Gamma^2$;

(b) $(\Gamma^1 \underline{\otimes} \Gamma^2) \underline{\otimes} \Gamma^3 \sim \Gamma^1 \underline{\otimes} (\Gamma^2 \underline{\otimes} \Gamma^3)$ <u>and, provided the composites are defined,</u> $(\Lambda \circ \Omega) \circ \Gamma \sim \Lambda \circ (\Omega \circ \Gamma)$;

(c) <u>provided the composites are defined,</u>

$$(\Omega^1 \circ \Gamma^1) \underline{\otimes} (\Omega^2 \circ \Gamma^2) = (\Omega^1 \underline{\otimes} \Omega^2) \circ (\Gamma^1 \underline{\otimes} \Gamma^2).$$

<u>Proof.</u> (a) Given deformations $h_i : \Gamma^i \times T_i \longrightarrow [-1,1]^2$ of centred planar graphs for $i = 1, 2$, let $\Gamma^1 + \Gamma^2$ be the topological sum of Γ^1, Γ^2 and define

$$h_1 \underline{\otimes} h_2 : (\Gamma^1 + \Gamma^2) \times T_1 \times T_2 \longrightarrow [-1,1]^2$$

by

$$(h_1 \underline{\otimes} h_2)(x, t_1, t_2) = \begin{cases} \tau(h_1(x, t_1) + e_2) & \text{for } x \in \Gamma^1 \\ \tau(h_2(x, t_2) - e_2) & \text{for } x \in \Gamma^2, \end{cases}$$

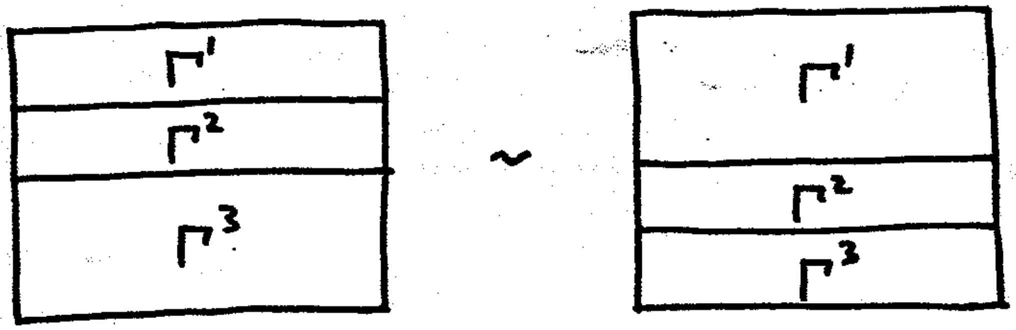
which is a deformation with $(\Gamma^1 + \Gamma^2)(t_1, t_2) = \Gamma^1(t_1) \underline{\otimes} \Gamma^2(t_2)$. This proves the first part of (a).

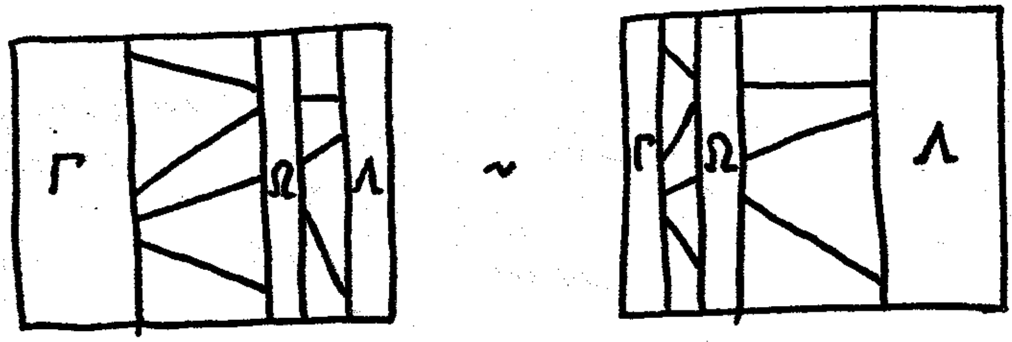
With h_1, h_2 as above, suppose the codomain of each $\Gamma^1(t_1)$ and the domain of each $\Gamma^2(t_2)$ is n. Let $a_1, \ldots, a_n \in \Gamma^1$ be the points corresponding under $\Gamma^1 \simeq \Gamma^1(t_1)$ to the outer nodes $a_1(t_1) > \ldots > a_n(t_1)$ of $\Gamma^1(t_1)$ on the slice $+1$. Similarly, let $b_1, \ldots, b_n \in \Gamma^2$ correspond to the outer nodes $b_1(t_2) > \ldots > b_n(t_2)$ of $\Gamma^2(t_2)$ on the slice -1. Let Ω be the quotient space of the topological sum $\Gamma^1 + (\{1, \ldots, n\} \times [0,1]) + \Gamma^2$ under the identifications $a_i = (i, 0)$, $(i, 1) = b_i$ for $i = 1, \ldots, n$. Define a deformation $h_2 \circ h_1 : \Omega \times T_1 \times T_2 \to [-1,1]^2$ by putting

$$(h_2 \circ h_1)(x, t_1, t_2) = \begin{cases} \gamma(h_1(x, t_1) - 2e_1) & \text{for } x \in \Gamma^1 \\ \gamma(h_2(x, t_2) + 2e_1) & \text{for } x \in \Gamma^2 \\ (1-\lambda)a_i(t_1) + \lambda b_i(t_2) & \text{for } x = (i, \lambda). \end{cases}$$

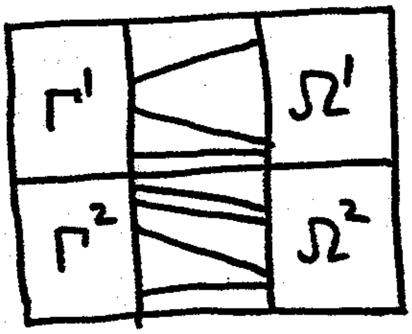
Then $\Omega(t_1, t_2) = \Gamma^2(t_2) \circ \Gamma^1(t_1)$

(b) We leave this to the reader. The deformations are constructed in much the same way as in the proof that multiplication in the fundamental group of a space is associative. The following diagrams indicate the constructions; the regions contain graphs similar to those shown therein.





(c) This is easy; both sides are indicated by the diagram below.



□

The concept of valuation of a recumbent graph in a tensor category $\mathbb{V}$ (Definition 1.4) did not involve composition in $\mathbb{V}$ and did not involve tensor products of morphisms (these were used, of course, in calculating the value). We therefore define a valuation $v : \Gamma \to \mathbb{D}$ of a recumbent graph Γ in a tensor scheme $\mathbb{D}$

just as for $\mathbb{V}$ except that the tensor products in the domain and codomain of $v_1(x)$ are replaced by words in the elements of ob $\mathbb{D}$. Thus we also have the notion of a <u>recumbent diagram</u> (Γ, v) <u>in a tensor scheme</u>. The <u>domain</u> (resp. <u>codomain</u>) of the diagram (Γ, v) is defined to be the word $v_0(e_1) \dots v_0(e_m)$ in elements of ob $\mathbb{D}$ where $e_1, \dots, e_m$ are the unique cells of Γ whose closures contain the outer nodes $x_1 > \dots > x_m$, in order, on the left (resp. right) bounding slice of Γ.

It is now clear how to define the <u>tensor product and composite of centred recumbent diagrams in a tensor scheme</u>. The tensor product $\Gamma^1 \underline{\otimes} \Gamma^2$ has a vertical decomposition into subgraphs isomorphic to Γ^1, Γ^2; so valuations on Γ^1, Γ^2 transport to the subgraphs and these together give a valuation on Γ^1, Γ^2. If the codomain of (Γ, v) agrees with the domain of (Ω, w) then there is a unique valuation on $\Omega \circ \Gamma$ whose restriction to $(\Omega \circ \Gamma)[-1, -1/3]$ transports to v under the canonical isomorphism with Γ and whose restriction to $(\Omega \circ \Gamma)[\frac{1}{3}, 1]$ transports to w under the canonical isomorphism with Ω

(note that $(\Omega \circ \Gamma)[-1/3, 1/3]$ has no inner nodes).

It is also clear how to define the concept of deformation of centred recumbent diagrams in a tensor scheme, and what it means for two centred recumbent diagrams to be isotopic. These are just as for the underlying planar graphs subject to compatibility with the valuations.

Given a tensor scheme $\mathbb{D}$, there is a strict tensor category $\mathcal{F}(\mathbb{D})$ defined as follows. The objects of $\mathcal{F}(\mathbb{D})$ are words in the elements of ob $\mathbb{D}$. The morphisms of $\mathcal{F}(\mathbb{D})$ are isotopy classes of centred recumbent diagrams in $\mathbb{D}$. The domain, codomain and composition of morphisms are induced on isotopy classes by the corresponding operations for centred recumbent diagrams. Identity morphisms are isotopy classes of diagrams whose underlying graphs have no inner nodes. The tensor product is given on objects by juxtaposition of words and on morphisms is induced by the tensor product of centred recumbent diagrams. That this gives a strict tensor category follows from Proposition 1.4.

There is an object N of the category $[\mathbb{D}, \mathcal{F}(\mathbb{D})]$ defined as follows. For $X \in \mathrm{ob}\,\mathbb{D}$, let $NX = X$ as a one-letter word. For $d : X_1 \cdots X_m \to Y_1 \cdots Y_n$ in $\mathrm{mor}\,\mathbb{D}$, take $Nd : X_1 \cdots X_m \to Y_1 \cdots Y_n$ to be the isotopy class of the diagram (Γ, v) where Γ is the union of the line segments

$$[(-1, 1 - \tfrac{2h-1}{m}), (0,0)], \quad [(0,0), (1, 1 - \tfrac{2k-1}{n})]$$

for $1 \leq h \leq m$, $1 \leq k \leq n$, in the plane $\mathbb{R}^2$ with the origin $(0,0)$ as the only inner node, where v_0 takes the line segments to X_h, Y_k, respectively, and where $v_1(0,0) = d$.

Theorem 5. The free tensor category on a tensor scheme $\mathbb{D}$ is given by $\mathcal{F}(\mathbb{D})$ together with N.

Proof. Without loss of generality we may suppose $\mathbb{V}$ to be a strict ~~tensor~~ category and prove that

$$- \circ N : \mathrm{Ten}(\mathcal{F}(\mathbb{D}), \mathbb{V}) \to [\mathbb{D}, \mathbb{V}]$$

is an equivalence of categories. In this case we shall show that $- \circ N$ is surjective on objects and fully faithful. Take any object K of $[\mathbb{D}, \mathbb{V}]$. In fact we shall produce the unique strict tensor functor $T : \mathcal{F}(\mathbb{D}) \longrightarrow \mathbb{V}$ with $T \circ N = K$. Since ob $\mathcal{F}(\mathbb{D})$ is the free monoid on ob $\mathbb{D}$, certainly T is uniquely determined on objects if it is to preserve tensor. Let (Γ, v) be a centred recumbent diagram in $\mathbb{D}$. Then (Γ, Kv) is a recumbent diagram in $\mathbb{V}$. Define T to take the isotopy class of (Γ, v) to the value $(Kv)(\Gamma)$ of (Γ, Kv). This is well defined by Theorem 1.3; it preserves domain and codomain; it preserves composition by Proposition 1.1; and, it preserves tensor product by Proposition 1.2. Moreover, $T \circ N = K$. Uniqueness of T follows from the fact that each (Γ, v) can be built up under composition and tensor product from diagrams with at most one inner node (see the Proof of Theorem 1.3).

Suppose $F, G : \mathcal{F}(\mathbb{D}) \longrightarrow \mathbb{V}$ are arbitrary tensor functors and suppose $\theta : F \circ N \longrightarrow G \circ N$ is a morphism of $[\mathbb{D}, \mathbb{V}]$. If we are to have $\alpha : F \longrightarrow G$ with $\alpha \circ N = \theta$, we are forced to define $\alpha_{X_1 \ldots X_m} : F(X_1 \ldots X_m) \longrightarrow G(X_1 \ldots X_m)$ to be

$$\varphi^m \circ (\theta_{X_1} \otimes \cdots \otimes \theta_{X_m}) \circ (\varphi^m)^{-1}.$$

Compatibility of α with φ^n is automatic. Naturality reduces to the case of morphisms represented by diagrams with at most one inner node; for this we invoke the condition satisfied by θ as a morphism of $[\mathbb{D}, \mathbb{V}]$. This proves $-\circ N$ is fully faithful. $\square$

Remark. The above proof shows that $-\circ N$ gives an isomorphism of categories between the full subcategory of $\mathrm{Ten}(\mathcal{F}(\mathbb{D}), \mathbb{V})$ consisting of the strict tensor functors (that is, those with φ^0, φ^2 identities) and the category $[\mathbb{D}, \mathbb{V}]$. This is an expected consequence of the "flexibility" of the tensor category concept.

CHAPTER 2

DUALITY AND POLARISED DIAGRAMS

§1. Duality

Let us recall the basic concepts of duality theory in the context of a general tensor category $\mathbb{V}$.

A *pairing* between two objects A and B of $\mathbb{V}$ is a map $\varepsilon : A \otimes B \longrightarrow I$. For any objects X, Y and map $f : X \longrightarrow B \otimes Y$, let $\varepsilon^{\#}(f)$ denote the composite map

$$A \otimes X \xrightarrow{1 \otimes f} A \otimes B \otimes Y \xrightarrow{\varepsilon \otimes 1} Y.$$

We also point out that $\varepsilon^{\#}(f)$ is the value of the diagram in Fig. 3.

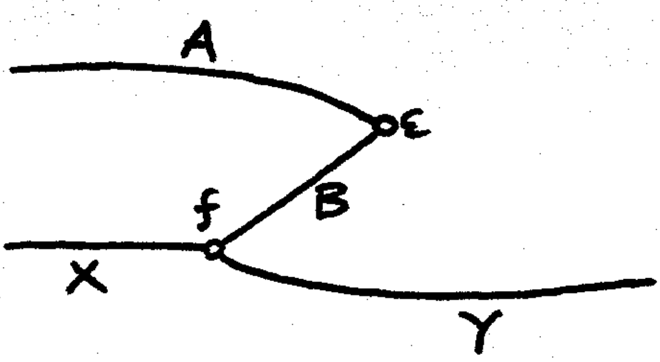


Fig. 3

The pairing ε is *exact* when the function

$$\varepsilon^{\#} : \mathrm{Hom}(X, B \otimes Y) \longrightarrow \mathrm{Hom}(A \otimes X, Y)$$

is bijective for all $X, Y \in \mathbb{V}$. This condition is equivalent to the existence of an arrow $\eta : I \to B \otimes A$ such that the following two triangles commute.

$$\begin{tikzcd} A \arrow[rr, "1\otimes\eta"] \arrow[dr, "1"'] & & A\otimes B\otimes A \arrow[dl, "\varepsilon\otimes 1"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "\eta\otimes 1"] \arrow[dr, "1"'] & & B\otimes A\otimes B \arrow[dl, "1\otimes\varepsilon"] \\ & B & \end{tikzcd}$$

Note that commutativity of the above triangles means that we have the equalities (of values) of the diagrams depicted in Fig. 4 and Fig. 5.

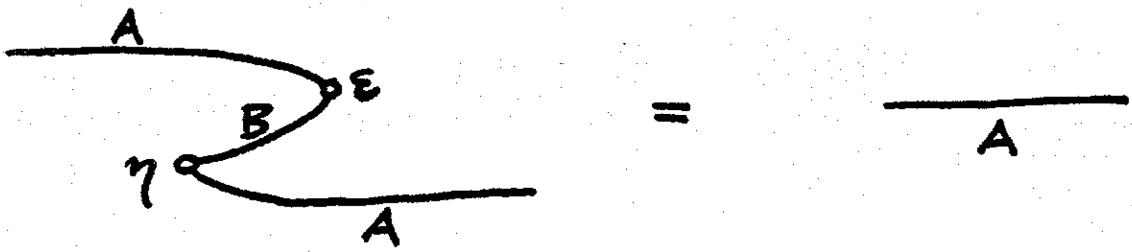


Fig. 4

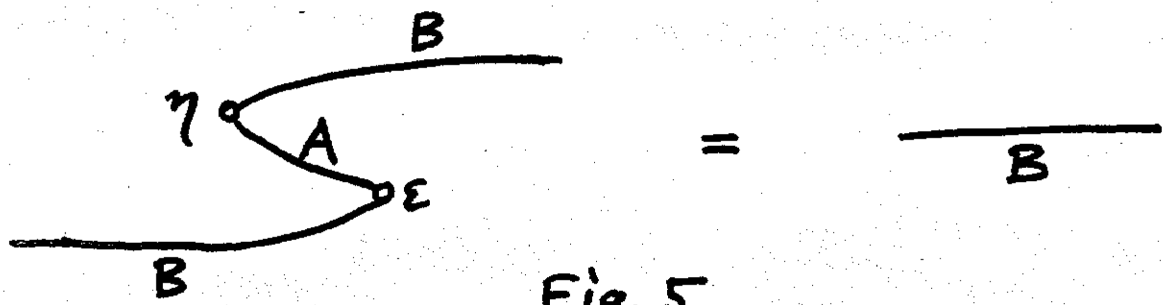


Fig. 5

Clearly, if η exists the function

$$g \longmapsto (1 \otimes g) \circ (\eta \otimes 1)$$

is the inverse of $\varepsilon^{\#}$. Conversely, let us assume $\varepsilon^{\#}$ is bijective. Then there is a map $\eta : I \to B \otimes A$ such that $\varepsilon^{\#}(\eta) = 1_A$; so the first triangle commutes. To see that the second triangle commutes, it suffices to show that the two legs (regarded as maps $B \longrightarrow B \otimes I$) are equal after applying $\varepsilon^{\#}$. This amounts to the equality depicted in Fig. 6;

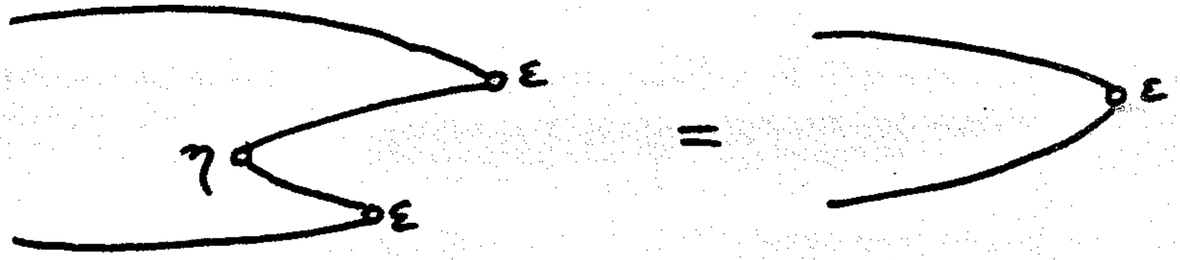


Fig. 6

but this follows from Theorem 1.3 and Fig. 4.

We say that the pair (η, ε) is an <u>adjunction</u> between A and B, and that A (respectively, B) is <u>left adjoint</u> or <u>left dual</u> to B (respectively, <u>right adjoint</u> or <u>right dual</u> to A) and write

$$(\eta, \varepsilon) : A \dashv B .$$

The fact that $\varepsilon^{\#}$ is bijective can be reformulated by saying that the functor $A \otimes - : \mathbb{V} \longrightarrow \mathbb{V}$ is left adjoint to the functor $B \otimes - : \mathbb{V} \longrightarrow \mathbb{V}$. The adjointness $A \dashv B$ between objects ~~[illegible]~~ ~~[illegible]~~ implies an adjointness $- \otimes B \dashv - \otimes A$ between functors. Also, $A \dashv B$ ~~is equivalent to~~ implies an adjointness $A\otimes - \dashv B\otimes -$ between functors (since η is a pairing in $\mathbb{V}^{op}$ with reverse tensor product).

A right adjoint (respectively, left adjoint) is unique up to unique isomorphism: for any two exact pairings $\varepsilon : A\otimes B \to I$, $\varepsilon' : A\otimes B' \to I$, there exists a unique $u : B' \to B$ such that $\varepsilon^*(u) = \varepsilon'$; that is, such that the following triangle commutes.

$$\begin{tikzcd}
A\otimes B' \arrow[rr, "A\otimes u"] \arrow[dr, "\varepsilon'"'] & & A\otimes B \arrow[dl, "\varepsilon"] \\
& I &
\end{tikzcd}$$

When A has a right dual, we see that the family

$$(B \mid \varepsilon : A\otimes B \to I \text{ is exact})$$

has the structure of a clique (§1.1).

Let $\varepsilon_1 : A^*\otimes A \to I$, $\varepsilon_2 : B^*\otimes B \to I$ be two pairings. We say that a map $f^* : B^* \to A^*$ is <u>left adjoint</u> to a map $f : A \to B$ (or that f is <u>right adjoint</u> to f^*), and write $f^* \dashv f$, when

$$\varepsilon_1 \circ (f^*\otimes 1) = \varepsilon_2 \circ (1\otimes f).$$

If ε_1 is exact, the relation $f^* \dashv f$ is equivalent to the commutativity of the square

$$\begin{tikzcd}
B^*\otimes A\otimes A^* \arrow[r, "1\otimes f\otimes 1"] \arrow[d, "1\otimes\eta_1"'] & B^*\otimes B\otimes A^* \arrow[d, "\varepsilon_2\otimes 1"] \\
B^* \arrow[r, "f^*"'] & A^*
\end{tikzcd}$$

as a simple computation shows. In this case f^* is unique. Similarly, if ε_2 is exact, and f^* has a unique right adjoint f.

If $f^* \dashv f$ and $g^* \dashv g$ then $f^* \circ g^* \dashv g\circ f$ whenever the composites $g\circ f$ and $f^*\circ g^*$ make sense.

Suppose that we have two adjunctions

$$(\eta_1, \varepsilon_1) : A^* \dashv A \quad \text{and} \quad (\eta_2, \varepsilon_2) : B^* \dashv B.$$

The <u>mate</u> [ ] of a map $f : X\otimes A \to B\otimes Y$ under these adjunctions is the map $f^M : B^*\otimes X \to Y\otimes A^*$ obtained as the composite

$$B^*\otimes X \xrightarrow{1\otimes 1\otimes \eta_1} B^*\otimes X\otimes A\otimes A^* \xrightarrow{1\otimes f\otimes 1} B^*\otimes B\otimes Y\otimes A^* \xrightarrow{\varepsilon_2\otimes 1\otimes 1} Y\otimes A^*.$$

The function

$$(\ )^M : \mathrm{Hom}(X\otimes A, B\otimes Y) \longrightarrow \mathrm{Hom}(B^*\otimes X, Y\otimes A^*)$$

is a bijection as can be seen from the two adjunctions $-\otimes A \dashv -\otimes A^*$, $B^*\otimes - \dashv B\otimes -$ of functors. It is easy to see that f^M can be characterized as the only map $B^* \otimes X \to Y \otimes A^*$ rendering either of the following two squares commutative.

$$
\begin{tikzcd}
X \arrow[r, "\eta_2 \otimes 1"] \arrow[d, "1 \otimes \eta_1"'] & B \otimes B^* \otimes X \arrow[d, "1 \otimes f^M"] \\
X \otimes A \otimes A^* \arrow[r, "f \otimes 1"'] & B \otimes Y \otimes A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
B^* \otimes X \otimes A \arrow[r, "f^M \otimes 1"] \arrow[d, "1 \otimes f"'] & Y \otimes A^* \otimes A \arrow[d, "1 \otimes \varepsilon_1"] \\
B^* \otimes B \otimes Y \arrow[r, "\varepsilon_2 \otimes 1"'] & Y
\end{tikzcd}
$$

We have the special cases: $f^M = \varepsilon_1^{\#}(f)$ when $B = B^* = I$; and $f^M \dashv f$ when $X = Y = I$.

<u>Proposition 1.</u> *Suppose* $A^* \dashv A$, $B^* \dashv B$, $C^* \dashv C$ *and* $f : X \otimes A \to B \otimes Y$, $g : X' \otimes B \to C \otimes Y'$. *If* h *denotes the composite*

$$X' \otimes X \otimes A \xrightarrow{1 \otimes f} X' \otimes B \otimes Y \xrightarrow{g \otimes 1} C \otimes Y' \otimes Y$$

then its mate h^M *is the composite*

$$C^* \otimes X' \otimes X \xrightarrow{g^M \otimes 1} Y' \otimes B^* \otimes X \xrightarrow{1 \otimes f^M} Y' \otimes Y \otimes A^*.$$

<u>Proof.</u> We leave this as an exercise. $\square$

<u>Definition 1.</u> A tensor category is <u>autonomous</u> when every object has both a right and a left adjoint. //

When $\mathbb{V}$ is autonomous, for each $A \in \mathbb{V}$, we can choose a pair of adjunctions

$$(\eta_A, \varepsilon_A) : A^* \dashv A, \qquad (\eta'_A, \varepsilon'_A) : A \dashv A^\vee,$$

from which two contravariant functors can be obtained

$$(\ )^* : \mathbb{V}^{op} \longrightarrow \mathbb{V}, \qquad (\ )^\vee : \mathbb{V}^{op} \longrightarrow \mathbb{V},$$

where, for any $f : A \longrightarrow B$, we have the relations

$$f^* \dashv f \dashv f^\vee.$$

There are canonical isomorphisms

$$(A^*)^\vee \simeq A \simeq (A^\vee)^*$$

making the functors $(\ )^*$ and $(\ )^\vee$ mutually quasi-inverse.

__Remark.__ The functors $(\ )^*$ and $(\ )^\vee$ are not uniquely defined since they depend on the choice of left and right adjoints. In some autonomous categories it is possible to choose these functors so that they are _inverses_ of each other and not merely quasi-inverse. This only requires $(\ )^*$ to be _invertible_ since we can choose $(\ )^\vee$

to be the inverse. It is obvious that the functor $(\ )^*$ is invertible if and only if the function $(\ )^*: \mathrm{ob}\,\mathbb{V} \to \mathrm{ob}\,\mathbb{V}$ is bijective. If $\mathbb{V}$ is skeletal, this certainly is the case since isomorphic objects are equal in a skeletal category. Using the axiom of choice we can prove that any category is equivalent to a skeletal one (we pick one representative object in each isomorphism class of objects). In this way we see that any autonomous category is equivalent to one in which the duality functors $(\ )^*$ and $(\ )^\vee$ are inverse to each other.

Example. Let k be a commutative ring and let A be a Hopf algebra in the sense of Sweedler [ ] over k with comultiplication Δ, counit ε and antipode S. The antipode is an algebra homomorphism from A to the opposite algebra A^o. In Chapter 1 we saw that the category $\mathrm{Mod}(A)$ of left A-modules is equipped with a tensor product defined from the comultiplication Δ. For all $V, W \in \mathrm{Mod}(A)$, an A-module structure on $\mathrm{Hom}_k(V, W)$ is obtained by restriction along the composite

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{1 \otimes S} A \otimes A^o$$

of the $A\otimes A^{\circ}$-module structure on $\mathrm{Hom}_k(V,W)$. A simple calculation shows that the canonical maps

$$\mathrm{Hom}_k(V,W)\otimes V \longrightarrow W, \qquad T \longrightarrow \mathrm{Hom}_k(V, T\otimes V),$$

are A-module morphisms. It follows that the functor

$$\mathrm{Hom}_k(V,-) : \mathcal{M}od(A) \longrightarrow \mathcal{M}od(A)$$

is right adjoint to the functor

$$-\otimes_k V : \mathcal{M}od(A) \longrightarrow \mathcal{M}od(A).$$

Similarly, let A' be the <u>opposite bialgebra</u> obtained by giving to the algebra A the opposite coalgebra structure $\Delta' = \sigma\circ\Delta$ where $\sigma : A\otimes A \longrightarrow A\otimes A$ is the switch map $a\otimes b \mapsto b\otimes a$. It can be shown [ ] that an antipode S' exists on A' if and only if S is bijective, in which case S and S' are mutual inverses. Denoting by $\mathrm{Hom}'_k(V,W)$ the A-module structure on $\mathrm{Hom}_k(V,W)$ obtained from S' instead of S, we obtain a pair of adjoint functors

$$V\otimes_k - \dashv \mathrm{Hom}'_k(V,-).$$

Denote by $\mathrm{Pr}_k(A)$ the full subcategory of

$\mathrm{Mod}(A)$ consisting of those A-modules which are finitely generated and projective as k-modules. The category $\mathrm{Pr}_k(A)$ is autonomous where, for all $V \in \mathrm{Pr}_k(A)$, we have

$$V^* = \mathrm{Hom}_k(V, k) \text{ and } V^\vee = \mathrm{Hom}'_k(V, k).$$

§2. Polarised diagrams

For simplicity we shall assume from this point on that every planar graph is piecewise linear. A rigorous treatment of the subject without this assumption would require the consideration of delecate aspects of the topology of the plane thereby considerabley affecting the length of the paper. Also, there is no clear advantage in treating the general case since the results turn out to be formally identical in both cases. From the practical view-point of using diagrams for calculations, the piecewise linear case is general enough since any planar figure is practically indistinguishable from a piecewise linear one.

We shall denote by $[p,q]$ the straight line segment connecting two points $p, q \in \mathbb{R}^2$. More generally, $[p_0, p_1, \ldots, p_n]$ will denote the polygonal segment defined by a sequence of points $p_0, p_1, \ldots, p_n \in \mathbb{R}^2$. We shall say that $[p_0, p_1, \ldots, p_n]$ is <u>reduced</u> if $p \notin [p_{i-1}, p_{i+1}]$ for all $0 < i < n$. Up to reparametrisation, a simple piecewise linear curve $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ is determined by its image $\gamma[a,b] = [p_0, \ldots, p_n]$. Suppose that $[p_0, \ldots, p_n]$ is reduced. We shall say that $[p_i, p_{i+1}]$ is a <u>line segment</u> of γ and that $[p_0, p_1]$ (resp. $[p_{n-1}, p_n]$) is the <u>initial segment</u> (resp. <u>terminal segment</u>). We also say that $[p_i, p_{i+1}]$ is <u>directed right</u> (resp. <u>directed left</u>, resp. <u>vertical</u>) if the first coordinate of p_i is the less than (resp. greater than, resp. equal to) the first coordinate of p_{i+1}.

A graph Γ embedded in $\mathbb{R}^2$ is <u>piecewise linear</u> when all its 1-cells are polygonal segment. A <u>line segment</u> of Γ is a line segment of an edge of Γ.

<u>Definition 2</u>. Let Γ be a piecewise linear planar graph between the slices $a<b$ in $\mathbb{R}$. We shall say that Γ is <u>polarised</u> if no initial or terminal segment of an edge of Γ is vertical.//

The <u>output</u> $\mathrm{out}(x)$ of an inner node x is the set of $\gamma \in \Gamma_1$ having ~~its~~ initial segment directed right and such that $\gamma(0) = x$. If a number $u > p_1(x)$ is chosen close enough to $p_1(x)$, the vertical line $\{u\} \times \mathbb{R}$ intersects the initial segment of each edge $\gamma \in \mathrm{out}(x)$ exactly once. This defines a bijection between $\mathrm{out}(x)$ and a subset of $\{u\} \times \mathbb{R}$ and hence defines a linear order on $\mathrm{out}(x)$. Similarly, the <u>input</u> $\mathrm{in}(x)$ is the set of $\gamma \in \Gamma_1$ having a terminal segment directed right and such that $\gamma(1) = x$. The set $\mathrm{in}(x)$ is also linearly ordered by intersecting by an appropriate vertical line.

For each edge $\gamma \in \Gamma_1$ we shall define a <u>rotation number</u> $\rho(\gamma) \in \mathbb{Z}$ with the property that

$$\rho(\gamma^0) = -\rho(\gamma).$$

To do this, we could replace γ by a "suitable" approximation $\tilde{\gamma}$ in which the tangent vectors

$\tilde{\gamma}'(0)$, $\tilde{\gamma}'(1)$ were horizontal. The rotation number $\rho(\gamma)$ would then be defined as the number of anticlockwise half turns experienced by the tangent vector $\tilde{\gamma}'(t)$ as t varied from 0 to 1; that is, as the degree of the loop $t \longmapsto \tilde{\gamma}'(t)^2$ in $\mathbb{C}-\{0\} \simeq \mathbb{R}^2-\{0\}$. Instead of us making precise what a "suitable" approximation is, we shall give another description of $\rho(\gamma)$ better adapted for use in this paper. Let $[q_0, q_1, \dots, q_n]$ be the image of γ as a reduced polygonal segment. We shall say that q_i $(0<i<n)$ is a *redirection point* when one of $[q_{i-1}, q_i]$, $[q_i, q_{i+1}]$ is directed left and the other is directed right. We define the *turning number* $t(q_i)$ to be zero unless q_i is a redirection point in which case it is $+1$ when the frame $(q_i - q_{i-1}, q_{i+1} - q_i)$ has orientation agreeing with the usual orientation of $\mathbb{R}^2$ and it is -1 otherwise. Similarly, we shall say that a vertical segment $[q_i, q_{i+1}]$ $(0<n<n-1)$ is a *redirection segment* when one of $[q_{i-1}, q_i]$, $[q_{i+1}, q_{i+2}]$ is directed left and the other is diected right. We define the *turning number* $t[q_i, q_{i+1}]$ of a segment to be zero unless $[q_i, q_{i+1}]$ is a vertical redirection segment in which case it is $+1$ when the frame

$(q_i - q_{i-1}, q_{i+1} - q_i)$ has orientation agreeing with the usual orientation of $\mathbb{R}^2$ and it is -1 otherwise. (For example, the points q_i in Fig. 7 have turning number -1 while the vertical segments $[q_i, q_{i+1}]$ in Fig. 8 have turning number $+1$.) Note that the turning numbers change sign if we reverse the orientation of γ. We now define

$$\rho(\gamma) = \sum_{0<i<n} t(q_i) + \sum_{0<i<n} t[q_i, q_{i+1}].$$

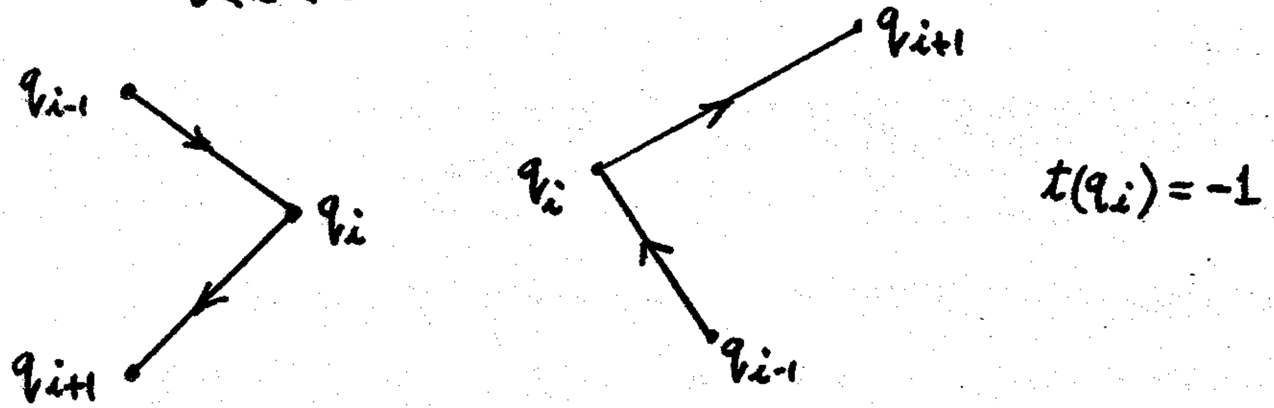


Fig. 7

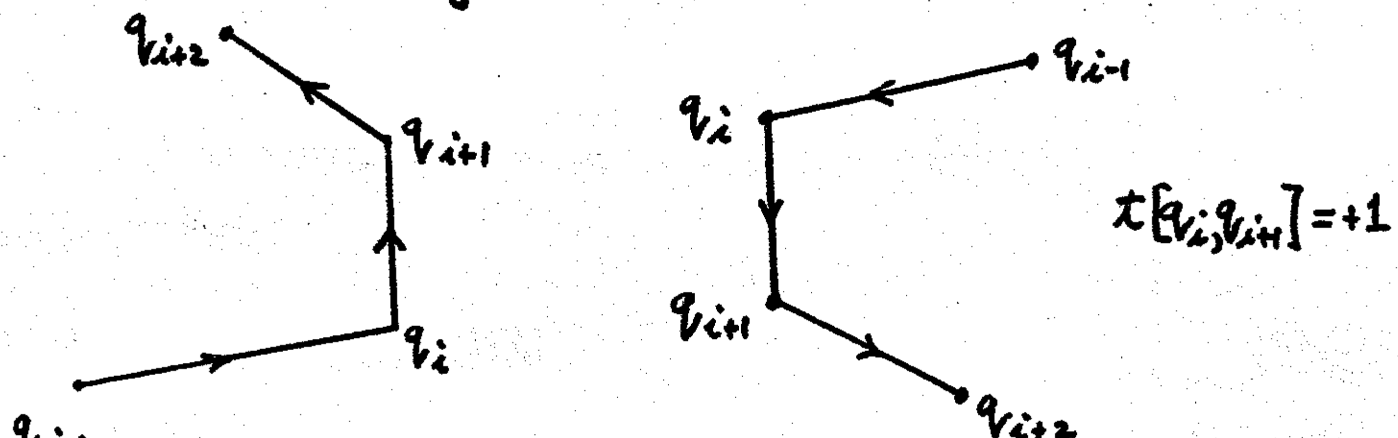


Fig. 8

We can now define the concept of <u>valuation</u> on a polarised graph Γ. Let $\mathbb{V}$ be an autonomous category equipped with an invertible duality functor $D : \mathbb{V}^{op} \longrightarrow \mathbb{V}$, $D(A) = A^*$, $(\eta_A, \varepsilon_A) : D(A) \dashv A$. Let Γ_0' be the set of inner nodes of Γ.

<u>Definition 3.</u> A <u>valuation</u> $v : \Gamma \longrightarrow \mathbb{V}$ is a pair of functions

$$v_0 : \Gamma_1 \longrightarrow \mathrm{ob}\,\mathbb{V}, \qquad v_1 : \Gamma_0' \longrightarrow \mathrm{mor}\,\mathbb{V}$$

satisfying the conditions

(i) for all $x \in \Gamma_0'$,

$$v_1(x) : v_0(\gamma_1^0) \otimes \cdots \otimes v_0(\gamma_m^0) \longrightarrow v_0(\delta_1) \otimes \cdots \otimes v_0(\delta_n)$$

where $\gamma_m < \cdots < \gamma_1$, $\delta_n < \cdots < \delta_1$ are listings of the elements of $in(x)$, $out(x)$, respectively,

(ii) for all $\gamma \in \Gamma_1$,

$$v_0(\gamma^0) = D^{\rho(\gamma)} v_0(\gamma).$$

The pair (Γ, v) is called a (<u>polarised</u>) <u>diagram</u> and denoted simply by Γ when no confusion arises. //

The valuation can be used to <u>label</u> the line segments of Γ, excluding the vertical redirection segments, by objects of $\mathbb{V}$. Here is the rule. Let $[q_0, q_1, \ldots, q_n]$ be the reduced polygonal segment representing an edge $\gamma \in \Gamma_1$. The initial segment $[q_0, q_1]$ is labelled by $v_0(\gamma) = A$. If $[q_i, q_{i+1}]$ is not a vertical redirection segment then its label is $D^k(A)$ where

$$k = \sum_{0<j\leq i} t(q_j) + \sum_{0<j\leq i} t[q_j, q_{j+1}].$$

Note that (ii) in Definition 3 implies that the labelling of a line segment of γ will be the same if we replace γ by γ^0.

Let $\Gamma[c,d]$ be a strip of the polarised graph Γ. A valuation v on Γ restricts as follows to a valuation, also denoted by v, on $\Gamma[c,d]$: its value on an inner node x of $\Gamma[c,d]$ is equal to $v_1(x)$; its value on an edge δ of $\Gamma[c,d]$ is the object labelling the initial line segment of δ. The diagram $(\Gamma[c,d], v)$ is called a <u>slice</u> of Γ.

Similarly, we can define the concept of <u>vertical decomposition</u> $\Gamma = \Gamma^1 \otimes \Gamma^2$ of the polarised diagram $\Gamma = (\Gamma, v)$.

The definition of the <u>value</u> $v(\Gamma)$ of the polarised diagram (Γ, v) will first be given in the case where Γ has no vertical line segments at all. To do this, we subdivide the 1-cells of Γ by adding to the 0-cells all the redirection points. In this way we obtain a new graph $\tilde{\Gamma}$ which is recumbent since Γ has no vertical line segments. A valuation $\tilde{v}$ can be defined on $\tilde{\Gamma}$ as follows:

(i) for $\delta \in \tilde{\Gamma}_1$, $\tilde{v}_0(\delta)$ is the label of any line segment of δ;

(ii) for a redirection point $x \in \Gamma_0'$ with $in(x) = \emptyset$, $out(x) = \{\delta_1, \delta_2\}$ in Γ, $\delta_2 < \delta_1$, and $A = v_0(\delta_1)$, put
$$\tilde{v}_1(x) = \eta_A : I \longrightarrow A \otimes D(A);$$

(iii) for a redirection point $x \in \Gamma_0'$ with $out(x) = \emptyset$, $in(x) = \{\delta_1, \delta_2\}$ in Γ, $\delta_2 < \delta_1$, and $A = v_0(\delta_2)$, put
$$\tilde{v}_1(x) = \varepsilon_A : D(A) \otimes A \longrightarrow I;$$

(iv) $\tilde{v}_1$ coincides with v_1 on the inner nodes of Γ.

Then we define the value of Γ by

$$v(\Gamma) = \tilde{v}(\tilde{\Gamma}).$$

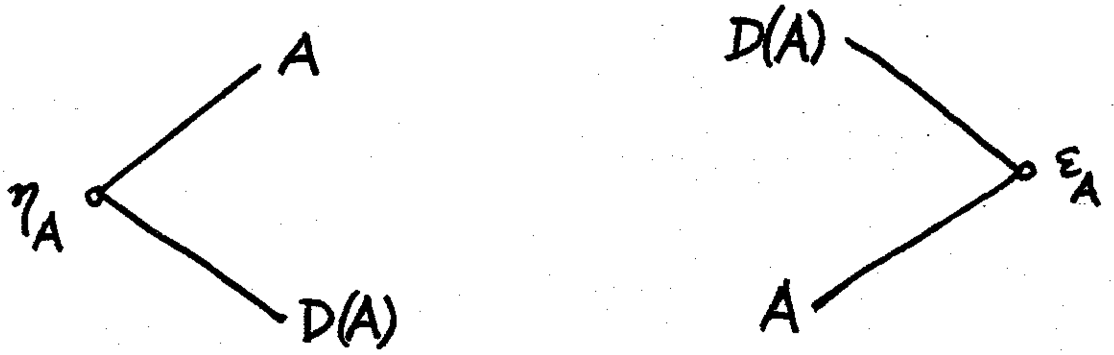


Fig. 9

The case where Γ has some vertical segments will be reduced to the case with none by replacing Γ by a perturbed graph Γ^ε as follows. Let $e_1 = (1, 0)$ be the unit vector and take $\varepsilon > 0$. A vertical line segment $[p, q]$ of Γ is replaced in Γ^ε by:

(i) the polygonal segment $[p, \frac{1}{2}(p+q) + \varepsilon e_1, q]$ (resp. $[p, \frac{1}{2}(p+q) - \varepsilon e_1, q]$) when $[p,q]$ is a redirection segment and the segment preceding $[p,q]$ in Γ is directed right (resp. left) (see Fig. 10);

(ii) the polygonal segment $[p, p+\varepsilon e_1, q-\varepsilon e_1, q]$

(resp. $[p, p-\varepsilon e_1, q+\varepsilon e_1, q]$) when $[p,q]$ is not a redirection segment and the segment preceding $[p,q]$ in Γ is directed right (resp. left) (see Fig. 11).

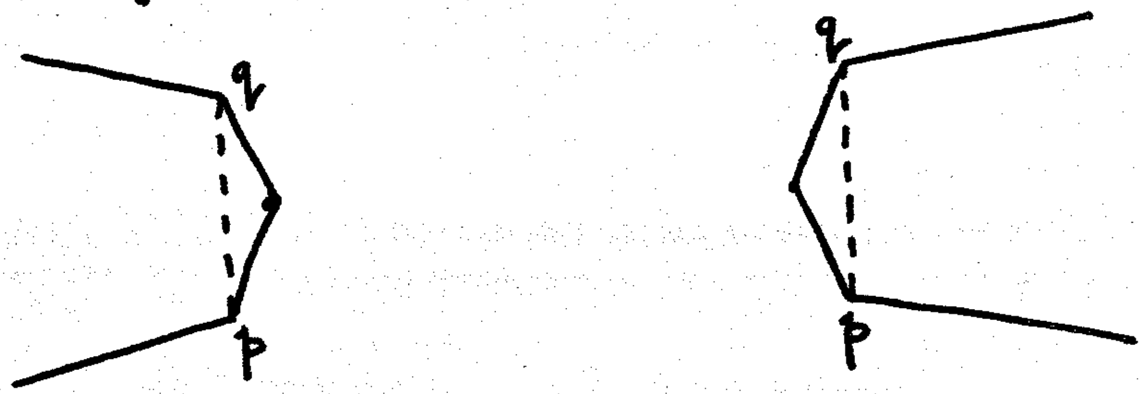


Fig. 10

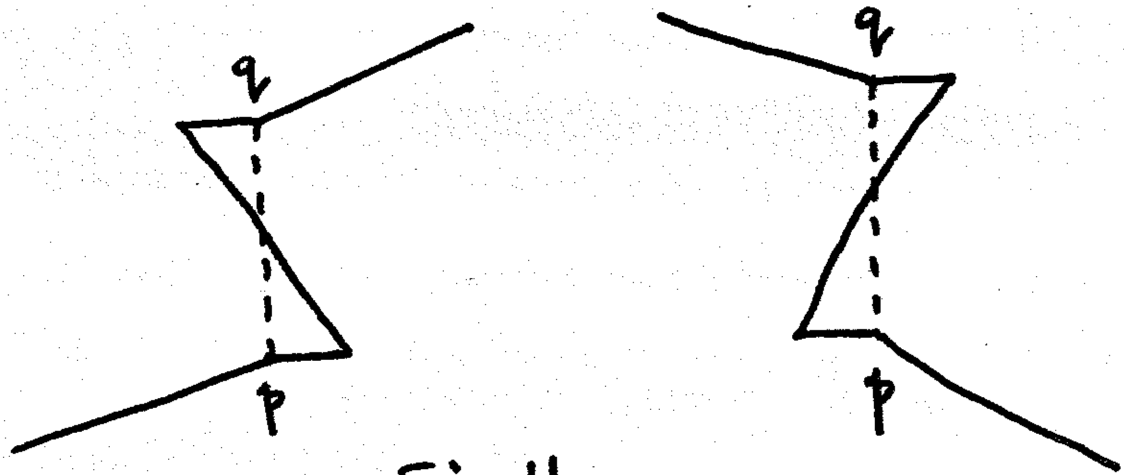


Fig. 11

When $\varepsilon > 0$ is small enough, the graphs Γ^ε are polarised and isomorphic to Γ. The obvious isomorphism $\Gamma \simeq \Gamma^\varepsilon$ (of topological graphs) respects the inner nodes, the ordered sets in(x) and out(x), and the rotation numbers of the edges. The valuation v on Γ can therefore be transported along $\Gamma \simeq \Gamma^\varepsilon$ to a valuation on Γ^ε. By Theorem 1.3, the value $v(\Gamma^{\varepsilon})$ is independent of ε, since the subdivided graphs Γ^{ε} are deformations of each other. Thus we define

$$v(\Gamma) = v(\Gamma^{\varepsilon}).$$

Of course, the particular choice of the perturbation will turn out to be irrelevant since we shall prove later (Theorem 2.4) that the value of a polarised diagram remains constant under any (regular) deformation, small or large.

<u>Proposition 2.</u> <u>If</u> $a = a_0 < \dots < a_n = b$ <u>are regular slices for the polarised diagram</u> Γ <u>then</u>

$$v(\Gamma) = v(\Gamma[a_{n-1}, a_n]) \circ \dots \circ v(\Gamma[a_0, a_1]).$$

<u>Proof.</u> The result follows from Proposition 1.1 since we have $\Gamma[a_i, a_{i+1}]^{\varepsilon} = \Gamma^{\varepsilon}[a_i, a_{i+1}]$ for $\varepsilon > 0$ small enough. $\square$

<u>Proposition 3.</u> <u>Let</u> $\Gamma = \Gamma^1 \otimes \dots \otimes \Gamma^n$ <u>be a vertical decomposition of the polarised diagram</u> Γ. <u>Then</u>

$$v(\Gamma) = v(\Gamma^1) \otimes \dots \otimes v(\Gamma^n).$$

<u>Proof.</u> The result follows from Proposition 1.2 since $\Gamma^{\varepsilon} = (\Gamma^1)^{\varepsilon} \otimes \dots \otimes (\Gamma^n)^{\varepsilon}$ for $\varepsilon > 0$ small enough. $\square$

Let T be a connected space. We shall say that a continuous function $h : T \times [0,1] \longrightarrow \mathbb{R}^2$ is a <u>regular deformation</u> of piecewise linear curves if there is a subdivision $0 = a_0 < a_1 < \ldots < a_n = 1$ of $[0,1]$ such that, for each $t \in T$, the curve $h(t,-) : [0,1] \longrightarrow \mathbb{R}^2$ is linear on each interval $[a_i, a_{i+1}]$.

Let Γ be a topological graph. Assume that each edge $\gamma \in \Gamma_1$ is equipped with an order-preserving parametrization $\gamma(t)$ $(0 \leq t \leq 1)$ such that $\gamma^0(t) = \gamma(1-t)$. Let

$$h : \Gamma \times T \longrightarrow [a,b] \times \mathbb{R}$$

be a deformation of planar graphs between the slices a and b (Definition 1.2). For every $t \in T$, let $\Gamma(t)$ be the image of $h(-,t) : \Gamma \longrightarrow [a,b] \times \mathbb{R}$.

<u>Definition 4.</u> We call h a <u>regular deformation of polarised graphs</u> when

(i) for all edges $\gamma \in \Gamma$, the map

$$h(-, \gamma(\sim)) : T \times [0,1] \longrightarrow [a,b] \times \mathbb{R}$$

is a regular deformation of piecewise linear curves,

and (ii) for all $t \in T$, the graph $\Gamma(t)$ is polarised.

A structure on $\Gamma(t)$ can be transported along the isomorphism $\Gamma \simeq \Gamma(t)$ to a structure on Γ. Some of the structures on Γ defined in this way are independent of the parameter $t \in T$. For example, the inner nodes, the ordered sets $in(x)$ and $out(x)$, and the rotation numbers $\rho(\gamma)$ are independent of $t \in T$ (while the nature and pattern of redirection points is not independent). Using this, we see that a valuation defined on $\Gamma(t)$ for some t will extend to all $t \in T$. The map h becomes a deformation of <u>diagrams</u>.

<u>Theorem 4</u>. <u>For any regular deformation</u>

$$h : \Gamma \times T \longrightarrow [a,b] \times \mathbb{R}$$

<u>of polarised diagrams, the value</u> $v(\Gamma(t))$ <u>is independent of</u> $t \in T$.

The proof of Theorem 4 is based on the following lemma. Let Γ be a connected polarised graph with no inner nodes. In this case Γ has a single 1-cell which is the image of a simple piecewise linear curve $\gamma : [0,1] \longrightarrow [a,b] \times \mathbb{R}$ such that $\gamma(0), \gamma(1) \in \{a,b\} \times \mathbb{R}$. A valuation v on Γ is entirely determined when we know $v_0(\gamma) = A$. The value of the diagram (Γ, v) is then given by cases (i), (ii), (iii) of the Lemma below. We shall prove the Lemma under the extra hypothesis that γ has no vertical segments and consists of three pieces γ_1, γ_2, γ_3 obtained by restriction of γ to $[0, a_1]$, $[a_1, a_2]$, $[a_2, 1]$, respectively, $(0 < a_1 < a_2 < 1)$ such that

(a) the first projection $p_1 : [a,b] \times \mathbb{R} \longrightarrow [a,b]$ is injective on γ_1 and γ_3, and

(b) the second projection $p_2 : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is injective on γ_2.

<u>Lemma</u>. <u>Under the hypothesis stated above,</u>

(i) <u>if</u> $\gamma(0) \in \{a\} \times \mathbb{R}$ <u>and</u> $\rho(\gamma) = 0$ <u>then</u> $v(\Gamma) = 1_A$;

(ii) <u>if</u> $\gamma(0) \in \{a\} \times \mathbb{R}$ <u>and</u> $\rho(\gamma) = 1$ <u>then</u>
$v(\Gamma) = \varepsilon_A : D(A) \otimes A \longrightarrow I$;

(iii) <u>if</u> $\gamma(0) \in \{b\} \times \mathbb{R}$ <u>and</u> $\rho(\gamma) = 1$ <u>then</u>
$v(\Gamma) = \eta_A : I \longrightarrow A \otimes D(A)$.

<u>Proof</u>. We shall give the proof only in case (i) since the other cases are similar. We suppose $a = 0$ and $b = 1$. The idea is to construct a deformation of γ to a staircase-like curve β.

The deformation will preserve the redirection point pattern of γ and therefore leave the value of $v(\Gamma) = v(\tilde{\Gamma})$ unchanged by Theorem 1.3. The graph of β will admit a decomposition into strips whose values are all identity maps so that the result will follow from Proposition 2.2.

We start by deforming γ_1 and γ_3 so that the become horizontal line segments. To do this we can express the image of γ_1 as the graph of a function $y = f(x)$ defined on some interval $[0, e]$; the homotopy $(1-t)f + t f(e)$ will deform f to a constant function $x \mapsto f(e)$ whose graph is a horizontal line segment. It is clear that the corresponding deformation of Γ will leave the value $v(\Gamma)$ unchanged. A similar deformation is used for γ_3.

To describe the next deformation we express the curve γ_2 as the graph of a function $x = g(y)$ defined on some interval $[c, d]$. The condition $\rho(\gamma) = 0$ implies that g has an equal number of maxima and minima. Let N be this number and let $c \leq x_1 < x_2 < \dots < x_{2N} < d$ be the ordered sequence of maxima and minima (they alternate).

The first element x_1 is a maximum or a minimum depending on whether $\gamma(a_1) = (g(c), c)$ or $\gamma(a_1) = (g(d), d)$. We consider only the first case. Suppose $N=1$. Let $x_1 < s < x_2$ and $u = g(s)$. The number u is a regular slice for Γ and the values z_0, z_1 of the strips $\Gamma[0,u]$, $\Gamma[u,1]$ can be computed directly from the definition of v to be given by

$$z_0 = \eta_B \otimes A \, , \qquad z_1 = A \otimes \varepsilon_B$$

where $B = D(A)$. It follows that

$$v(\Gamma) = z_1 \circ z_0 = 1_A$$

by the adjunction triangle. This proves the result for $N=1$. When $N>1$, let $x_{2i} < s_i < x_{2i+1}$ for $i = 1, 2, \ldots, N-1$ and let $s_0 = c$, $s_N = d$. Choose $\varepsilon > 0$ small enough so that none of the intervals $[s_i - \varepsilon, s_i + \varepsilon]$ $(0 < i < N)$ contains a maximum or a minimum. Consider the function $k: [c,d] \to [0,1]$ such that $k(c) = 0$, $k(d) = (N-1)/N$, $k(y) = i/N$ for $y \in [s_i + \varepsilon, s_{i+1} - \varepsilon]$ $(i = 0, \ldots, N-1)$, and which is continuous, non-decreasing and linear on the complementary intervals $[s_i - \varepsilon, s_i + \varepsilon]$. The function g can be deformed into the function $w = \frac{1}{N} g + k$

in two steps: first the homotopy $t \mapsto tg$ ($1/N \leq t \leq 1$) and second the homotopy $t \mapsto 1/N\, g + t k$ ($0 \leq t \leq 1$). These two deformations preserve the pattern of maxima and minima of g. If we adjoin the horizontal line segments γ_1, γ_3 to the curve $x = w(y)$ ($c \leq y \leq d$), we obtain a polarised graph $\Gamma(w)$ with a single 1-cell. The subdivided graph $\tilde{\Gamma}(w)$ is a deformation of the recumbent graph $\tilde{\Gamma}$ and we thus have $v(\Gamma) = v(\Gamma(w))$. It is easy to see that the numbers $1/N, 2/N, \ldots, N-1/N$ are regular slices for $\Gamma(w)$. Each strip $\Gamma(w)[i/N, i+1/N]$ ($0 \leq i < N$) has exactly one redirection point of each kind and its value can be computed as above in the case $N=1$ showing that it is equal to 1_A. $\square$

<u>Proof of Theorem 4</u>. This is similar to the proof of Theorem 1.3. We show that, for any $t_0 \in T$, the value $v(\Gamma(t))$ is equal to $v(\Gamma(t_0))$ provided t is close enough to t_0. To do this, we decompose $\Gamma(t_0)$ first into strips having at most one critical slice. Each strip is then decomposed horizontally into connected components

$$\Gamma(t_0)[a_i, a_{i+1}] = C_1 \otimes \cdots \otimes C_r .$$

By continuity, this vertical decomposition corresponds to a decomposition

$$\Gamma(t)[a_i, a_{i+1}] = C_1(t) \otimes \cdots \otimes C_r(t)$$

valid for t close enough to t_0. It suffices to prove that $v(C_j(t))$ is constant for t near t_0. If $C_j(t_0)$ has no vertical segment, the subdivided graph $\tilde{C}_j(t)$ is a deformation of $\tilde{C}_j(t_0)$ and the result follows from Theorem 1.3. If $C_j(t_0)$ contains a vertical segment, the diagram $C_j(t)^\varepsilon$ will satisfy the hypothesis of the Lemma for $\varepsilon > 0$ small enough and t close enough to t_0. The conclusion of the Lemma then show that

$$v\left(C_j(t)^\varepsilon\right) = v\left(C_j(t_0)^\varepsilon\right),$$

which means that

$$v\left(C_j(t)\right) = v\left(C_j(t_0)\right). \square$$

~~§3. Autonomous tensor schemes~~

~~Definition 5. An autonomous tensor scheme $(\mathbb{D}, D)$ a tensor scheme $\mathbb{D}$ together with a bijective function $D : \mathrm{ob}\,\mathbb{D} \to \mathrm{ob}\,\mathbb{D}$ such that $D^n X = X$ implies $n = 0$. We write $\mathbb{D}$ for $(\mathbb{D}, D)$ when there is no confusion likely.~~

§3. Autonomous tensor schemes.

Suppose $F : \mathbb{V} \to \mathbb{W}$ is a tensor functor. Each pairing $\varepsilon : A \otimes B \to I$ in $\mathbb{V}$ yields a pairing $\varepsilon^F : FA \otimes FB \to I$ in $\mathbb{W}$ given by the composite

$$FA \otimes FB \xrightarrow{\varphi^2} F(A \otimes B) \xrightarrow{F\varepsilon} FI \xrightarrow{(\varphi^0)^{-1}} I .$$

If ε is exact then so is ε^F : in fact, if $(\eta, \varepsilon) : A \dashv B$ then $(\eta^F, \varepsilon^F) : FA \dashv FB$ where η^F is the composite

$$I \xrightarrow{\varphi^0} FI \xrightarrow{F\eta} F(B \otimes A) \xrightarrow{(\varphi^2)^{-1}} FB \otimes FA .$$

So tensor functors preserve duals. They also preserve mates under adjunctions: if $f : X \otimes A \to B \otimes Y$ in $\mathbb{V}$ with $A^* \dashv A$, $B^* \dashv B$ then $(Ff)^M = F(f^M)$. In particular, $f^* \dashv f$ in $\mathbb{V}$ implies $Ff^* \dashv Ff$.

<u>Proposition 5.</u> <u>Suppose $\alpha : F \to G$ is a tensor transformation between tensor functors</u> $F, G : \mathbb{V} \to \mathbb{W}$. <u>If $A^* \dashv A$ in $\mathbb{V}$ then the morphisms α_{A^*} and $(\alpha_A)^*$ are mutually inverse in</u> $\mathbb{W}$.

Proof. Suppose $(\eta,\varepsilon) : A^* \dashv A$. Then

$$(\varepsilon^F)^\#((\alpha_A)^* \circ \alpha_{A^*}) = \varepsilon^F \circ ((\alpha_A)^* \alpha_{A^*}) \otimes 1$$

$= \varepsilon^F \circ ((\alpha_A)^* \otimes 1) \circ (\alpha_{A^*} \otimes 1)$ since $\otimes$ is a functor,

$= \varepsilon^G \circ (1 \otimes \alpha_A) \circ (\alpha_{A^*} \otimes 1)$ by definition of $(\alpha_A)^*$,

$= (\varphi^0)^{-1} \circ G\varepsilon \circ \varphi^2 \circ (\alpha_{A^*} \otimes \alpha_A)$ by functoriality and definition,

$= (\varphi^0)^{-1} \circ G\varepsilon \circ \alpha_{A^* \otimes A} \circ \varphi^2$ since α is a tensor trans.,

$= (\varphi^0)^{-1} \circ \alpha_I \circ F\varepsilon \circ \varphi^2$ since α is natural,

$= (\varphi^0)^{-1} \circ F\varepsilon \circ \varphi^2$ since α is a tensor trans.,

$= (\varepsilon^F)^\#(1_{FA^*})$ by definition.

Hence $(\alpha_A)^* \circ \alpha_{A^*} = 1_{FA^*}$ by exactness of ε^F. A similar calculation using η yields $\alpha_{A^*} \circ (\alpha_A)^* = 1_{GA^*}$. $\square$

<u>Corollary 6. If $\mathbb{V}$ is an autonomous tensor category then</u> $\mathrm{Ten}(\mathbb{V}, \mathbb{W})$ <u>is a groupoid.</u>

<u>The inverse of a morphism</u> $\alpha : F \to G$ <u>is given by</u>

$$\alpha_A^{-1} = (\alpha_{A^\vee})^*. \square$$

<u>Definition 5</u>. An <u>autonomous tensor scheme</u> is a tensor scheme $\mathbb{D}$ enriched with a bijective function $D : \mathrm{ob}\,\mathbb{D} \to \mathrm{ob}\,\mathbb{D}$ such that $D^n X = X$, for some X, implies $n = 0$. //

To provide an example of an autonomous tensor scheme it suffices to provide a complete set S of representatives for the orbits of D; then $\mathrm{ob}\,\mathbb{D} = \{ D^n X \mid X \in S,\ n \in \mathbb{Z} \} \simeq \mathbb{Z} \times S$.

For an autonomous tensor scheme $\mathbb{D}$, we write $[\mathbb{D}, \mathbb{V}]_D$ for the subcategory of $[\mathbb{D}, \mathbb{V}]$ consisting of those objects K such that

$$KDX \dashv KX \quad \text{for all } X \in \mathrm{ob}\,\mathbb{D},$$

and those morphisms θ which are invertible and are such that

$$\theta_{DX}^{-1} \dashv \theta_X \quad \text{for all } X \in \mathrm{ob}\,\mathbb{D}.$$

By Propositions 2.5 and 2.1, we have a restricted "composition" functor

$$\mathrm{Ten}(\mathbb{V}, \mathbb{W}) \times [\mathbb{D}, \mathbb{V}]_D \longrightarrow [\mathbb{D}, \mathbb{W}]_D .$$

<u>Definition 6</u>. A <u>free autonomous tensor category</u> on the autonomous tensor scheme $\mathbb{D}$ is an autonomous

tensor category $\mathbb{F}$ together with an object N of $[\mathbb{D},\mathbb{F}]_{\mathbb{D}}$ such that, for all tensor categories $\mathbb{V}$, the functor

$$-\circ N : \mathrm{Ten}(\mathbb{F},\mathbb{V}) \longrightarrow [\mathbb{D},\mathbb{V}]_{\mathbb{D}}$$

is an equivalence of categories.//

A <u>valuation</u> $v : \Gamma \longrightarrow \mathbb{D}$ of a polarised graph Γ in an autonomous tensor scheme can be defined just as in Definition 2.3 with $\mathbb{D}$ in place of $\mathbb{V}$ and $\otimes$ replaced by juxtaposition. The pair (Γ, v) is called a <u>polarised diagram</u> in $\mathbb{D}$. The <u>domain</u> (resp. <u>codomain</u>) of the diagram (Γ, v) is defined to be the word $X_1 \ldots X_m$ in elements of ob $\mathbb{D}$ where X_i is the label on the line segment of Γ whose initial (resp. final) point x_i is an outer node of Γ on the left (resp. right) bounding slice of Γ and $x_1 > \ldots > x_m$.

As the tensor product and composite of centred polarised graphs are polarised, it is clear how to define tensor product and composite of centred polarised diagrams in an autonomous tensor scheme. It is also clear how to define the concept of deformation of centred polarised diagrams in an autonomous tensor scheme, and the associated notion of isotopy.

Given an autonomous tensor scheme $\mathbb{D}$, there is a strict autonomous tensor category $\mathcal{F}_D(\mathbb{D})$ defined as follows. The objects of $\mathcal{F}_D(\mathbb{D})$ are words in the elements of ob $\mathbb{D}$. The morphisms of $\mathcal{F}_D(\mathbb{D})$ are isotopy classes of centred polarised diagrams in $\mathbb{D}$. The domain, codomain and composition of morphisms are induced on isotopy classes by the corresponding operations for centred polarised diagrams. The identity of a one-letter word X is the isotopy class of the diagram whose graph is the line segment $[(-1,0),(1,0)]$ with no inner nodes and whose valuation is X at the only cell. The tensor product on objects is given by juxtaposition of words and on morphisms is induced by the tensor product of centred polarised diagrams. The identity of a word $X_1 \dots X_m$ is the tensor product of the identities of $X_1, \dots, X_m$. This gives a strict tensor category by Proposition 1.4.

We shall now see that each one-letter word X has a left dual, namely, DX. The pairing $\varepsilon : (DX)X \longrightarrow I$ is the isotopy class of the centred diagram consisting of the polygonal segment

$$[(-1,-\tfrac{1}{2}),(0,0),(-1,\tfrac{1}{2})]$$

with no inner nodes and with valuation having value X on the one edge oriented positively (Fig. 12).

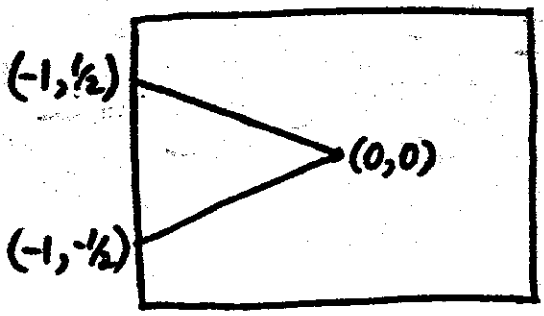


Fig. 12

Let $\eta : I \longrightarrow X(DX)$ be obtained similarly using the graph obtained by reflexion in the y-axis. The deformations needed to prove exactness (see Fig. 4 and Fig. 5) are easily exhibited. Replacing X by $D^{-1}X$, we see also that X has a right dual. In any tensor category, if $A \dashv B$ and $C \dashv D$ then $C \otimes A \dashv B \otimes D$. Hence every object of $\mathcal{F}_D(\mathbb{D})$ has both a left and a right dual.

As every recumbent diagram is certainly polarised, there is a canonical tensor functor

$$\mathcal{F}(\mathbb{D}) \longrightarrow \mathcal{F}_D(\mathbb{D})$$

which is the identity on objects. Composing with $N \in \mathrm{ob}[\mathbb{D}, \mathcal{F}(\mathbb{D})]$, we obtain an object $N_D \in \mathrm{ob}[\mathbb{D}, \mathcal{F}_D(\mathbb{D})]_I$

<u>Theorem 7.</u> *<u>The free autonomous tensor category on an autonomous tensor scheme</u> $\mathbb{D}$ <u>is given by</u> $\mathcal{F}_D(\mathbb{D})$ <u>together with</u> N_D.*

<u>Proof</u>. We may suppose that $\mathbb{V}$ is a skeletal autonomous strict tensor category. As in the proof of Theorem 1.5, for each $K \in [\mathbb{D}, \mathbb{V}]_D$, there is a unique strict tensor functor $T : \mathcal{F}(\mathbb{D}) \longrightarrow \mathbb{V}$ with $T \circ N_D = K$: for objects we need the equality $KDX = DKX$ and for morphisms we define $T[\Gamma, v] = (Kv)(\Gamma)$. Uniqueness follows from the decomposition of each (Γ, v), using tensor product and composition, into diagrams with at most one inner node (see the proof of Theorem 2.4).

As in the proof of Theorem 1.5, suppose $\theta : F \circ N_D \longrightarrow G \circ N_D$ is a morphism of $[\mathbb{D}, \mathbb{V}]_D$ and define $\alpha : F \rightarrow G$ as there. Compatibility of α with φ^n is automatic. Naturality reduces to the case of morphisms represented by diagrams with at most one inner node : the one-node case is as before ; the no-node cases (as described in the proof of Theorem 2.4) amount to (i) identity morphisms where naturality is immediate, (ii) counit morphisms where naturality amounts to $(\theta_X)^* \circ \theta_{X^*} = 1$, and (iii) unit morphisms where naturality amounts to $\theta_{X^*} \circ (\theta_X)^* = 1$. $\square$

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