Limits in 2–categories of locally–presented categories

Ph.D. Thesis, G.J. Bird, 1984

Sydney Category Seminar Report

LIMITS IN 2-CATEGORIES OF LOCALLY-PRESENTABLE CATEGORIES

by

Gregory J. Bird

A thesis submitted for the degree of
Ph.D. in the University of Sydney

May 1984

Since Greg Bird, a former student of mine, is pursuing interests other than mathematics, the Sydney Category Theory Seminar has had 100 copies made of his thesis. The first chapter of the thesis, to which Ross Street and Max Kelly contributed, and which was improved by later suggestions from John Power, will appear in two articles by Bird, Kelly, Power and Street entitled "Flexible limits for 2-categories" and "Explicit formulas for the strict reflexions of pseudo and lax natural transformations". Some of the arguments in the thesis are simplified in the light of hindsight.

The Seminar is sending copies of Bird's thesis to those on our mailing list; since that list usually contains only one name in each major centre, we ask the recipients to make copies available to their colleagues.

CONTENTS

Acknowledgements

During the research for, and preparation of, this thesis, I have been greatly assisted and encouraged by a number of people. Foremost among these is, of course, my supervisor Professor G.M. Kelly who has, with patience and understanding, guided my research throughout the term of my candidature. When he was absent during the months of May, June, and July 1983, I was supervised by Dr R.F.C. Walters and Professor R.H. Street. I thank them, in particular Professor Street, for their assistance during those three months and at other times when they contributed useful suggestions. As a member of the Sydney Category Theory Seminar I have been encouraged and influenced by the other members, both permanent and visiting, and I would like to express my appreciation for the ideal environment the seminar has provided. My research has been supported financially by the Commonwealth Postgraduate Research Award Scheme. Finally, I thank Helen Rubin for her efforts spent in typing this thesis.

Chapter 0, the introductory chapter, contains a summary of established facts with which we assume the reader has some acquaintance. Much of Chapter 1 is the result of work done in collaboration with Professors Kelly and Street. This work will appear soon as a joint paper. Otherwise, unless specifically stated in the text, results are original and have not, to the author's knowledge, appeared elsewhere.

Abstract

This thesis has its origins in responding to some unpublished work of Ulmer [26], [27], [28]. There, Ulmer proves that certain constructions on locally-presentable categories yield locally-presentable categories.

Let $\mathcal{C}$ be a small category and Γ a set of cones in $\mathcal{C}$. The category $[\mathcal{C},A]_\Gamma$ is the full subcategory of the functor category $[\mathcal{C},A]$ given by those functors T such that each $T\gamma$, where γ is in Γ, is a limit-cone. Gabriel and Ulmer [10] had already established that $[\mathcal{C},A]_\Gamma$ is reflective in $[\mathcal{C},A]$, and hence locally presentable, if A is locally presentable. The result about reflectivity was extended by Freyd and Kelly [9] to the case where A is a locally--bounded category and Γ is a (possibly) large set. Some results on the coreflectivity of subcategories determined by functors taking (inductive) cones to colimit-cones existed, but were unpublished, before the work of Ulmer. One major thrust of this work was to establish coreflectivity for the case of A being a locally-presentable category.

It now seems that the appropriate setting for the formulation of Ulmer's results is a 2-categorical one. What Ulmer calls "prealgebras" and "bialgebras" are but special instances of *inserters* and *equifiers*; specifying structure maps may be interpreted as the "insertion" of 2-cells in the 2-category *CAT*, and the imposition of relations as "equifying" pairs of 2-cells.

In this new 2-categorical setting it becomes imperative to identify a reasonable class of limits which include, at least, products, inserters, and equifiers. Chapter 1 is given over to this task.

For a small 2-category A let $Lax[A,Cat]$ denote the 2-category whose objects are the 2-functors $F: A \to Cat$, whose 1-cells are the lax-natural transformations, and whose 2-cells are modifications. Let $Psd[A,Cat]$ be the sub-2-category given by the pseudo-natural transformations. Then the inclusions $J: [A,Cat] \to Lax[A,Cat]$ and $K: [A,Cat] \to Psd[A,Cat]$ both have left adjoints $(\ )^{\dagger}: Lax[A,Cat] \to [A,Cat]$ and $(\ )': Psd[A,Cat] \to [A,Cat]$ (but, in general, the inclusion $Psd[A,Cat] \to Lax[A,Cat]$ does not). We give an explicit description of these left adjoints using 2-categorical versions of the Grothendieck construction el(F), *the category of elements* for a functor $F: B \to Set$. Any indexed limit $\{M,G\}$ whose indexing type $M: A \to Cat$ is a retract of a functor of the form $F': A \to Cat$ is said to be an *indexed limit of retract type*.

The indexed limits of retract type do include products, inserters and equifiers. Moreover, they include all indexed lax-limits and indexed pseudo-limits, and they may be constructed from certain basic ones, namely inserters, equifiers, products, cotensor products, and splittings of idempotents. Thus, to show the existence of indexed limits of retract type in a 2-category A it is sufficient to prove that A admits these five basic types, and to prove that a sub-2-category B of A is closed under the formation of indexed limits of retract type it is sufficient to prove that B is closed under formation of these basic ones.

In this context, Ulmer's early results are subsumed in the statement that Loc and $Ladj$, the sub-2-categories of CAT determined, respectively, by the locally-presentable categories, right adjoint functors and natural transformations and by the locally-presentable categories, left adjoint functors and natural transformations, are closed in *CAT* under the formation of indexed limits of retract type. However, this statement expresses substantially more than what is expressly proved by Ulmer. The closedness of *Loc* is the main result of Chapter 2, while the main result of Chapter 3 is the closedness of *Ladj*.

Not only is the sub-2-category *Loc* closed in *CAT* under the formation of indexed limits of retract type, but so also are the sub-2-categories α-*Loc* given by the locally α-presentable categories and the right adjoint functors with rank α. The corresponding sub-2-categories α-*Ladj* of *Ladj* are not, however, closed under the formation of indexed limits of retract type. If the regular cardinal α is uncountable, then α-*Ladj* is closed in *CAT* for all indexed limits of retract type where the indexing type is *of size* α. Thus, for instance, α-*Ladj*, for an uncountable α, admits products with fewer than α components, and these products are formed as in *CAT*. For $\alpha = \aleph_0$, we provide an interesting counterexample to show that $\aleph_0$-*Ladj* is not closed in *CAT* under inserters.

In the process of establishing these conclusions we also prove a number of results of independent interest. For a class J of limits let *J-Comp* be the 2-category determined by all categories admitting these limits, all functors between them preserving these limits, and all natural transformations between these functors. Then *J-Comp* is closed in *CAT* under indexed limits of retract type. Under mild restrictions on the 2-category A, the sub-2-category *Radj*(A) determined by the right adjoint 1-cells, but with the same objects and 2-cells, admits splitting of idempotents if A does.

To obtain the cocontinuous analogue, in the case of locally--presentable categories, for the results of Freyd and Kelly mentioned earlier we employ a notion of *purity*. Fakir's definition of purity, given in [8], is used in Chapter 4. It is different from that used by Ulmer [28] and is a more natural extension of the usual module-theoretic equational descritpion of purity. We prove that if a full subcategory $\mathcal{B}$ of a locally α-presentable category A is closed under colimits and pure subobjects, then $\mathcal{B}$ is itself locally presentable, and hence coreflective.

If A and $\mathcal{B}$ are locally-presentable categories then, by the results in Chapter 3, *Ladj*$(A,\mathcal{B})$ is also locally presentable. In fact, as we show in Chapter 5, this defines the internal hom of a symmetric monoidal closed structure, in the appropriate 2-categorical sense, on the 2-category *Ladj*; the tensor product of A and $\mathcal{B}$ is the category *Cocont*$(A,\mathcal{B}^{op})^{op}$, the dual of the category of cocontinuous functors from A to $\mathcal{B}^{op}$.

To conclude the thesis we show how the results of Chapters 2 and 3 readily extend to locally-presentable enriched categories, provided that the base-category V is locally presentable as a symmetric monoidal category (see Kelly [18]). The 2-categories V-*Loc* consisting of locally-presentable V-categories, right adjoint V-functors and V-natural transformations and its companion V-*Ladj*, where now the 1-cells are left adjoint V-functors, are closed in V-*CAT* under the formation of indexed limits of retract type.

To prove the closure of V-*Loc* we prove that the category of algebras for a monad with rank α on a locally α-presentable

V-category is itself a locally α-presentable *V*-category. When the base-category *V* is *Set*, our proof provides an alternative to that given in Gabriel and Ulmer [10]. To prove the closure of *V-Ladj* we appeal to a characterization of locally α-presentable *V*-categories given in Kelly [18].

Chapter 0. Some introductory remarks

As the title indicates, this chapter is concerned with some introductory comments about terminology, notation and basic facts used throughout the thesis.

0.1 Some remarks about categories and 2-categories

Questions of a set-theoretic nature do not play an important role in this thesis. By a "set" we shall usually mean an element in a particular category *Set* which is a model of set theory. However, particularly in Chapter 4, we often consider "large sets" or "classes" which are elements in some larger model *SET*, and we then refer to the elements of *Set* as "small sets".

Except when dealing with functor categories, all categories are locally small. The objects of the 2-category *CAT* are locally-small categories, so that *CAT*[A,B] need not be locally small.

By a small category we mean a category which is *equivalent* to a category with a small set of objects. These are the objects of a 2-category *Cat*. Unless otherwise stated, the domain category of a functor of which we are considering the limit or colimit is small. Similarly, when we speak of a collection of objects in a category, for instance the α-presentable objects in a locally α-presentable category, as being a small set, we mean that their isomorphism classes form a small set.

Associated with any 2-category A are the 2-categories A^{op} and A^{co}. They have the same objects as A but $A^{co}(A,B) = A(A,B))^{op}$ and $A^{op}(A,B) = A(B,A)$.

For 2-categories there is a notion weaker than that of adjunction; for pseudo-functors, that is, homomorphisms of bicategories between 2-categories, $S: \mathcal{B} \to \mathcal{A}$ and $T: \mathcal{A} \to \mathcal{B}$, a *biadjunction* $S \dashv T: \mathcal{A} \to \mathcal{B}$ is given by a pseudo-natural *equivalence* $\mathcal{B}(SA,B) \simeq \mathcal{A}(A,TB)$ (see Street [24]). Note the use of the symbol "$\simeq$" for an equivalence, and "$\cong$" for an isomorphism. Similarly, two 2-categories $\mathcal{A}$ and $\mathcal{B}$ may be *biequivalent*, in which case we write $\mathcal{A} \sim \mathcal{B}$.

0.2 Monadicity

Any adjunction $F \dashv G: \mathcal{A} \to \mathcal{B}$, with unit $\eta: 1 \to GF$ and counit $\varepsilon: FG \to 1$, gives rise to a monad $(T = GF, \eta, G\varepsilon F)$ on $\mathcal{B}$. If the comparison functor $K: \mathcal{A} \to \mathcal{B}^T$ is an equivalence we say that G is *monadic*. The following version of the Beck monadicity theorem appears in Mac Lane [21], p.151. Recall that a colimit is absolute if it is preserved by all functors.

Theorem 0.1. *Given an adjunction* $F \dashv G: \mathcal{A} \to \mathcal{B}$ *the functor* G *is monadic if and only if every pair* $f,g: X \to Y$ *in* $\mathcal{B}$, *such that* Gf, Gg *have an absolute coequalizer in* $\mathcal{A}$, *has a coequalizer and* G *preserves and reflects coequalizers of such pairs.* □

0.3 Locally-presentable categories

Throughout this section α is a fixed regular cardinal. An *α-category* is a category with fewer than α morphisms, and hence with fewer than α objects. A category $\mathcal{B}$ is *α-filtered* if every functor $T: \mathcal{A} \to \mathcal{B}$ whose domain is an α-category admits a cone $\rho: T \to \Delta X$, where X is an object of $\mathcal{B}$. The colimit of a functor $T: A \to \mathcal{B}$ whose domain A is α-filtered is said to be *α-filtered*. A functor preserving α-filtered colimits is said to *have rank* α. In particular, if $\mathcal{B}$ admits α-filtered colimits and if the representable functor $\mathcal{B}(B,-): \mathcal{B} \to \mathit{Set}$ has rank α, then B is said to be an *α-presentable object* of $\mathcal{B}$.

A *locally α-presentable category* $\mathcal{B}$ is a cocomplete category with a strong generator consisting of α-presentable objects. The full subcategory of α-presentable objects is denoted by $\mathcal{B}_\alpha$; it is, in fact, small. If A is locally β-presentable for some regular cardinal β, then A is *locally presentable*. Gabriel and Ulmer [10] and Kelly [18] give a thorough account of locally-presentable categories. We shall continually use basic properties of locally-presentable categories, often without explicit comment. We list some of these basic properties in the following portmanteau theorems. The proofs of these statements may be found in Gabriel and Ulmer [10] or, for the case of enriched categories, in Kelly [18].

Theorem 0.2. *Let A be a locally α-presentable category.*

(1) *The category A is complete.*

(2) *In A, α-limits (that is, limits whose indexing category is an α-category) and α-filtered colimits commute.*

(3) *Let $\mathcal{D}$ be any strong generator consisting of α-presentable objects in A. Then A_α is the closure of $\mathcal{D}$ under α-colimits.*

(4) *For any object* A *the comma category* A_α/A *is* α*-filtered. Moreover,* A *is the colimit of the canonical functor* $A_\alpha/A \to A$. □

Theorem 0.3. *Let* A *and* B *be locally presentable.*

(1) *A functor* $T: A \to B$ *has a right adjoint if and only if it is cocontinuous (that is, preserves small colimits).*

(2) *A functor* $T: A \to B$ *has a left adjoint if and only if it is continuous and has rank* β *for some regular cardinal* β.

(3) *Moreover, if* A *and* B *are locally* α*-presentable and if* $S \dashv T: A \to B$, *then* T *has rank* α *if and only if* S *preserves* α*-presentable objects.* □

The objects of the 2-category α-*Rank* are the locally α-presentable categories, its 1-cells the functors with rank α, and its 2-cells the natural transformations between these functors. Restricting the 1-cells to those which have a left adjoint - that is, to those which are continuous and have rank α - gives the 2-category α-*Loc*. If, instead, we restrict to the 1-cells which are cocontinuous and preserve α-presentable objects, we obtain α-*Ladj*. From Theorem 0.3, we have a biequivalence $(\alpha\text{-}Loc)^{coop} \sim \alpha\text{-}Ladj$. The 2-categories *Rank*, *Loc* and *Ladj* are the unions, taken over all regular cardinals α, of the 2-categories α-*Rank*, α-*Loc* and α-*Ladj* respectively. So the objects of *Rank*, *Loc* and *Ladj* are the locally-presentable categories. The 1-cells of *Rank* are the functors having some rank, the 1-cells of *Loc* are the right-adjoint functors, and those of *Ladj* are the left-adjoint functors.

Let α-Th be the 2-category whose objects are the *α-theories*, that is, the small α-complete categories. Its 1-cells are the α-continuous functors and its 2-cells are the natural transformations between them. For each locally α-presentable category A the category $(A_\alpha)^{op}$ is an object of α-Th. Given any small α-complete category B, the full subcategory α-$Cont[B,Set]$ of $[B,Set]$ determined by the α-continuous functors is locally α-presentable.

Theorem 0.4. (Gabriel and Ulmer [10], Kelly [18]). *There is a biequivalence*

$$\alpha\text{-}Th \sim (\alpha\text{-}Loc)^{op}$$

whose action on objects is as described above.

Finally, we note a useful fact about monads on locally-presentable categories.

Theorem 0.5. (Gabriel and Ulmer [10]). *Let* (T,η,μ) *be a monad on the locally* α*-presentable category* A. *If the functor* T *has rank* α *then the category of algebras* A^T *is locally* α*-presentable.*

CHAPTER 1. Limits in 2-categories

In enriched category theory the classical notion of limit, involving universal cones, proved to be inadequate for developing results parallel to those in ordinary category theory, and accordingly indexed limits were introduced. In the case of 2-categories, that is *Cat*-categories, these limits accommodate 2-dimensional aspects; and they include such notions as the lax-limits of Gray [12]. However, many 2-categories, including most of those examined below, do not admit all of these strict indexed limits, while in a general bicategory this notion of limit does not even make sense. The limit notions appropriate to a bicategory are those which involve representation to within equivalence rather than isomorphism, and to which we refer generically by the name *bilimit*. Thus the appropriate limits which take into account the 2-dimensional aspects of a bicategory are the *indexed bilimits* (Street [24]). For the 2-categories of primary interest in this thesis the indexed bilimits can, in fact, be chosen to be *indexed pseudo-limits*.

This chapter explores how various notions of limit relate to each other and how, as ordinary limits are formed from products and equalizers, these various limits may be constructed from certain basic ones.

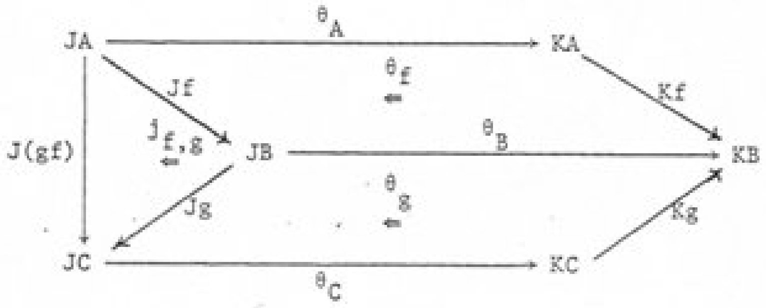
1.1 Definitions relevant to bicategories

For a background to bicategories we refer the reader to Bénabou [3], and for 2-categories to Kelly and Street [19]. If there is a need to distinguish between the two types of composition of 2-cells, horizontal composition is denoted by $\alpha.\beta$ and vertical composition by $\gamma\delta$. For convenience we write as if bicategories were 2-categories, suppressing the various 2-cells relating to associativity and identities.

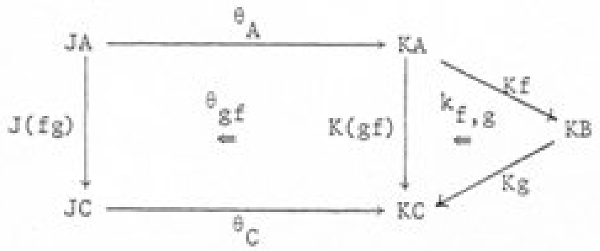
Recall that a *morphism of bicategories* $J: A \to B$ is given by a map of objects together with functors $J_{A,B}: A(A,B) \to B(JA,JB)$ for each pair of objects in A (as usual we ignore subscripts if no ambiguity is inherent), along with 2-cells $j_A: 1_{JA} \to J1_A$, indexed by the objects of A, and $j_{f,g}: J_g \cdot J_f \to J(g,f)$, indexed by the pairs of composable 1-cells in A; these are subject to appropriate "functoriality" and coherence conditions. (For details see Bénabou [3]). If the 2-cells j_A and $j_{f,g}$ are invertible then J is a *homomorphism of bicategories*. A morphism of bicategories between 2-categories is often called a *lax-functor*, and a homomorphism a *pseudo-functor*.

A *lax-natural transformation* $\theta: J \to K$ between morphisms of bicategories has data given by 1-cells $\theta_A: JA \to KA$ and 2-cells $\theta_f: Kf \cdot \theta_A \to \theta_B \cdot J_f$, indexed by the objects A and by the 1-cells $f: A \to B$ of A respectively, subject to the conditions:

(a) The composite 2-cells



and



are equal. (Note the suppression of associativity).

(b) The composite 2-cells $\theta_{1_A}(k_A \cdot \theta_A)$ and $\theta_A \cdot j_A$ are equal.

(c) For a 2-cell $\alpha\colon f \to g$ in A the composites $(\theta_B \cdot J\alpha)\theta_f$ and $\theta_g(K\alpha \cdot \theta_A)$ are equal.

If the 2-cells θ_f are invertible θ is a *pseudo-natural transformation*

A *modification* $\rho\colon \theta \to \psi\colon J \to K$ of lax-natural transformations consists of 2-cells $\rho_A\colon \theta_A \to \psi_A$ such that $(\rho_B \cdot Jf)\theta_f$ and $\psi_f(Kf \cdot \rho_A)$ are equal.

For any pair of bicategories A and B there are 2-categories $Bicat[A,B]$ and $Hom[A,B]$. The objects of $Bicat[A,B]$ are morphisms $J\colon A \to B$, its 1-cells are lax-natural transformations, and the 2-cells are modifications. For $Hom[A,B]$ the objects are homomorphisms, the 1-cells are pseudo-natural transformations and the 2-cells are modifications. The fact is, however, that lax-natural transformations are relevant even between 2-functors and so, when A and B are 2-categories, we introduce $Lax[A,B]$ and $Psd[A,B]$ which are the full sub-2-categories of $Bicat[A,B]$ and $Hom[A,B]$ respectively whose objects are 2-functors. Each of these in turn contains $[A,B]$, the 2-category of 2-functors, 2-natural transformations and modifications. In addition, we have cause to consider briefly $Oplax[A,B]$ whose 1-cells are now oplax-natural transformations (called "right natural transformations" in Street [22]).

When it is obvious that J and K are 2-functors, we often use "natural" to mean "2-natural" for $\phi\colon J \to K$.

1.2 Limit notions

Although many of the definitions we give below are relevant, with perhaps slight alteration, to bicategories, it is sufficient for our purposes to consider only 2-categories.

Since 2-categories are categories enriched over *Cat* we have the usual notion of indexed limit (see Kelly [17]). For a small 2-category A and 2-functors $G\colon A \to B$ and $F\colon A \to Cat$ the *F-indexed limit of* G is the representing object $\{F,G\}$ in the 2-natural isomorphism

$$B(B,\{F,G\}) \cong [A,Cat](F,B(B,G-)) \tag{1.1}$$

Similarly one can introduce the *indexed pseudo-limit* $\{F,G\}_{psd}$ and the *indexed lax-limit* $\{F,G\}_{lax}$, namely as the representing objects for the 2-natural isomorphisms

$$B(B,\{F,G\}_{psd}) \cong Psd[A,Cat](F,B(B,G-)) \tag{1.2}$$

and

$$B(B,\{F,G\}_{lax}) \cong Lax[A,Cat](F,B(B,G-)) \tag{1.3}$$

respectively. If a 2-category B admits all small indexed limits we say it is *complete*. Likewise, if B admits all small indexed pseudo-limits it is *pseudo-complete*, and it is *lax-complete* when it admits all indexed lax-limits. The representing object $\{F,G\}_{oplax}$ for the natural isomorphism

$$B(B,\{F,G\}_{oplax}) \cong Oplax[A\ Cat](F,B(B,G-)) \tag{1.4}$$

is called the *indexed oplax-limit*.

Indexed pseudo-limits in $\mathcal{B}$ are related to those in $\mathcal{B}^{co}$, the 2-category obtained from $\mathcal{B}$ by reversing the sense of the 2-cells. (The 2-category $\mathcal{B}^{op}$ is obtained by reversing the sense of the 1-cells). One of the distinguishing properties of Cat is the duality $D = (\ )^{op}: Cat^{co} \to Cat$ which assigns to each category A the dual category A^{op}. Hence to any 2-functor $J: A \to Cat$ we may associate the composite 2-functor $J^{\#} = DJ^{co}: A^{co} \to Cat$.

Proposition 1.5. *Let* $J: A \to Cat$ *and* $S: A \to \mathcal{B}$ *be 2-functors. Then*

(1) $\{J,S\} \cong \{J^{\#},S^{co}\}$,

(2) $\{J,S\}_{psd} \cong \{J^{\#},S^{co}\}_{psd}$,

and

(3) $\{J,S\}_{lax} \cong \{J^{\#},S^{co}\}_{oplax}$.

Proof. Note that when we state that two limits are isomorphic it is always implied that one of them exists if and only if the other does.

(1) The indexed limit $\{J,S\}$, if it exists, is the representing object for the natural isomorphism

$$[A,Cat](J,\mathcal{B}(B,S-)) \cong \mathcal{B}(B,\{J,S\})$$

$$= (\mathcal{B}^{co}(B,\{J,S\}))^{op}$$

and $\{J^{\#},S^{co}\}$ is the representing object for

$$[A^{co},Cat](J^{\#},\mathcal{B}^{co}(B,S^{co}-)) \cong \mathcal{B}^{co}(B,\{J^{\#},S^{co}\}) .$$

For $J,T\colon A \to Cat$ the duality above yields

$$[A,Cat](J,T) \cong [A^{co},Cat]^{co}(J^{\#},T^{\#})$$

$$= ([A^{co},Cat](J^{\#},T^{\#}))^{op} .$$

Taking $T = \mathcal{B}(B,S-)\colon A \to Cat$ then $T^{\#} = \mathcal{B}^{co}(B,S^{co}-)\colon A^{co} \to Cat$. Combining these observations gives $\{J,S\} \cong \{J^{\#},S^{co}\}$.

(2) and (3) are proved similarly, noting that $Psd[A,Cat] \cong (Psd[A^{co},Cat])^{co}$ and $Lax[A,Cat] \cong (Oplax[A^{co},Cat])^{co}$. □

Thus $\mathcal{B}$ is pseudo-complete if and only if $\mathcal{B}^{co}$ is.

Street [24] introduces the more general notion of indexed limit appropriate for a bicategory, namely the *indexed bilimit* $\{F,G\}_{bi}$, which is the representing object for the *equivalence*

$$\mathcal{B}(B,\{F,G\}_{bi}) \simeq Hom[A,Cat](F,\mathcal{B}(B,G-)) \qquad (1.6)$$

when $F\colon A \to Cat$ and $G\colon A \to \mathcal{B}$ are homomorphisms of bicategories. (Note the use of $\cong$ for an isomorphism and $\simeq$ for an equivalence). When F and G are 2-functors, we may replace $Hom[A,Cat]$ in (1.6) by $Psd[A,Cat]$, so that $\{F,G\}_{psd}$, if it exists, is an indexed bilimit $\{F,G\}_{bi}$; however $\{F,G\}_{bi}$ may well exist with no choice of the representing object rendering (1.6) an actual isomorphism.

The diagonal 2-functor $\Delta\colon \mathcal{B} \to [A,\mathcal{B}]$ assigns to $B \in \mathcal{B}$ the 2-functor $\Delta B\colon A \to \mathcal{B}$ which is constant at B. For the special case where the indexing type is $\Delta 1\colon A \to Cat$ we set

$$\lim G = \{\Delta 1, G\},$$

$$\text{psdlim } G = \{\Delta 1, G\}_{psd} ,$$

$$\text{laxlim } G = \{\Delta 1, G\}_{lax} ,$$

$$\text{oplaxlim } G = \{\Delta 1, G\}_{oplax} .$$

Thus $\lim G$ is the representing object in

$$\mathcal{B}(B, \lim G) \cong [A, \mathcal{Cat}](\Delta B, G) \tag{1.7}$$

and so $\mathcal{B}(B, \lim G)$ is the category of cones over G with vertex B. Similarly psdlim G and laxlim G represent the pseudo-cones and lax-cones over G.

When $\mathcal{B} = \mathcal{Cat}$ we have

$$\{F, G\}_{lax} = \mathcal{L}ax[A, \mathcal{Cat}](F, G) , \tag{1.8}$$

and corresponding expressions for the F-indexed pseudo-limit and F-indexed limit.

Accordingly, the representable 2-functors $\mathcal{B}(B,-): \mathcal{B} \to \mathcal{Cat}$ preserve, and jointly reflect, all the types of limits mentioned above. Thus, for example, $\mathcal{B}(B, \text{laxlim } G) \cong \text{laxlim } \mathcal{B}(B, G-)$. It follows easily that if $\mathcal{B}$ admits F-indexed lax-limits, then $[K, \mathcal{B}]$ admits them and they are formed pointwise.

Dual to these notions of limit are the corresponding notions of colimit. For 2-functors $F: A^{op} \to \mathcal{Cat}$ and $G: A \to \mathcal{B}$ the *indexed lax-colimit* $F \star_{lax} G$ is the representing object in

$$\mathcal{B}(F \star_{lax} G, B) \cong \mathcal{L}ax[A^{op}, \mathcal{Cat}](F, \mathcal{B}(G-, B)) . \tag{1.9}$$

Similarly we have the *indexed colimit* $F * G$, the *indexed oplax-colimit* $F \underset{psd}{*} G$ and the *indexed oplax-colimit* $F \underset{oplax}{*} G$. Again, specializing to the case of $F = \Delta 1: A^{op} \to Cat$ gives the *colimit, pseudo-colimit, lax-colimit* and *oplax-colimit*, namely $\operatorname{colim} G = \Delta 1 * G$, $\operatorname{psdcolim} G = \Delta 1 \underset{psd}{*} G$, $\operatorname{laxcolim} G = \Delta 1 \underset{lax}{*} G$ and $\operatorname{oplaxcolim} G = \Delta 1 \underset{oplax}{*} G$. Note that some authors call the lax-colimit by the name "oplax-colimit", since

$$B(\operatorname{laxcolim} G, B) \cong Oplax[A,B](G,\Delta B).$$

Proposition 1.10. *For a 2-functor* $G: A \to B$ *we have a natural isomorphism*

$$B(\operatorname{laxcolim} G, B) \cong \operatorname{laxlim} B(G-,B) ,$$

the lax-colimit existing precisely when the right-hand side is representable in B. □

1.3 Examples of indexed limits

Throughout the thesis we shall be continually referring to particular indexed limits from which we can construct, for instance, indexed pseudo--limits. We present these as illustrations.

(a) *Products.* If A is a set, considered as a discrete and locally--discrete 2-category, then a 2-functor $G: A \to B$ is a set of objects of B, indexed by A. The limit of G is the usual product $\prod GA$. Note, however, that $B(B,\prod GA) \cong \prod B(B,GA)$ is required to be an isomorphism of categories, and not merely of the underlying sets.

(b) *Cotensor products.* When $A = 1$ is the terminal 2-category with one object $*$, a 2-functor $G: 1 \to B$ is given completely by the object $X = G(*)$. By abuse of notation we shall write X for the 2-functor. For $X: 1 \to B$ and $F: 1 \to Cat$ the indexed limit is the cotensor product $F \pitchfork X$, which is the representing object from the isomorphism $B(B, F \pitchfork X) \cong Cat(F, B(B,X))$.

When the 2-category A has only identity 1-cells the 2-categories $[A, Cat]$, $Psd[A, Cat]$ and $Lax[A, Cat]$ are identical. Hence the notions of product, pseudo-product and lax-product coincide, as do those of cotensor product, cotensor pseudo-product and cotensor lax-product.

(c) *Inserters.* The *inserter*, also called the subequalizer by Lambek [20], was originally defined as a "Cartesian quasi-limit" (see Gray [12]). However, using Street [23] we prefer to define the inserter as an indexed limit.

Let C be the category, considered as a 2-category, consisting of two objects A and B and two non-identity morphisms $h,k: A \to B$. Thus a 2-functor $G: C \to B$ is a pair of parallel 1-cells Gh and Gk in B. The indexing type for the inserter is $L: C \to Cat$ where $LA = 1$, $LB = 2$, Lh sends the unique object of 1 to the initial object of 2 and Lk sends it to the terminal object. The inserter $Ins(Gh/Gk) = \{L,G\}$ is equipped with a 1-cell $f: \{L,G\} \to GA$ and a 2-cell $\lambda: (Gh)f \to (Gh)f$ which are universal with respect to "inserting" a 2-cell from Gh to Gk.

The inserter $Ins(F/G)$ in Cat, for the pair of functors $F,G: F \to G$, is readily described. It is the category H whose objects are pairs (A,ϕ), where $A \in F$ and $\phi: FA \to GA$, and whose morphisms $\bar{g}: (A,\phi) \to (B,\psi)$

are morphisms $g: A \to B$ of F such that $(Gg)\phi = \psi(Fg)$. The associated functor $J: H \to F$ is given by $J(A,\phi) = A$ and $J\bar{g} = g$, and the associated natural transformation $\lambda: FJ \to GJ$ has ϕ as its (A,ϕ)-component. Note that J is conservative.

(d) *Equifiers.* Let $\mathcal{D}$ be the 2-category having the same underlying category as $\mathcal{C}$ above and with two extra 2-cells $\sigma,\rho: h \to k$. Thus a 2-functor $G: \mathcal{D} \to \mathcal{B}$ is a parallel pair of 2-cells $G\sigma$ and $G\rho$ in $\mathcal{B}$. Let $M: \mathcal{D} \to \mathcal{C}at$ have the same underlying functor as L, so that $M\sigma$ and $M\rho$ are uniquely determined. The *equifier* $\mathrm{Equif}(G\sigma,G\rho)$ of the pair $G\sigma$ and $G\rho$ is the indexed limit $\{M,G\}$. This indexed limit is endowed with a 1-cell $f: \{M,G\} \to GA$, which is universal with respect to "equifying" $G\sigma$ and $G\rho$, that is $(G\rho)f = (G\sigma)f$.

Given $\lambda,\mu,\lambda',\mu': f \to g: X \to Y$ in $\mathcal{B}$, we can equify *both pairs* λ,μ *and* λ',μ' by first forming the equifier $j: Z \to X$ of λ and μ and then the equifier $k: W \to Z$ of $\lambda' j$ and $\mu' j$.

Again, in $\mathcal{C}at$, the equifier has a simple description. For $\lambda,\mu: F \to G: F \to G$ the equifier is the full subcategory H of F consisting of the objects X such that $\lambda_X = \mu_X$, and the 1-cell associated with the equifier is the inclusion $J: H \to F$. Note that J is once again conservative (being, in fact, fully faithful).

In Chapter 2 we shall see how equifiers and inserters may be used in constructing locally-presentable categories.

(e) *Inverters.* Let $\mathcal{D}$ again have the same underlying category as $\mathcal{C}$, but now with only one extra 2-cell $\sigma: h \to k$. So a 2-functor $G: \mathcal{D} \to \mathcal{B}$

is a 2-cell $G\sigma$ in $\mathcal{B}$. Let $\mathcal{I}so$ be the category with two objects C and D and two non-identity morphisms $c: C \to D$ and $d: D \to C$. (Hence c is the inverse of d). The indexing type for the *inverter* is $F: \mathcal{D} \to \mathcal{C}at$ with $FA = 1$, $FB = \mathcal{I}so$, Fh sending the unique object of 1 to C and Fk sending it to D. Thus, for $\{F,G\} = \mathrm{Inv}(G\sigma)$ there is a 1-cell $f: \{F,G\} \to GA$ such that $(G\sigma)f$ is invertible and f is universal with this property. Dually we have the *coinverter* of a 2-cell.

If $\mathcal{B}$ admits equifiers and inserters then it also admits inverters. To form the inverter of the 2-cell $\mu: f \to g: X \to Y$ in $\mathcal{B}$ first form the inserter $Z = \mathrm{Ins}(g/f)$ with the associated 1-cell $m: Z \to X$ and the associated 2-cell $\lambda: gm \to fm$. Equifying both pairs $\lambda(\mu.m)$, 1_{fm} and $(\mu.m)\lambda$, 1_{gm} gives the inverter.

The inverter in $\mathcal{C}at$ of a natural transformation $\rho: H \to K: \mathcal{F} \to \mathcal{G}$ is the full subcategory of $\mathcal{F}$ given by the objects X such that ρ_X is an isomorphism. The coinverter of ρ is the category of fractions $\mathcal{G}[\Sigma^{-1}]$ where Σ is the set of all components ρ_X of the natural transformation.

(f) *Iso-inserters*. Let $\mathcal{C}$ be the category mentioned above in connection with describing the inserter. Let $L: \mathcal{C} \to \mathcal{C}at$ now have the same underlying functor as that for the indexing type of the inverter. The *iso-inserter* of Gh and Gk, where $G: \mathcal{C} \to \mathcal{B}$, is $\{L,G\}$. Associated with it are a 1-cell $f: \{L,G\} \to GA$ and an invertible 2-cell $\lambda: (Gh)f \to (Gk)f$, which are universal with respect to inserting an invertible 2-cell from Gh to Gk.

Again the iso-inserter may be formed using equifiers and inserters; first insert a 2-cell and then invert it.

We may use inserters and equifiers to construct *objects of algebras*. Let $T = (t,\eta,\mu)$ be a monad on the object B of a 2-category $\mathcal{B}$. A T-*algebra* is a 1-cell $s: A \to B$ and an *action* of t on s, that is a 2-cell $\nu: ts \to s$ such that $\nu(\eta.s) = 1$ and $\nu(t.\nu) = \nu(\mu.s)$. The object of T-algebras B^T is the universal such T-algebra. If $\mathcal{B}$ admits inserters and equifiers this may be formed by taking the inserter $X = \mathrm{Ins}(t/1)$ with associated 1-cell $r: X \to B$ and associated 2-cell $\rho: tr \to r$, and then jointly equifying the pairs $\rho(\eta.r)$, 1 and $\rho(t.\rho)$, $\rho(\mu.r)$. (See Kelly and Street [19] for further details). Note that for any adjunction $f \dashv g: C \to A$ inducing the monad T on A there is a unique comparison 1-cell $C \to A^T$ with the usual properties.

1.4 The Grothendieck constructions

Comma categories have 2-categorical analogues; they are called *lax-comma categories* by Gray [12]. For a pair of 2-functors $G: \mathcal{A} \to \mathcal{B}$ and $H: \mathcal{C} \to \mathcal{B}$ the objects of $G//H$ are of the form (A,h,C), where $A \in \mathcal{A}$, $C \in \mathcal{C}$ and $h: GA \to HC$. A 1-cell $(f,\phi,g): (A,h,C) \to (A',h',C')$ of $G//H$ has $f: A \to A'$, $g: C \to C'$ and $\phi: (Hg)h \to h'(Gf)$, and a 2-cell $(\alpha,\beta): (f,\phi,g) \to (f',\phi',g')$ is a pair of 2-cells $\alpha: f \to f'$ and $\beta: g \to g'$ satisfying $(h'G(\alpha))\phi = \phi'(H(\beta)h)$. There are 2-functors $d_o: G//H \to \mathcal{A}$ and $d_1: G//H \to \mathcal{B}$ and a lax-natural transformation $\tau: Gd_o \to Hd_1$, with the universal property given in Kelly [15].

The objects of the *oplax-comma category* $G\backslash\backslash H$ are again of the form (A,h,C), where $A \in \mathcal{A}$, $C \in \mathcal{C}$ and $h: GA \to HC$, but now a 1-cell $(f,\phi,g): (A,h,C) \to (A',h',C')$ has $f: A \to A'$, $g: C \to C'$ and $\phi: h'(Gf) \to (Hg)h$. A 2-cell $(\alpha,\beta): (f,\phi,g) \to (f',\phi,g)$ is a pair of

2-cells $\alpha: f \to f'$ and $\beta: g \to g'$ satisfying $(H(\beta)h)\phi = \phi'(h'G(\alpha))$. There are 2-functors $k_0: G\backslash\backslash H \to A$ and $k_1: G\backslash\backslash H \to C$ and an oplax-natural transformation $\sigma: Gk_0 \to Hk_1$, with a universal property similar to that for the lax-comma category.

Our interest lies in a particular instance of the lax-comma category, the *Grothendieck lax-construction* $E\ell F = \Delta 1//F$ for $F: A \to \mathcal{C}at$ and $\Delta 1: 1 \to \mathcal{C}at$. Adopting a convenient notation, the objects of $E\ell F$ are of the form (A,x) where $A \in A$ and $x: 1 \to FA$ is an object of FA. A 1-cell $(f,\phi): (A,x) \to (B,y)$ has $f: A \to B$ and $\phi: (Ff)x \to y$. The 2-cells of $E\ell F$ have the form $\lambda: (\phi,f) \to (\theta,g)$ where $\lambda: f \to g$ is a 2-cell of A for which $\phi((Ff)x) = \theta$. Associated with $E\ell F$ are the canonical projection $d_F: E\ell F \to A$ sending (A,x) to A, (f,ϕ) to f and λ to λ, and the lax-natural transformation

$$\begin{array}{ccc} E\ell F & \xrightarrow{\quad d_F \quad} & A \\ {\scriptstyle !}\downarrow & \overset{\tau}{\Rightarrow} & \downarrow{\scriptstyle F} \\ 1 & \xrightarrow[\quad \Delta 1 \quad]{} & \mathcal{C}at \end{array}$$

with $\tau_{(A,x)} = x$ and $\tau_{(f,\phi)} = \phi$. The universal property here becomes:

Proposition 1.11. (Kelly [15]). *For a 2-functor* $F: A \to \mathcal{C}at$ *and a 2-category* C *there is a bijection between 2-functors* $G: C \to E\ell F$ *and pairs* (H,α) *consisting of a 2-functor* $H: C \to A$ *and a lax-natural transformation* $\alpha: \Delta 1 \to FH$. *Under the bijection* $\alpha = \tau G$ *and* $H = d_F G$. *If* $G': C \to E\ell F$ *corresponds to* (H',α') *then there is a bijection between 2-natural transformations* $\gamma: G \to G'$ *and pairs* (ρ,μ) *consisting of a 2-natural transformation* $\rho: H \to H'$ *and a modification* $\mu: (F\rho)\alpha \to \alpha'$, *given by* $\rho = d_F\gamma$ *and* $\mu_C = \tau_{\gamma_C}$. □

When necessary the lax-natural transformation $\tau: \Delta 1 \to Fd_F$ associated with the 2-functor $F: A \to Cat$ is denoted by τ_F.

For a lax-natural transformation $\gamma: F \to G: A \to Cat$ the associated lax-natural transformation $(\gamma d_F)\tau_F$ must be of the form $\tau_G T$, where $d_G T = d_F$, for a unique 2-functor $T: E\ell F \to E\ell G$. We denote T by d_γ. Using the 2-dimensional aspect of the universal property a modification $\xi: \gamma \to \beta: F \to G$ gives a modification $\tau_G d_\gamma \to \tau_G d_\beta$, and hence a 2-natural transformation $d_\xi: d_\gamma \to d_\beta$ such that $d_G d_\xi = 1$. It is easily checked that these definitions give a 2-functor $d: Lax[A,Cat] \to 2\text{-}Cat/A$. For instance, if $\gamma: F \to G$ and $\mu: G \to H: A \to Cat$ are lax-natural transformations then $\tau_H d_\mu d_\gamma = ((\mu.d_G)\tau_G)d_\gamma = ((\mu\gamma).d_F)\tau_F$ and $d_H d_\mu d_\gamma = d_G d_\gamma = d_F$, implying $d_\mu d_\gamma = d_{\mu\gamma}$.

The *Grothendieck oplax-construction*, for a 2-functor $F: A \to Cat$, is the 2-category $GrF = \Delta 1 \backslash\backslash F$ with the projection $k_F: GrF \to A$ and the oplax-natural transformation $\sigma_F: \Delta 1 \to Fk_F$. As with the Grothendieck lax-construction, k has an obvious extension to a 2-functor $k: Oplax[A,Cat] \to (2\text{-}Cat/A)^{co}$.

An element A of a 2-category A may be considered as a 2-functor $A: 1 \to A$. For a 2-functor $H: C \to A$ we have

Proposition 1.12. *The 2-categories* $(H//A)^{op}$ *and* $Gr(A(H-,A))$ *are isomorphic.*

Proof. Note that $A(H-,A): C^{op} \to Cat$. The 1-cell of C^{op} corresponding to the 1-cell $g: C \to D$ of C is denoted by $\bar{g}: D \to C$.

Let $(c,x) \in Gr(A(H-,A))$, where $C \in C^{op}$ and $x\colon 1 \to A(HC,A)$. Thus $(C,x,*) \in (H//A)^{op}$, where $*$ is the unique object of the terminal 2-category 1. A 1-cell $(\bar{f},\phi)\colon (C,x) \to (C',x')$ of $Gr(A(H-,A))$ is a 1-cell $\bar{f}\colon C \to C'$ of C^{op} and a 2-cell $\phi\colon x' \to A(Hf,A)x$, that is, a 1-cell $f\colon C' \to C$ of C and a 2-cell $\phi\colon x' \to x(Hf)$. These are precisely the data for a 1-cell $(f,\phi,1_*)\colon (C',c',*) \to (C,x,*)$ of $H//A$. The 2-cell $\alpha\colon (\bar{f},\phi) \to (\bar{f}',\phi')$ of $Gr(A(H-,A))$ gives the 2-cell $(\alpha,1)\colon (f,\phi,1_*) \to (f',\phi',1_*)$ of $(H//A)^{op}$. □

Note the construction $H//A$ is 2-functorial in both A and H, in an evident way, and it is easy to verify the 2-naturality of the isomorphism here.

For 2-functors $H\colon C \to A$ and $F\colon A \to \mathcal{C}at$ the universal property of Proposition 1.11 yields an isomorphism

$$\pi\colon 2\text{-}\mathcal{C}at/A(H,dF) \to \mathcal{L}ax[C,\mathcal{C}at](\Delta 1, FH)$$

given by composition with τ_F; that is, $\pi(T) = \tau_F T$ for $T\colon C \to \mathcal{E}\ell F$ such that $d_F T = H$, and $\pi(\lambda) = (\tau_F)\lambda$ for $\lambda\colon T \to T'$ such that $d_F\lambda = 1_H$. Thus

Proposition 1.13. *For 2-functors* $H\colon C \to A$ *and* $F\colon A \to \mathcal{C}at$, *we have*

$$2\text{-}\mathcal{C}at/A(H,d_F) \cong \text{laxlim}(FH).$$

Proof. Recall that $\text{laxlim}\,(FH) \cong \mathcal{L}ax[C,\mathcal{C}at](\Delta 1, FH)$. □

The pasting composition with τ_F gives, for 2-functors $F,G\colon A \to \mathcal{C}at$, a functor

$$p\colon \mathcal{L}ax[A,\mathcal{C}at](F,G) \to \mathcal{L}ax[\mathcal{E}\ell F,\mathcal{C}at](\Delta 1, Gd_F)\ ;$$

if $\lambda: F \to G$ is a lax-natural transformation then $p(\lambda) = (\lambda d_F)\tau_F$, and if $\xi: \lambda \to \lambda'$ is a modification then $p(\xi) = (\xi d_F)\tau_F$.

Proposition 1.14. *The functor* p *is an isomorphism.*

Proof. We give the inverse $q : Lax[E\ell F, Cat](\Delta 1, Gd_F) \to Lax\ A, Cat\ (F,G)$. For a lax-cone $\beta: \Delta 1 \to Gd_F$ the lax-natural transformation $\alpha = q(\beta): F \to G$ has A-component $\alpha_A: FA \to GA$ such that $\alpha_A(x) = \beta_{(A,x)}$ for $x \in FA$ and $\alpha_A(\phi) = \beta_{(1,\phi)}$ for a morphism $\phi: x \to x'$ in FA; for a 1-cell $f: A \to B$ of A the natural transformation α_f has x-component $(\alpha_f)_x = \beta_{(f,1)}$, where $(f,1): (A,x) \to (B,(Ff)x)$. For a modification $\xi: \beta \to \beta'$ the A-component of the modification $q(\xi): \alpha \to \alpha'$ is the natural transformation $\rho: \alpha_A \to \alpha'_A$ whose x-component is $\xi_{(A,x)}: \beta_{(A,x)} \to \beta'_{(A,x)}$. □

Thus indexed lax-limits in Cat reduce to lax-limits, since $\{F,G\}_{lax} \cong \text{laxlim}(Gd_F)$ from the proposition above; and, since the definition of limit is representable, this isomorphism holds even when the codomain of G is an arbitrary 2-category B admitting either (and hence both) of the limits.

Propositions 1.13 and 1.15 together give

Theorem 1.15. *The 2-functor* $d: Lax[A, Cat] \to 2\text{-}Cat/A$ *is fully faithful.*

Proof. For 2-functors $F,G: A \to Cat$ we have, as already noticed, isomorphisms $p: Lax[A,Cat](F,G) \to Lax[E\ell F, Cat](\Delta 1, Gd_F)$ and $\pi: 2\text{-}Cat/A(d_F, d_G) \to Lax[E\ell F, Cat](\Delta 1, Gd_F)$. But p is the composite of π

and $d_{F,G}: Lax[A,Cat](F,G) \to 2\text{-}Cat/A(d_F,d_G)$. So $d_{F,G}$ is an isomorphism, as required. □

Similarly, it may be shown that

Theorem 1.16. *The 2-functor* $k: Oplax[A,Cat] \to (2\text{-}Cat/A)^{co}$ *is fully faithful.* □

1.5 Lax-colimits in *Cat*

The functor $\pi: Cat \to Set$ assigns to each category its set of connected components; it is the left adjoint of $D: Set \to Cat$, which sends each set to the corresponding discrete category. Then (see Kelly [17]), for an ordinary functor $F: A \to Set$, the colimit of F is given by colim $F = \pi(\mathit{el}F)$. A similar formula is available for the lax-colimit of a 2-functor $F: A \to Cat$.

The functor $D: Set \to Cat$ induces a 2-functor $D_*: Cat \to 2\text{-}Cat$, given by treating each category as a locally-discrete 2-category (see Eilenberg and Kelly [7]). Since D_* is a full embedding it is often suppressed, so that B may denote, depending on the context, a category or its corresponding locally-discrete 2-category $D_*(B)$. The 2-functor D_* also has a left adjoint $\pi_*: 2\text{-}Cat \to Cat$, where $\pi_*(A)$ and A have the same objects and $(\pi_*(A))(A,B) = \pi(A(A,B))$.

Proposition 1.17. *For a 2-functor* $F: A \to Cat$ *we have a 2-natural isomorphism*

$$\text{laxcolim } F \cong \pi_*(G\wr F)^{op} .$$

Proof. By Theorem 1.16, the 2-functor $k: \mathit{Oplax}[A,\mathcal{C}at] \to (2\text{-}\mathcal{C}at/A)^{co}$ is fully faithful. Thus (see Section 1.2)

$$\begin{aligned} \mathcal{C}at(\mathrm{laxcolim}\ F,B) &\cong \mathit{Oplax}[A,\mathcal{C}at](F,\Delta B) \\ &\cong (2\text{-}\mathcal{C}at/A)^{co}(k_F,k_{\Delta B}) \\ &= ((2\text{-}\mathcal{C}at/A)(k_F,k_{\Delta B}))^{op} . \end{aligned}$$

The Grothendieck oplax-construction $k_{\Delta B}: Gr(\Delta B) \to A$ from $\Delta B: A \to \mathcal{C}at$ is isomorphic to $pr_1: A \times B^{op} \to A$. So

$$\begin{aligned} 2\text{-}\mathcal{C}at/A(k_F,k_{\Delta B}) &\cong 2\text{-}\mathcal{C}at(GrF,B^{op}) \\ &\cong \mathcal{C}at(\pi_*(GrF),B^{op}) . \end{aligned}$$

Thus

$$\mathcal{C}at(\mathrm{laxcolim}\ F,B) = \mathcal{C}at(\pi_*(GrF)^{op},B) ,$$

the isomorphism being 2-natural in both B and F. □

1.6 The left adjoint to the inclusion $[A,\mathcal{C}at] \to \mathit{Lax}[A,\mathcal{C}at]$

Since $d: \mathit{Lax}[A,\mathcal{C}at] \to 2\text{-}\mathcal{C}at/A$ is fully faithful we have a left adjoint to the inclusion $J: [A,\mathcal{C}at] \to \mathit{Lax}[A,\mathcal{C}at]$ if we have a left adjoint to the composite $H = dJ: [A,\mathcal{C}at] \to 2\text{-}\mathcal{C}at/A$.

Theorem 1.18. (1) *The 2-functor* $\Psi: 2\text{-}\mathcal{C}at/A \to [A,\mathcal{C}at]$ *such that* $\Psi(G) = \pi_*(G//-)$ *is left adjoint to* $H: [A,\mathcal{C}at] \to 2\text{-}\mathcal{C}at/A$.

(2) *The inclusion* $J: [A, Cat] \to Lax[A, Cat]$ *has a left adjoint* $(\ )^{\dagger}: Lax[A, Cat] \to [A, Cat]$ *such that* $F^{\dagger} = \pi_*(d_F//-)$.

Proof. (1) Proposition 1.13 established that

$$2\text{-}Cat/A(G, H(F)) = 2\text{-}Cat/A(G, d_F) \cong \text{laxlim}(FG) ,$$

for 2-functors $G: C \to A$ and $F: A \to Cat$. But, by the 2-categorical Yoneda lemma,

$$\begin{aligned}(FG)C &\cong [A, Cat](A(GC,-)F) \\ &= [A, Cat](\overline{G}C, F) ,\end{aligned}$$

where $\overline{G} = YG^{op}: C^{op} \to [A, Cat]$ and $Y: A^{op} \to [A, Cat]$ is the Yoneda embedding. Hence

$$\begin{aligned}\text{laxlim}(FG) &\cong \text{laxlim}[A, Cat](\overline{G}-, F) \\ &\cong [A, Cat](\text{laxcolim}\ \overline{G}, F) ,\end{aligned}$$

by Proposition 1.10. But by Proposition 1.12 and 1.17,

$$\begin{aligned}\text{laxcolim}\ \overline{G} &= \pi_*(G \wr \overline{G})^{op} \\ &= \pi_*(G//-) \\ &= \Psi(G) .\end{aligned}$$

Hence

$$2\text{-}Cat/A(G, H(F)) \cong [A, Cat](\Psi(G), F) ,$$

the isomorphism being 2-natural in G and F.

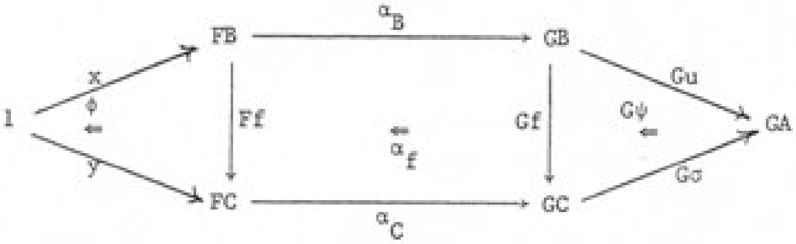
(2) follows immediately. □

In later sections it will prove convenient to have a more explicit description of the isomorphism

$$Lax[A,Cat](F,G) \cong [A,Cat](F^{\dagger},G) \tag{1.19}$$

For $F: A \to Cat$ and $A \in A$ the objects of $d_F//A$ are triples (x,B,u), where $B \in A$, $x: 1 \to FB$ is an object of FB and $u: B \to A$ is a 1-cell of A. A 1-cell $(\phi,f,\psi): (x,B,u) \to (y,c,v)$ is a triple such that $f: B \to C$, $\phi: (Ff)x \to y$ and $\psi: u \to vf$.

The image of a lax-natural transformation $\alpha: F \to G$ under the isomorphism (1.19) is the 2-natural transformation $\beta: F^{\dagger} \to G$; the A-component $\beta_A: \pi_*(d_F//A) \to GA$ sends (x,B,u) to $Gu(\alpha_\beta(x))$ and sends $[(\phi,f,\psi)]$, the equivalence class represented by (ϕ,f,ψ), to the composite



(1.20)

1.7 The left adjoint to the inclusion $[A,Cat] \to Psd[A,Cat]$

To describe the left adjoint to the inclusion $K: [A,Cat] \to Psd[A,Cat]$ we shall specify those elements of $[A,Cat](F^{\dagger},G)$ which are the images of *pseudo*-natural transformations under the isomorphism (1.19). From the explicit description of the image $\beta: F^{\dagger} \to G$ of the lax-natural transformation $\alpha: F \to G$ it follows that each α_f is invertible if and only if each $\beta_A: F^{\dagger}A \to GA$ inverts those 1-cells $[(\phi,f,\psi)]$ for which

ϕ and ψ are invertible. This set of 1-cells will be denoted by F_cA and be regarded as a discrete category. If ϕ, ψ and α_f are invertible then so is the composite (1.20). If each $\beta_A: F^\dagger A \to GA$ inverts the 1-cells in F_cA then, taking $\phi = 1$, $C = A$, $v = 1$ and $\psi = 1$, we see that each natural transformation α_f is invertible.

There are obvious functors $\partial_o A, \partial_1 A: F_cA \to F^\dagger A$ such that $(\partial_o A)(\xi) = \mathrm{dom}(\xi)$ and $(\partial_1 A)(\xi) = \mathrm{codom}(\xi)$. The natural transformations $\rho_A: \partial_o A \to \partial_1 A$ such that $(\rho_A)_\xi = \xi$ are the 2-cells for a modification $\rho: \partial_o \to \partial_1: F_c \to F^\dagger: A \to \mathcal{C}at$. Let $F'A$ be the category of fractions of $F^\dagger A$ giving the coinverter of ρ_A. In fact $F^\dagger A$ is a value of a 2-functor $(\ )^\dagger(\ ): \mathcal{L}ax[A,\mathcal{C}at] \times A \to \mathcal{C}at$, while F_cA is a value of a 2-functor $(\ )_c(\ ): \mathcal{P}sd[A,\mathcal{C}at] \times A \to \mathcal{C}at$. By restricting the first of these to $\mathcal{P}sd[A,\mathcal{C}at] \times A$, ∂_o and ∂_1 become 2-natural transformations between these 2-functors, with ρ as a modification. So $(\ )'$ becomes a 2-functor $(\ )': \mathcal{P}sd[A,\mathcal{C}at] \to [A,\mathcal{C}at]$

Theorem 1.21. *The inclusion* $K: [A,\mathcal{C}at] \to \mathcal{P}sd[A,\mathcal{C}at]$ *has a left adjoint* $(\ )'$, *where* $F'A$ *is as described above.*

Proof. The coinverter of ρ is $t: F^\dagger \to F'$. So, for each G, the induced functor $[A,\mathcal{C}at](t,G): [A,\mathcal{C}at](F',G) \to [A,\mathcal{C}at](F^\dagger,G)$ is the inverter of $[A,\mathcal{C}at](\rho,G)$. But, from the observations above, this inverter is precisely the full subcategory of $[A,\mathcal{C}at](F^\dagger,G)$ whose objects are the images of the pseudo-natural transformations under (1.19). So $[A,\mathcal{C}at](F',G) \cong \mathcal{P}sd[A,\mathcal{C}at](F,G)$, the isomorphism being 2-natural in F and G.

In the remainder of this chapter a pseudo-natural transformation will be denoted by $F \rightsquigarrow G$, while the notation $F \to G$ will be reserved for a 2-natural transformation.

We henceforth use $\eta_F \colon F \rightsquigarrow F'$ for the unit of the adjunction in Theorem 1.21 and $\varepsilon_G \colon G' \to G$ for the counit. The unit has the property that any pseudo-natural transformation $s \colon F \rightsquigarrow G$ factorizes as $t\eta_F$ for a unique 2-natural transformation $t : F' \to G$, and the counit has the property that any 2-natural transformation $f \colon F' \to G$ is $\varepsilon_G g'$ for a unique pseudo-natural transformation $g \colon F \rightsquigarrow G$.

Note that, in general, there is no left adjoint to the inclusion $\mathcal{Psd}[A,\mathcal{Cat}] \to \mathcal{Lax}[A,\mathcal{Cat}]$. This inclusion does, however, have a biadjoint, namely the composite $K(\ )^{\dagger}$. The counit $\varepsilon_{F^{\dagger}} \colon F^{\dagger\prime} \to F^{\dagger}$ is an equivalence in $[A,\mathcal{Cat}]$, and this equivalence induces, by composition, the equivalence

$$\begin{aligned} \mathcal{Lax}[A,\mathcal{Cat}](F,G) &\cong [A,\mathcal{Cat}](F^{\dagger},G) \\ &\simeq [A,\mathcal{Cat}](F^{\dagger\prime},G) \\ &\cong \mathcal{Psd}[A,\mathcal{Cat}](F^{\dagger},G)\ , \end{aligned}$$

which is 2-natural in F and G.

Let $A = 2$ the category, which is also considered as a locally-discrete 2-category, with two objects and one non-identity morphism $f \colon A \to B$. If the inclusion $L \colon \mathcal{Psd}[2,\mathcal{Cat}] \to \mathcal{Lax}[2,\mathcal{Cat}]$ has a left adjoint $(\ )^{*} \dashv L$, then $F^{\dagger} \cong (F^{*})'$ for each 2-functor $F \colon 2 \to \mathcal{Cat}$. When $F = \Delta 1 \colon 2 \to \mathcal{Cat}$ the 2-functor $F^{\dagger} \colon 2 \to \mathcal{Cat}$ is such that $F^{\dagger}A = 1$, $F^{\dagger}B = 2$ and $F^{\dagger}f$ sends the unique object $*$ of 1 to A. If there is a 2-functor $G \colon 2 \to \mathcal{Cat}$ such that $F^{\dagger} = G'$ then, as is easily seen,

$GA = 1$ and the morphism $(1,f,1)\colon (*,A,f) \to ((Gf)(*),B,1)$ in $G^{\dagger}B$ gives an invertible non-identity morphism of $G'B$. But $F^{\dagger}B$ has no such invertible non-identity morphism. Hence L has no left adjoint.

1.8. Reduction of indexed pseudo-limits to indexed limits

Note that indexed pseudo-limits are but particular instances of indexed limits.

Proposition 1.22. *Let* $F\colon A \to Cat$ *and* $G\colon A \to B$ *be 2-functors. Then*

(1) $$\{F,G\}_{lax} \cong \{F^{\dagger},G\}$$

and

(2) $$\{F,G\}_{psd} \cong \{F',G\} .$$

Proof. (1) As a particular instance of the isomorphism (1.19) we have

$$Lax[A,Cat](F,B(B,G-)) \cong [A,Cat](F^{\dagger},B(B,G-)) .$$

The representing objects for each side are $\{F,G\}_{lax}$ and $\{F^{\dagger},G\}$ respectively.

(2) is proved similarly. □

1.9 Indexed limits of retract type

As shown in the previous section indexed pseudo-limits are particular instances of indexed limits. Indeed they can be constructed from the basic indexed limits mentioned in Section 1.3.

Proposition 1.23. (Street [24]). *If the 2-category* $\mathcal{B}$ *admits products, cotensor products and iso-inserters, then* $\mathcal{B}$ *admits indexed pseudo-limits.*

Proof. For 2-functors $F: \mathcal{A} \to \mathcal{C}at$ and $G: \mathcal{A} \to \mathcal{B}$ the indexed pseudo-limit $\{F,G\}_{psd}$ is the iso-inserter for a pair of 1-cells $f,g: \prod_{A} (FA \pitchfork GA) \to \prod_{A,B} (\mathcal{A}(A,B) \times FA) \pitchfork GB$, the products being taken over objects of $\mathcal{A}$. □ (Note June 1988: this proof is inadequate; for a correct one see Street, Corrigendum to "Fibrations in bicategories", *Cahiers de Top. et Géom. Diff.* 28(1987), 53-56.)

Similarly, if $\mathcal{B}$ admits products, cotensor products and inserters, then it admits indexed lax-limits. Thus $\mathcal{B}$ admits indexed pseudo-limits and indexed lax-limits if it admits products, cotensor products, inserters and equifiers. The equifier, inserter and iso-inserter are not indexed pseudo-limits. They are, however, of "retract type"; a 2-functor $F: \mathcal{A} \to \mathcal{C}at$ and the allied indexed limits $\{F,G\}$ are *of retract type* if the counit $\varepsilon_F: F' \to F$ of the adjunction in Theorem 1.21 is a retraction in $[\mathcal{A},\mathcal{C}at]$. (Note June 1988: the epithet "retract-type" has since been replaced by "flexible".)

Proposition 1.24. *For a 2-functor* $F: \mathcal{A} \to \mathcal{C}at$ *the following are equivalent:*

(1) *There is a 2-functor* $G: \mathcal{A} \to \mathcal{C}at$ *and a retraction* $G' \to F$ *in* $[\mathcal{A},\mathcal{C}at]$.

(2) *There is a retraction* $F' \to F$ *in* $[\mathcal{A},\mathcal{C}at]$.

(3) *The counit* $\varepsilon_F\colon F' \to F$ *is a retraction in* $[A,Cat]$.

Proof. Obviously (3) implies (2) and (2) implies (1).

Assume $f\colon G' \to F$ is a retraction. Then $f = \varepsilon_F t'$ for a unique $t\colon G \rightsquigarrow F$, and so ε_F is a retraction. □

Recall that a category, or a 2-category, K is *Cauchy complete* if all idempotent 1-cells in K split (see Kelly [17]). The Cauchy completion of the full image of $(\ )'\colon Psd[A,Cat] \to [A,Cat]$ is the full sub-2-category of 2-functors of retract type.

The splitting of an idempotent may be viewed as a limit. Let *Idem* be the category, considered also as a locally-discrete 2-category, with one object X and two morphisms, 1_X and $e = e^2$. A 2-functor $F\colon Idem \to B$ is determined entirely by the idempotent $f = Fe\colon FX \to FX$. The limit of F has two associated 1-cells $i\colon \lim F \to FX$ and $r\colon FX \to \lim F$, such that $ri = 1$ and $ir = f$, which are called the *splitting* of f. Obviously such limits are absolute limits: they are preserved by all 2-functors.

In *Cat*, the splitting of an idempotent endofunctor $F\colon A \to A$ is given by $R\colon A \to B$ and $I\colon B \to A$ as follows. The category B is the "image" of F; it is the subcategory of A given by the objects A with $FA = A$ and the morphisms $f\colon A \to B$ with $Ff = f$. The functor I is the inclusion, and R is F seen as taking values in B. The hom-sets $B(FA,FB)$ of B are, in fact, given by the splitting of the idempotent $F_{FA,FB}\colon A(FA,FB) \to A(FA,FB)$, where the morphism $F_{C,D}\colon A(C,D) \to A(FC,FD)$ is that determined by the functor F.

Using Cauchy completeness - that is, completeness with respect to the splitting of idempotents - we can characterize the 2-categories which admit all indexed limits of retract type.

Theorem 1.25. *For a 2-category $\mathcal{B}$ the following are equivalent:*

(1) $\mathcal{B}$ *is Cauchy complete and pseudo-complete.*

(2) $\mathcal{B}$ *admits all indexed limits of retract type.*

(3) $\mathcal{B}$ *is Cauchy complete and admits inserters, equifiers, products, and cotensor products.*

(4) $\mathcal{B}$ *is Cauchy complete and admits iso-inserters, products, and cotensor products.*

Proof. (1) ⇒ (2). Assume $\mathcal{B}$ is Cauchy complete and admits indexed pseudo-limits. For $F: A \to \mathcal{C}at$ of retract type there is $\rho: F \to F'$ such that $\varepsilon_F \rho = 1_F$. (Remember that ε_F is the counit of the adjunction in Section 1.7). The indexed limit $\{F',G\} = \{F,G\}_{psd}$ exists for all 2-functors $G: A \to \mathcal{B}$. The idempotent $e = \rho\varepsilon_F: F' \to F'$ induces an idempotent on $\{F',G\}$, which splits as $t: \{F',G\} \to L$ and $s: L \to \{F',G\}$. Thus, for each $B \in \mathcal{B}$, there is a splitting $\mathcal{B}(B,t): \mathcal{B}(B,\{F',G\}) \to \mathcal{B}(B,L)$ and $\mathcal{B}(B,s): \mathcal{B}(B,L) \to \mathcal{B}(B,\{F',G\})$. But composition with ρ and ε_F respectively give a splitting, $[A,\mathcal{C}at](F',\mathcal{B}(B,G-)) \to [A,\mathcal{C}at](F,\mathcal{B}(B,G-))$ and $[A,\mathcal{C}at](F, (B,G-)) \to [A,\mathcal{C}at](F',\mathcal{B}(B,G-))$, of the idempotent given by composition with e. Hence $L = \{F,G\}$.

(2) ⇒ (3). Obviously products and cotensor products are of retract type; for if $\mathcal{D}$ is a 2-category with no non-identity 2-cells then the inclusion $[\mathcal{D},\mathcal{C}at] \to \mathcal{P}sd[\mathcal{D},\mathcal{C}at]$ is an isomorphism, and so all 2-functors

$\mathcal{D} \to \mathcal{Cat}$ are of retract type. Inserters, equifiers and splittings of idempotents are also of retract type, as may be checked by considering the counit ε_F for the indexing type F in each case. It is easy to see what the counit ε_F: F' → F is in these simple cases without using the general formula of Theorem 1.21. For instance, if L: $C \to \mathcal{Cat}$ is the indexing type for the inserter (given in Section 1.3(c)), then L': $C \to \mathcal{Cat}$ is identical with L, except that L'B differs from LB in that each object is replaced by a pair of isomorphic objects.

(3) ⇒ (4). This implication follows from the observations in Section 1.3.

(4) ⇒ (1). This is a direct consequence of Proposition 1.23. □

A 2-category which satisfies the equivalent conditions of the proposition is *retract-type complete*.

Corollary 1.26. *If B is retract-type complete then it is lax-complete.*

Proof. If B is retract-type complete then it admits products, cotensor products and inserters, and hence indexed lax-limits.

Alternatively, note that each $F^\dagger$ is of retract type. Let ν and δ be the unit and counit respectively of the adjunction $(\ )^\dagger \dashv J$: $[A,\mathcal{Cat}] \to Lax[A,\mathcal{Cat}]$. The triangular identities for an adjunction give $\delta_F \nu_F = 1_F$ and $\delta_{F^\dagger}(\nu_F)^\dagger = 1_{F^\dagger}$. Let $\rho = \delta_{F^{\dagger\prime}}(\eta_{F^\dagger})^\dagger(\nu_F)^\dagger$, where η is, as before, the unit of the adjunction $(\ )' \dashv K$: $[A,\mathcal{Cat}] \to Psd[A,\mathcal{Cat}]$. Then

$$\begin{aligned}\varepsilon_{F^\dagger}\rho &= \varepsilon_{F^\dagger}\delta_{F^\dagger},(\eta_F)^\dagger(\nu_F)^\dagger \\ &= \delta_{F^\dagger}(\varepsilon_{F^\dagger})^\dagger(\eta_{F^\dagger})^\dagger(\nu_F)^\dagger \\ &= \delta_{F^\dagger}(\nu_F)^\dagger \\ &= 1_{F^\dagger}\,,\end{aligned}$$

that is, $\varepsilon_{F^\dagger}$ is a retraction in $[A, \mathcal{C}at]$. □

A 2-category A is *of size* α, where α is a regular cardinal, if the number of 2-cells in A, and hence the number of 1-cells and the number of objects, is less than α. A 2-functor $F\colon A \to \mathcal{C}at$ is *of size* α if A and each FA is of size α.

Lemma 1.27. (1) *If* $F\colon A \to \mathcal{C}at$ *is of size* α *then so is* $F^\dagger$.

(2) *If, moreover,* α *is uncountable, then* F' *is also of size* α.

Proof. (1) For each $A \in A$ the number of 2-cells in the 2-category $d_F//A$ is less than α, and hence $F^\dagger(A) = \pi_\star(d_F//A)$ is of size α.

(2) Since $F'(A) = F^\dagger(A)[\Sigma^{-1}]$, for a set of morphisms Σ in $F^\dagger(A)$, it is of size α, provided α is uncountable. □

If $\mathcal{B}$ admits all indexed limits with indexing type of size α [and of retract type] we say that $\mathcal{B}$ is α-*complete* [*retract-type* α-*complete*]. If $\mathcal{B}$ admits all indexed pseudo-limits with indexing type of size α it is *pseudo-*α*-complete*. The proof of the next theorem is analogous to that for Theorem 1.25.

Theorem 1.28. *Let* α *be an uncountable regular cardinal. For a 2-category* $\mathcal{B}$ *the following are equivalent:*

(1) $\mathcal{B}$ *is Cauchy complete and pseudo-*α*-complete.*

(2) $\mathcal{B}$ *is retract-type* α*-complete.*

(3) $\mathcal{B}$ *is Cauchy complete and admits inserters, equifiers, products of size* α*, and cotensor products of size* α*.*

(4) $\mathcal{B}$ *is Cauchy complete and admits iso-inserters, products of size* α*, and cotensor products of size* α*.* □

Some examples of retract-type α-complete 2-categories are examined in the next two chapters.

Chapter 2. The retract-type completeness of *Loc*

The intention of this chapter is to show that α-*Loc* admits all indexed limits of retract type. Moreover, these limits are formed as in *CAT*, and so may be readily calculated.

In an unpublished paper Ulmer [28] uses the existence of inserters and equifiers in *Loc* and *Ladj* to prove, for instance, that a category of cosheaves on a locally-presentable category is itself locally presentable. We prefer to adopt a different approach to his and instead stress the 2-categorical framework for these results.

Throughout the chapter α is an arbitrary regular cardinal.

2.1. Cauchy completeness and adjunctions

We wish to establish the Cauchy completeness of α-*Loc*. To do this we first prove some results about idempotent right adjoints. For a 2-category K let $Radj(K)$ denote the locally-full sub-2-category of K with the same objects, but whose 1-cells are right adjoints in K. Thus α-*Loc* = *Radj*(α-*Rank*) and *Loc* = *Radj*(*Rank*). Similarly, $Ladj(K)$ denotes the sub-2-category of K whose 1-cells are left adjoints in K. Since for adjunctions $S_1 \dashv T_1$ and $S_2 \dashv T_2$, natural transformations $\rho\colon T_1 \to T_2$ between right adjoints correspond bijectively to natural transformations $\sigma\colon S_2 \to S_1$ between left adjoints, it is clear that $Ladj(K)$ and $Radj(K)^{coop}$ are biequivalent.

The following proposition is an immediate generalization of an observation by Paré (see Mac Lane [21], p.84).

Proposition 2.1. (Paré). *Let* $G: A \to B$ *and* $F: B \to A$ *be 1-cells in a 2-category* K, *and let* $\rho: 1_B \to GF$ *and* $\varepsilon: FG \to 1_A$ *be 2-cells such that* $(G\varepsilon)(\rho G) = 1_G$. *Then the 2-cell* $(\varepsilon F)(F\rho)$ *is idempotent. This idempotent splits if and only if* G *has a left adjoint.*

Proof. Denoting $(\varepsilon F)(F\rho)$ by σ we have $\sigma^2 = (\varepsilon F)(F((G\varepsilon)(\rho G))F)(F\rho)$ $= (\varepsilon F)(F\rho) = \sigma$.

Assume σ splits as $\tau: F \to K$ and $\lambda: K \to F$, so that $\lambda\tau = \sigma$ and $\tau\lambda = 1_K$. Then $K \dashv G$, with unit $(G\tau)\rho$ and counit $\varepsilon(\lambda G)$. Conversely, if $K \dashv G$, with unit η and counit β, then $\tau = (\varepsilon K)(F\eta): F \to K$ and $\lambda = (\beta F)(K\rho): K \to F$ give a splitting of σ. □

A 2-category K is *locally Cauchy complete* if each category $K(A,B)$ is Cauchy complete.

Recall our convention that, for an adjunction $P \dashv Q: C \to A$ in a 2-category $\mathcal{B}$ admitting objects of algebras, the 1-cell Q is monadic if the comparison 1-cell $C \to A^T$ is an equivalence, T being the monad induced by the adjunction. The property of being monadic is representable. For let $f: B \to D$ be a 1-cell such that each functor $\mathcal{B}(X,f): \mathcal{B}(X,B) \to \mathcal{B}(X,D)$ is an equivalence. Then there is $g: D \to B$ such that $fg \cong 1_D$, and, since $fgf = f$ and $\mathcal{B}(B,f)$ is fully faithful, $gf \cong 1_B$. Moreover, by definition, $(X,A^T) \cong \mathcal{B}(X,A)^{\mathcal{B}(X,T)}$.

Proposition 2.2. (1) *If* K *is a locally Cauchy complete 2-category, then* $Radj(K)$ *is locally Cauchy complete.*

(2) *If* K *is a locally Cauchy complete 2-category which is also Cauchy complete, then* $Radj(K)$ *is Cauchy complete.*

(3) *If K is locally Cauchy complete and Cauchy complete, and admits objects of algebras, and if the idempotent right adjoint* $G: A \to A$ *in* K *splits as* $R: A \to B$ *and* $I: B \to A$, *then* I *is monadic.*

Proof. (1) Let $F \dashv G$ with unit η and counit ε. An idempotent 2-cell $\gamma: G \to G$ on the right adjoint gives an idempotent 2-cell $\beta = (\varepsilon F)(F\gamma F)(F\eta): F \to F$ on the left adjoint. These idempotent 2-cells split in K, as $\sigma: G \to \overline{G}$ and $\tau: \overline{G} \to G$ and as $\lambda: F \to \overline{F}$ and $\rho: \overline{F} \to F$ say. Then $\overline{F} \dashv \overline{G}$ with unit $(\sigma.\lambda)\eta$ and counit $\varepsilon(\rho.\tau)$.

(2) Again let $F \dashv G: A \to A$ with unit η and counit ε, and $G = G^2$ idempotent. Assume G splits in K as $R: A \to B$ and $I: B \to A$. Letting $\overline{R} = FI: B \to A$, the 2-cells $\rho = R\eta I: 1_B \to R\overline{R}$ and $\varepsilon: \overline{R}R = GF \to 1_A$ satisfy the hypothesis of Proposition 2.1. Thus R has a left adjoint. Similarly, letting $\overline{I} = RF$, the 2-cells $\eta: 1_A \to I\overline{I} = GF$ and $\beta = R\varepsilon I: \overline{I}I \to 1_B$ satisfy the hypothesis of Proposition 2.1. So I also has a left adjoint.

(3) From the remarks about representability above it is sufficient to consider adjunctions $R_* \dashv R: A \to B$ and $I_* \dashv I: B \to A$ in CAT such that R and I are a splitting of the idempotent $G = IR: A \to A$. As in Section 1.9, we may consider $I: B \to A$ to be an inclusion. Let $f,g: GX \to GY$ be a parallel pair of morphisms in B such that $If = f$ and $Ig = g$ have an absolute coequalizer $h: GY \to Z$ in A. Then $Gh: GY \to GZ$ is also a coequalizer of $Gf = f$ and $Gg = g$. Thus f and g have a coequalizer, $Rh: GY \to GZ$, in B, and I preserves and reflects coequalizers of such pairs f,g. By the modified version of the monadicity theorem of Beck, given in Mac Lane [21] p.151, the functor I is monadic in our sense. □

2.2 Retract-type completeness and complete categories

Throughout the remaining sections of this chapter we shall be concerned with various sub-2-categories of CAT, and with their closure with respect to indexed limits of retract type.

For a class F of small categories the objects of the locally-full sub-2-category F-*Comp* of CAT are the *F-complete categories*; that is, they admit the limits of all functors whose domain belongs to F. The 1-cells of F-*Comp* are the *F-continuous functors*, those which preserve all these limits. For instance, when F consists of the category *Idem* of Section 1.9, then F-*Comp* is *Cauchy*, the full and locally-full sub-2-category of CAT whose objects are the Cauchy complete categories. A 2-category K is *locally F-complete* if each category $K(A,B)$ is F-complete and if the functors $K(f,B)$ and $K(A,g)$, given by composition with $f: D \to A$ and $g: B \to C$, are F-continuous.

Proposition 2.3. *For any class F of small categories, F-Comp is locally F-complete.*

Proof. Let A and B be F-complete categories, let K be a category in the class F, and let $T: K \to F\text{-}Comp(A,B)$ be a functor. If $J: F\text{-}Comp(A,B) \to CAT(A,B)$ is the inclusion, then the composite $JT: K \to CAT(A,B)$ has a limit, $L = \lim JT$, constructed pointwise. Since limits commute with limits it is easily seen that L is F-continuous and that L is the limit of T in $F\text{-}Comp(A,B)$. Obviously, such limits are preserved by composition with F-continuous functors. □

Corollary 2.4. *If every F-complete category is Cauchy complete then F-Comp is locally Cauchy complete.* □

It is not true, in general, that (the underlying category of) *F-Comp* is F-complete. Let F be the set of finite categories and let $F,G\colon 1 \to Iso$ be the two distinct functors from 1 to *Iso*. The only functor which equalizes the functors F and G is $0 \to 1$, and the empty category 0 is not finitely complete.

Before showing that *F-Comp* is retract-type complete we note a result connecting the completeness of categories and the splitting of idempotent endofunctors.

Proposition 2.5. (Isbell [14]). *Let* A *be a Cauchy complete category and let* $F\colon A \to A$ *be an idempotent endofunctor which splits as* $R\colon A \to B$ *and* $I\colon B \to A$. *The functor* $T\colon K \to B$ *has a limit if* $IT\colon K \to A$ *does.*

Proof. First note that $\mathcal{B}$ is Cauchy complete. If $b\colon B \to B$ is an idempotent in $\mathcal{B}$ then Ib splits, as $f\colon IB \to A$ and $g\colon A \to IB$ say. So $Rf\colon B \to RA$ and $Rg\colon RA \to B$ give a splitting of $b = RI(b)$.

Assume $T\colon K \to \mathcal{B}$ is a functor such that $\lim IT = X$ exists, with limit-cone $\rho\colon \Delta X \to IT$. Now $F\rho\colon F(\Delta X) = \Delta(FX) \to FIT = IT$ is also a cone over IT. Hence there is a unique morphism $t\colon FX \to X$ in A such that $F\rho = \rho(\Delta t)$. But $F\rho = F^2\rho = F\rho(\Delta(Ft)) = \rho(\Delta t)(\Delta(Ft)) = \rho(\Delta(t(Ft)))$, and so $t = t(Ft)$. Thus $Rt = Rt(RFt) = (Rt)^2$ is an idempotent in $\mathcal{B}$, splitting as $r\colon RX \to L$ and $i\colon L \to RX$. It is easily checked that $(R\rho)(\Delta i)\colon \Delta L \to T$ is a limit-cone in $\mathcal{B}$. □

In the proof above, if F preserves $\lim IT$ then t is an isomorphism, and Rt is an identity. In this case the hypothesis that A is Cauchy complete is unnecessary. A similar proposition holds for colimits.

Theorem 2.6. *For any class F of small categories the 2-category F-Comp admits all indexed limits of retract type. Moreover, these limits are preserved by the inclusion F-Comp → CAT.*

Proof. By Theorem 1.25 it suffices to establish the results for (a) splittings of idempotents, (b) products, (c) cotensor products, (d) inserters, and (e) equifiers.

(a) Splittings of idempotents. Let $F = F^2: A \to A$ be an idempotent 1-cell in F-Comp, which splits as $R: A \to B$ and $I: B \to A$ in CAT For K in F and a functor $T: K \to B$, the limit $X = \lim IT$ exists, with limit-cone $\rho : \Delta X \to IT$. Now $F\rho : \Delta FX \to IT$ is also a limit-cone, since F is F-continuous. As in the proof of Proposition 2.5 there is a unique morphism $t: FX \to X$ in A such that $F\rho = \rho(\Delta t)$. Moreover t is an isomorphism and Rt is the identity on RX. The cone $R\rho: RX \to T$ is a limit-cone in B. So B is F-complete. Considering the formation of F-limits in B using those in A, it is easily seen that R and I are F-continuous.

(b) Products. For a set $\{A_j\}$ of F-complete categories the product $\prod A_j$ in CAT is F-complete and the projections $p_i: \prod A_j \to A_i$ are F-continuous. Since the projections are jointly conservative - that is, a morphism t in $\prod A_j$ is an isomorphism if and only if each $p_i(t)$ is - a functor $F: B \to \prod A_j$ from an F-complete category B is F-continuous if and only if each p_iF is.

(c) Cotensor products. If $\mathcal{C}$ is a small category then $CAT[\mathcal{C},\mathcal{D}]$ admits whatever limit-types $\mathcal{D}$ does, and these limits are formed pointwise. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{F}$-complete categories. Under the isomorphism $CAT[\mathcal{B},CAT[\mathcal{C},\mathcal{A}]] \cong CAT[\mathcal{C},CAT[\mathcal{B},\mathcal{A}]]$ the $\mathcal{F}$-continuous functors $F\colon \mathcal{B} \to CAT[\mathcal{C},\mathcal{A}]$ correspond to functors $G\colon \mathcal{C} \to CAT[\mathcal{B},\mathcal{A}]$ whose image actually lands in $\mathcal{F}\text{-}Comp(\mathcal{B},\mathcal{A})$. Hence there is a natural isomorphism $\mathcal{F}\text{-}Comp(\mathcal{B},CAT[\mathcal{C},\mathcal{A}]) \cong CAT[\mathcal{C},\mathcal{F}\text{-}Comp(\mathcal{B},\mathcal{A})]$, and $CAT[\mathcal{C},\mathcal{A}]$ is the cotensor product of $\mathcal{C}$ and $\mathcal{A}$ in $\mathcal{F}\text{-}Comp$.

(d) Inserters. Let $F,G\colon \mathcal{A} \to \mathcal{B}$ be a parallel pair of 1-cells in $\mathcal{F}\text{-}Comp$, for which the inserter $\mathrm{Ins}(F/G)$ in CAT is $J\colon \mathcal{P} \to \mathcal{A}$, with associated 2-cell $\mu\colon FJ \to GJ$. We use the explicit description of $\mathcal{P}$ given in Section 1.3(c). Let $\mathcal{K}$ be a category in $\mathcal{F}$ and $T\colon \mathcal{K} \to \mathcal{P}$ a functor. The limit $A = \lim(JT)$ exists, with limit-cone $\rho\colon \Delta A \to JT$. Now F and G preserve this limit. Hence there is a unique morphism $f = \lim(\mu T)\colon FA \to GA$ such that $\mu_{TK}(F\rho_K) = (G\rho_K)f$ for all $K \in \mathcal{K}$. The limit of J is (A,f), with limit-cone $\delta\colon \Delta(A,f) \to T$ such that $J\delta = \rho$. So $\mathcal{P} \in \mathcal{F}\text{-}Comp$ and J preserves, and creates, all $\mathcal{F}$-limits. Again since J is conservative, $\mathcal{P}$, with J and μ, is the inserter in $\mathcal{F}\text{-}Comp$.

(e) Equifiers. For a pair of 2-cells $\sigma,\tau\colon F \to G\colon \mathcal{A} \to \mathcal{B}$ in $\mathcal{F}\text{-}Comp$ let $J\colon \mathcal{P} \to \mathcal{A}$ be the equifier in CAT. We consider J to be the inclusion of a full subcategory. Let $\mathcal{K}$ be a category in $\mathcal{F}$ and $T\colon \mathcal{K} \to \mathcal{P}$ a functor. It is sufficient to prove that $A = \lim JT$ lies in $\mathcal{P}$ in order to prove that $\lim T$ exists. Now F and G preserve the limit of $JT\colon \mathcal{K} \to \mathcal{A}$. Since $\sigma JT = \rho JT$ the two morphisms $\sigma_A = \lim(\sigma JT)\colon FA \to GA$ and $\rho_A = \lim(\rho JT)\colon FA \to GA$ are equal. So $A \in \mathcal{P}$. Thus $\mathcal{P}$ is $\mathcal{F}$-complete and J is $\mathcal{F}$-continuous. Again, since J is conservative, $J\colon \mathcal{P} \to \mathcal{A}$ is the equifier in $\mathcal{F}\text{-}Comp$. □

The analogous result, with an analogous proof, holds for F-*Comp*, the 2-category of F-cocomplete categories, F-cocontinuous functors and natural transformations.

Theorem 2.7. *For any class* F *of small categories the 2-category* F-*Cocomp* *admits all indexed limits of retract type. Moreover, these limits are preserved by the inclusion* F-*Cocomp* $\to$ CAT. □

Using these results we have:

Proposition 2.8. *The 2-category Rank is locally Cauchy complete and Cauchy complete.*

Proof. Let F_α be the class of small α-filtered categories and let F be the union of these classes, the union being taken over all regular cardinals. Then, by Corollary 2.4, F-*Cocomp* is locally Cauchy complete. Thus its full and locally-full sub-2-category *Rank* is also locally Cauchy complete.

Let A be locally α-presentable and let $F: A \to A$ be an idempotent endofunctor of rank α. The functor F splits, in *CAT*, as $R: A \to B$ and $I: B \to A$. By Theorem 2.7, B admits α-filtered colimits and R and I have rank α. There is a regular cardinal $\beta \geq \alpha$ such that $F(A_\alpha) \subseteq A_\beta$. So, for a functor $T: K \to B$ whose domain is a small β-filtered category and for $A \in A_\alpha$, the canonical morphisms $\operatorname{colim} IT = \operatorname{colim} FIT \to F \operatorname{colim} IT$ and $\operatorname{colim} A(FA, IT-) \to A(FA, \operatorname{colim} IT)$ are isomorphisms. Hence, splitting the idempotent $F_{FA, F \operatorname{colim} IT}: A(FA, F \operatorname{colim} IT) \to A(FA, F \operatorname{colim} IT)$ shows that the canonical morphism $\operatorname{colim} B(RA, T-) \to B(RA, \operatorname{colim} T) = B(RA, R \operatorname{colim} IT)$

is an isomorphism. Thus, if $A \in A_\alpha$, then RA is a β-presentable object of $\mathcal{B}$. Let $\mathcal{D}$ be the full image of A under R. Every object in A is an α-filtered colimit of α-presentable objects. Since R is surjective and has rank α, every object in $\mathcal{B}$ is an α-filtered colimit of objects in $\mathcal{D}$. Hence $\mathcal{D}$ is a strong generator of β-presentable objects in $\mathcal{B}$. By Proposition 2.5 $\mathcal{B}$ is also cocomplete, and hence locally β-presentable. □

Corollary 2.9. *The 2-categories Loc and Ladj are Cauchy complete and locally Cauchy complete.*

Proof. We have *Loc* = *Radj*(*Rank*) and *Ladj* = *Ladj*(*Rank*). Hence, by Propositions 2.2 and 2.8, the 2-category *Loc* is Cauchy complete and locally Cauchy complete. The same holds for *Ladj*, using now a modification of Proposition 2.2 for left adjoints. □

While α-*Rank* is locally Cauchy complete it is not Cauchy complete. It is, nonetheless, true that α-*Loc* = *Radj*(α-*Rank*) is Cauchy complete.

2.3 The retract-type completeness of α-*Loc*

We shall use Theorem 1.25 to show that α-*Loc* admits all indexed limits of retract type.

Proposition 2.10. *The 2-category* α-*Loc* *is Cauchy complete and locally Cauchy complete.*

Proof. Since α-*Rank* is locally Cauchy complete by Corollary 2.4, so is α-*Loc* = *Radj*(α-*Rank*).

Let $F = F^2: A \to A$ be an idempotent 1-cell in α-*Loc*, which splits as $R: A \to B$ and $I: B \to A$ in *CAT*. Then, by Theorems 2.6 and 2.7, the category B admits all small limits and α-filtered colimits, and the functors I and R are continuous and have rank α. The 2-category *Cauchy* of Section 2.2 is, by Corollary 2.4, locally Cauchy complete. By Theorem 2.6 it is Cauchy complete and admits objects of algebras, which are formed as in *CAT*. So Proposition 2.2(3) implies that I is monadic; the category B is equivalent to the category of algebras for a monad with rank α on a locally α-presentable category, and thus is itself locally α-presentable. □

Let $\{A_i\}$ be a set of complete and cocomplete categories whose product has projections $P_j: \prod A_i \to A_j$. These projections are continuous and cocontinuous. Selecting initial objects, which we denote by 0, in each category A_j, a left adjoint $Q_j: A_j \to \prod A_i$ to P_j is given by $P_jQ_j = 1$ and, for $i \neq j$, $P_iQ_j = \Delta 0$. A right adjoint $R_j: A_j \to \prod A_i$ is given by considering terminal objects.

Proposition 2.11. *The 2-category* α-*Loc* *admits products. These products are preserved by the inclusion* α-*Loc* $\to$ *CAT*.

Proof. Let $\{A_i\}$ be a set of locally α-presentable categories. With the notation used above, the projections P_j are continuous and have rank α. The product $\prod A_i$ is cocomplete. Let C be an α-presentable object of A_j. Then $(\prod A_i)(Q_jC,-) \cong A_j(C,-)P_j$ has rank α; that is, Q_jC

is an α-presentable object of $\prod A_i$. Now an object B in $\prod A_i$ is the coproduct $\coprod Q_j P_j(B)$, and each $P_j(B)$ is an α-filtered colimit of objects in $(A_j)_\alpha$. Let M_j be the set of objects of the form $Q_j C$, where $C \in (A_j)_\alpha$, and let P be the full subcategory of $\prod A_i$ whose objects belong to the union $\cup M_j$. Since $\prod A_i$ is the closure of P under small colimits, P is a strong generator made of α-presentable objects, and so $\prod A_i$ is locally α-presentable. If $T: B \to \prod A_i$ is a functor whose domain is locally α-presentable, then T is continuous and has rank α if and only if each $P_j T$ is continuous and has rank α. So $\prod A_i$, with projections $P_j: \prod A_i \to A_j$, is also the product in α-*Loc*. □

We can easily identify the α-presentable objects in the product $\prod A_i$. Taking the closure under α-colimits of the subcategory P in the proof above gives the subcategory of α-presentable objects; an object $B \in \prod A_i$ is α-presentable if and only if each projection $P_j(B)$ is α-presentable and fewer than α of these are not initial objects.

Proposition 2.12. *The 2-category α-Loc admits cotensor products and the inclusion α-Loc $\to$ CAT preserves them.*

Proof. Let K be a small category and A a locally α-presentable category. The cotensor product $CAT(K,A)$, in CAT, is locally α-presentable (see Gabriel and Ulmer [10]). A functor $F: B \to C$ between locally α-presentable categories is in α-*Loc* if and only if it is continuous and has rank α. So, by Theorems 2.6 and 2.7, $CAT(K,A)$ is also the cotensor product in α-*Loc*. □

The corresponding results for inverters and equifiers require different proofs. We shall reduce them to instances of algebras, in the sense of Adámek and Trnková [1] (and the references given in their extensive bibliography), and then appeal to the results of Kelly [16].

Proposition 2.13. *Let* $F,G\colon A \to B$ *be a parallel pair of 1-cells in a 2-category* $\mathcal{B}$, *such that* G *has a left adjoint* $G_* \dashv G$, *the counit being* $\varepsilon\colon G_*G \to 1$. *Then* $J\colon C \to A$, *with associated 2-cell* $\mu\colon FJ \to GJ$, *is the inserter* $\mathrm{Ins}(F/G)$ *if and only if* $J\colon C \to A$, *with associated 2-cell* $(\varepsilon J)\mu\colon G_*FJ \to J$, *is* $\mathrm{Ins}(G_*F/1)$.

Proof. For a 1-cell $K\colon D \to A$ there is a bijection

$$K(D,B)(FK,GK) \cong K(D,A)(G_*FK,K)$$

sending $\delta\colon FK \to GK$ to $(\varepsilon K)(G_*\delta)\colon G_*FK \to K$; the inverse sends $\tau\colon G_*FK \to K$ to $(G\tau)(\eta FK)$, where $\eta\colon 1 \to GG_*$ is the unit of the adjunction $G_* \dashv G$. Using this bijection it is readily seen that $J\colon C \to A$, with $\mu\colon FJ \to GJ$, has the universal property of an inserter if and only if J, with $(\varepsilon J)(G_*\mu)\colon G_*FJ \to J$, has it. □

Now observe that an object of $\mathrm{Ins}(T/1)$ is an object A and a map $a\colon TA \to A$, that is, an algebra for the endofunctor T in the sense of Kelly [16] Section 18.

Proposition 2.14. *The 2-category* α-Loc *admits inserters, and they are preserved by the inclusion into* CAT.

Proof. Let $F,G: A \to B$ be a parallel pair of 1-cells in α-*Loc*. Thus $G_* \dashv G$ in *CAT*, and $\mathrm{Ins}(G_*F/1)$ is the category of algebras G_*F-*Alg* in the sense of Kelly. Since G_*F has rank α and A is locally α-presentable, the associated functor $J: G_*F\text{-}\mathcal{A}\ell g \to A$, which takes the underlying object of an algebra, has a left adjoint (see Kelly [16], Proposition 14.3 and Theorem 15.6). By Theorem 2.7, J has rank α and, since J is also the 1-cell associated with $\mathrm{Ins}(F/G)$, it is continuous.

If $f,g: A \to B$ are a pair of 1-cells in G_*F-*Alg* such that the pair Jf and Jg has an absolute coequalizer, then, as in the proof of Theorem 2.6, the pair f,g in G_*F-*Alg* has a coequalizer. Moreover, J preserves and creates coequalizers of such pairs. Thus J is monadic (the comparison functor being actually an isomorphism by the Beck monadicity theorem) and $G_*F\text{-}\mathcal{A}\ell g = \mathrm{Ins}(F/G)$ is locally α-presentable.

Again using Theorems 2.6 and 2.7, if $T: C \to \mathrm{Ins}(F/G)$ is a functor such that JT is in α-*Loc*, then T is in α-*Loc*. So the inserter $\mathrm{Ins}(F/G)$, with associated 1-cell and 2-cell, exist in α-*Loc* and is formed as in *CAT*. □

For the 2-category α-*Rank* each hom-category $\alpha\text{-}\mathit{Rank}(A,B)$ admits coequalizers. They are formed pointwise, and so, for each 1-cell $F: C \to A$, the functor $\alpha\text{-}\mathit{Rank}(F,B): \alpha\text{-}\mathit{Rank}(A,B) \to \alpha\text{-}\mathit{Rank}(C,B)$ preserves coequalizers.

Let K be any such 2-category, so that each hom-category $K(A,B)$ admits coequalizers and each functor $K(F,B)$ preserves them. Let $\sigma,\tau: F \to G: A \to B$ be a pair of 2-cells in K such that $G_* \dashv G$ with counit $\varepsilon: G\,G \to 1$. Also, let $\rho: 1 \to T$ be the coequalizer of $\varepsilon(G_*\sigma)$ and $\varepsilon(G_*\tau)$ in $K(A,A)$. Then

Proposition 2.15. *For a 1-cell* $J: C \to A$ *in* K *the following are equivalent:*

(1) J *is the equifier of* σ *and* τ.

(2) J *is the equifier of* $\varepsilon(G_*\sigma)$ *and* $\varepsilon(G_*\tau)$.

(3) J *is the inverter of* ρ.

Proof. Let $K: D \to A$ be a 1-cell in K.

(1) ⟺ (2). Using the adjunction $G_* \dashv G$ we have $\sigma K = \tau K$ if and only if $\varepsilon(G_*\sigma)K = \varepsilon(G_*\tau)K$.

(2) ⟺ (3). The 2-cell ρK is the coequalizer of τK and σK. So K inverts ρ if and only if it equifies σ and τ. □

Proposition 2.16. *The 2-category* α-*Loc admits equifiers. They are preserved by the inclusion* α-*Loc* $\to$ *CAT*.

Proof. For a parallel pair of 2-cells $\sigma,\tau: F \to G: A \to B$ in α-*Loc* there is an adjunction $G_* \dashv G$, with counit ε, in α-*Rank*. Thus, for the coequalizer $\rho: 1 \to T$ of $\varepsilon(G_*\sigma)$ and $\varepsilon(G_*\tau)$ in *CAT*(A,A) (which is also the coequalizer in α-*Rank*(A,A)), the functor T has rank α and ρ is epic. Thus (T,ρ) is a well-pointed endofunctor (see Kelly [16], Section 5) and the inverter of ρ is the category of (T,ρ)-algebras. From Kelly [16], Theorem 6.2, this full subcategory of A, which is also the equifier of σ and τ in *CAT*, is reflective. By Theorem 2.7 the inclusion J: Equif$(\sigma,\tau) \to A$ has rank α. Thus J is a 1-cell in α-*Loc*. Since J is conservative, the equifier of τ and σ exists in α-*Loc* and is formed as in *CAT*. □

Gathering together the results of this section we have

Theorem 2.17. *The 2-category* α-*Loc* *is retract-type complete and the inclusion* α-*Loc* $\to$ CAT *preserves all indexed limits of retract type.* □

We have already seen that *Loc* is Cauchy complete. However, the 2-category *Loc* is the union, taken over all regular cardinals α, of the increasing sequence of sub-2-categories α-*Loc*. Hence

Theorem 2.18. *The 2-category* *Loc* *admits all indexed limits of retract type and the inclusion* *Loc* $\to$ CAT *preserves them.* □

Chapter 3. Limits in *Ladj*

In this chapter we shall examine indexed limits of retract type in *Ladj*. This case is somewhat different from that afforded by *Loc*. Each of the sub-2-categories α-*Loc* admits indexed limits of retract type and the inclusion in *Loc* preserves them. However, for α-*Ladj* the size of the regular cardinal α places a definite restriction on the size of those limits which exist and are calculated as in *CAT*.

Because α-*Th*, being the small categories in *F-Comp* where *F* is the set of categories of size α, is pseudo-complete, and $\alpha\text{-}Th^{co}$ is biequivalent to α-*Ladj*, the 2-category α-*Ladj* admits all indexed bilimits. Moreover, α-*Ladj* is also biequivalent to $(\alpha\text{-}Loc)^{coop}$, and hence admits indexed bicolimits. Our concern, however, is with retract--type completeness, and to explore this we again appeal to the existence of the basic indexed limits of retract type, namely the splitting of idempotents, products, cotensor products, inserters and equifiers. Much of the section on inserters and equifiers is dependent on Ulmer [28] but we include it so that the account is self-contained.

The result in Section 3.5 below - that Ladj(*A*,*B*) is locally presentable when *A* and *B* are - is given as an application of the results in the main body of the chapter. It is also proved by Ulmer. We shall have cause to reconsider it in Chapter 5. Our other major application partially answers a question of Lawvere.

3.1 Splitting idempotents in α-*Ladj*

By Isbell's theorem, Proposition 2.5, the retract of a cocomplete category is itself cocomplete. We combine this fact with some observations about the behaviour of α-presentable objects under the splitting of idempotent endofunctors to establish the Cauchy completeness of α-*Ladj*.

Let A be a locally α-presentable category and $F: A \to A$ an idempotent functor with rank α which preserves α-presentable objects. In *CAT* the idempotent F splits as $R: A \to B$ and $I: B \to A$ where I is an inclusion.

Proposition 3.1. *The category* B *above is locally* α*-presentable and the functors* R *and* I *have rank* α.

Proof. The proof follows that of Proposition 2.8, but now $F(A_\alpha) \subseteq A_\alpha$. □

Proposition 3.2. *The 2-category* α-*Ladj* *is Cauchy complete.*

Proof. Let $F: A \to A$ be an idempotent 1-cell in α-*Ladj* which splits, in *CAT*, as $R: A \to B$ and $I: B \to A$. Then R and I are cocontinuous. The full image B' of A_α under R is contained in B_α, as shown in the proof of Proposition 2.8. But B' is closed in B under α-colimits, since A_α is closed in A under α-colimits and I preserves α-colimits. Hence R and I preserve α-presentable objects and the splitting occurs in α -*Ladj*. □

3.2 Products and cotensor products in α-*Ladj*

We have already seen in Section 2.3 that if the categories A_i are locally α-presentable then the product $\prod A_i$ is also locally α-presentable. The projection $P_j\colon \prod A_i \to A_j$ has both a left and a right adjoint. The right adjoint $R_j\colon A_j \to \prod A_i$ is not only continuous but preserves all filtered colimits (indeed the colimit of any functor whose domain is connected). So the projections are functors in α-*Ladj*.

Recall from Section 2.3 that an object $c \in \prod A_j$ is α-presentable if and only if each projection $P_j(c)$ is α-presentable and fewer than α of these are not initial objects. Thus, if $M_i\colon B \to A_i$ is a set of functors in α-*Ladj*, and if the product has α or more components, then the resulting functor $M\colon B \to \prod A_i$ need not preserve α-presentable objects. So $\prod A_i$ need not be the product in α-*Ladj*. But

Proposition 3.3. *The 2-category α-Ladj admits all products with fewer than α components. These products are preserved by the inclusion into* CAT.

Proof. In this case the resultant functor M above does preserve α-presentable objects, and is cocontinuous. □

The situation with cotensor products is similar. Whenever K is a small category and B is locally α-presentable the functor category $[K,B]$ is locally α-presentable. To ensure that this is the cotensor product, not just in CAT but also in α-*Ladj*, we impose the restriction that K be an α-category.

For each $K \in \mathcal{K}$ the evaluation functor $Ev_K: [\mathcal{K},\mathcal{B}] \to \mathcal{B}$ has a right adjoint $M_K: \mathcal{B} \to [\mathcal{K},\mathcal{B}]$ such that $(M_K(B))K' = \prod_{\mathcal{K}(K',K)} B$ and a left adjoint $L_K: \mathcal{B} \to [\mathcal{K},\mathcal{B}]$ such that $(L_K(B))K' = \mathcal{K}(K,K').B$ is the coproduct, indexed by the set $\mathcal{K}(K,K')$, of copies of B.

Lemma 3.4. *Let* $\mathcal{B}$ *be a locally* α*-presentable category and* $\mathcal{K}$ *an* α*-category. If* $F: \mathcal{K} \to \mathcal{B}$ *factorizes through the inclusion* $\mathcal{B}_\alpha \to \mathcal{B}$ *then* F *is an* α*-presentable object of* $[\mathcal{K},\mathcal{B}]$.

Proof. Since each $F(K)$ is α-presentable, and colimits in $[\mathcal{K},\mathcal{B}]$ are evaluated pointwise, each generalized representable $\mathcal{K}(K',-).F(K)$ is α-presentable. Now F, as the quotient in

$$\textstyle\sum_{K,K'} \mathcal{K}(K,K').\mathcal{K}(K',-).FK \rightrightarrows \sum_K \mathcal{K}(K,-).FK \to F \,,$$

is the α-colimit of such functors, $\mathcal{K}$ being an α-category. Thus F too is α-presentable. □

Proposition 3.5. *Let* $\mathcal{B}$ *be a locally* α*-presentable category and* $\mathcal{K}$ *an* α*-category. Then* $[\mathcal{K},\mathcal{B}]$ *is the cotensor product in* α*-Ladj*.

Proof. Let $\mathcal{A}$ be a locally α-presentable category. Let $(\ )^\#$ be the composite $\alpha\text{-}Ladj(\mathcal{A},[\mathcal{K},\mathcal{B}]) \to CAT[\mathcal{A},[\mathcal{K},\mathcal{B}]] \cong CAT[\mathcal{K},CAT[\mathcal{A},\mathcal{B}]]$ and $(\ )^*$ the composite $CAT[\mathcal{K},\alpha\text{-}Ladj(\mathcal{A},\mathcal{B})] \to CAT[\mathcal{K},CAT[\mathcal{A},\mathcal{B}]] \cong CAT[\mathcal{A},[\mathcal{K},\mathcal{B}]]$. It is sufficient to prove that if $T \in \alpha\text{-}Ladj(\mathcal{A},[\mathcal{K},\mathcal{B}])$ then its image $T^\#$ is actually in $CAT[\mathcal{K},\alpha\text{-}Ladj(\mathcal{A},\mathcal{B})]$, and if $N \in CAT[\mathcal{K},\alpha\text{-}Ladj(\mathcal{A},\mathcal{B})]$ then its image N^* lies in $\alpha\text{-}Ladj(\mathcal{A},[\mathcal{K},\mathcal{B}])$.

Consider T in $\alpha\text{-}Ladj(\mathcal{A},[\mathcal{K},\mathcal{B}])$ with $T \dashv S: [\mathcal{K},\mathcal{B}] \to \mathcal{A}$, the right adjoint S having rank α. Products of size α (that is, with fewer than

α factors) commute with α-filtered colimits. So, for each $K \in \mathcal{K}$, the functor M_K has rank α. Thus each functor SM_K has rank α. Moreover

$$
\begin{aligned}
\mathcal{B}((T^{\#}(K))A,B) &\cong \mathcal{B}(Ev_K(TA),B) \\
&\cong [\mathcal{K},\mathcal{B}](TA,M_K(B)) \\
&\cong \mathcal{A}(A,SM_K(B)) ,
\end{aligned}
$$

that is $T^{\#}(K) \dashv SM_K$. Hence, $T^{\#}(K) \in \alpha\text{-}\mathit{Ladj}(\mathcal{A},\mathcal{B})$ for each $K \in \mathcal{K}$ and $T^{\#} \in \mathit{CAT}[\mathcal{K},\alpha\text{-}\mathit{Ladj}(\mathcal{A},\mathcal{B})]$ as required.

Consider $N \in \mathit{CAT}[\mathcal{K},\alpha\text{-}\mathit{Ladj}(\mathcal{A},\mathcal{B})]$. From Proposition 2.7 the functor $N^*: \mathcal{A} \to [\mathcal{K},\mathcal{B}]$ is cocontinuous. Also, if $A \in \mathcal{A}_\alpha$, then $(N^*(A))(K) = (N(K))(A) \in \mathcal{B}_\alpha$ and hence, from Lemma 3.4, $N^*(A)$ is an α-presentable object of $[\mathcal{K},\mathcal{B}]$. So N^* preserves α-presentable objects and is in $\alpha\text{-}\mathit{Ladj}(\mathcal{A},[\mathcal{K},\mathcal{B}])$, as required. □

3.3 Inserters and equifiers in α-*Ladj*

We have seen already that the size of α has an impact on the size of products and cotensor products which exist in α-*Ladj* and are calculated as in *CAT*. This is also true for inserters and equifiers, but for a different reason - these limits are finite whereas products and cotensor products can be arbitrarily large. For α an uncountable regular cardinal, results parallel to those for α-*Loc* hold, namely that inserters and equifiers exist in α-*Ladj* and that they are constructed as in *CAT*. This is not true, as we shall see, for $\alpha = \aleph_0$. For the immediate sequel we assume that α is uncountable.

We deal first with the case of the equifier. Consider a parallel pair of 2-cells $\rho, \sigma: F \to G: A \to B$ in α-*Ladj*, and let $J: P \to A$ be the equifier of this pair in CAT. So P may be considered as a full subcategory of A. From Theorem 2.7 the category P is cocomplete and J is cocontinuous. Let P' be the full subcategory of P formed from those objects P such that JP is α-presentable in A.

Lemma 3.6. *If* $P \in P'$ *then* $P \in P_\alpha$.

Proof. Assume $P \in P'$. Then $P(P,-) \cong A(JP,-)J: P \to Set$ has rank α. □

The following lemma is in Ulmer [28]. Remember that α is uncountable.

Lemma 3.7. *If* $f: A \to JQ$ *is a morphism of* A *such that* $A \in A_\alpha$ *and* $Q \in P$, *then there is* $P \in P'$, *and morphisms* $s: A \to JP$ *and* $t: JP \to JQ$, *such that* $f = ts$.

Proof. We treat P as a full subcategory of A. Thus $\sigma_Q = \rho_Q$. The object Q is the α-filtered colimit of the canonical functor $T: A_\alpha/Q \to A$. So $GQ = \text{colim } GT$. Now $G(f)\sigma_A = \sigma_Q F(f) = \rho_Q F(f) = G(f)\rho_A$. Since FA is α-presentable there is an α-presentable object A_1, and morphisms $s_1: A \to A_1$ and $f_1: A_1 \to Q$, such that $f = f_1 s_1$ and $G(s_1)\sigma_A = G(s_1)\rho_A$. Replacing f by f_1 we obtain an α-presentable object A_2 and morphisms $s_2: A_1 \to A_2$ and $f_2: A_2 \to Q$ such that $f_1 = f_2 s_2$ and $G(s_2)\sigma A_1 = G(s_2)\rho A_1$. We may continue in this manner to obtain, for each $i < \omega$, an α-presentable object A_{i+1} and morphisms $s_{i+1}: A_i \to A_{i+1}$ and $f_{i+1}: A_{i+1} \to Q$ such that $f_i = f_{i+1} s_{i+1}$ and $G(s_{i+1})\sigma A_i = G(s_{i+1})\rho A_i$. Set $p = \operatorname{colim}_{i<\omega} A_i$. Then

$$\sigma_P = \operatorname*{colim}_{i<\omega} (Gs_{i+1})\sigma_{A_i}) = \rho_P$$

and P is α-presentable, being the countable colimit of α-presentable objects. □

Corollary 3.8. *The subcategory* P' *is dense in* P.

Proof. For $Q \in P$ the inclusion $P'/Q \to A_\alpha/Q$ is cofinal by the above lemma. Hence Q is the α-filtered colimit of $P'/Q \to P$. So, using the theorem on density in Chapter 0 and Lemma 3.6, the subcategory P' is dense. □

Proposition 3.9. *The 2-category* α-*Ladj* *admits equifiers and the inclusion* α-*Ladj* $\to$ *CAT* *preserves them.*

Proof. From the above, the subcategory P' is a dense subcategory of P consisting of α-presentable objects. The category P is thus locally α-presentable since it is cocomplete. Since J preserves colimits and A_α is closed under α-colimits, the subcategory P' of P is closed under α-colimits, implying that $P_\alpha = P'$. Thus $J: P \to A$ is a 1-cell in α-*Ladj*.

If $T: B \to P$ is a functor such that JT is in α-*Ladj* then T is cocontinuous. For $B \in B_\alpha$ we have $J(T(B)) \in A_\alpha$, so that $TB \in P' = P_\alpha$. Thus T also preserves α-presentable objects.

So $J: P \to A$ is the equifier in α-*Ladj*. □

The arguments need be only slightly modified for the inserter. Let $F,G: A \to B$ be a parallel pair of 1-cells in α-*Ladj* with $J: P \to A$ and $\mu: FJ \to GJ$ giving the inserter in *CAT*. Again P' denotes the full subcategory of $\mathcal{P}$ given by objects P such that $JP \in A_\alpha$. The category $\mathcal{P}$ is cocomplete and J is cocontinuous.

Lemma 3.10. *If* $P \in \mathcal{P}'$ *then* $P \in \mathcal{P}_\alpha$.

Proof. For objects C and D of $\mathcal{P}$ with $JC = A$, $JD = B$, $a = \mu_C: FA \to GA$ and $b = \mu_D: FB \to GB$ the hom-set $\mathcal{P}(C,D)$ is the equalizer of $\mathcal{B}(1,b)F_{A,B}$ and $\mathcal{B}(a,1)G_{A,B}: \mathcal{A}(A,B) \to \mathcal{B}(FA,GB)$. Thus, since $\mathcal{A}(JP,-)$, $\mathcal{B}(FJP,-)$ and G have rank α, and α-filtered colimits commute with equalizers in *Set*, the representable functor $\mathcal{P}(P,-): \mathcal{P} \to Set$ has rank α. □

The following lemma also appears in Ulmer [28].

Lemma 3.11. *If* $f: A \to JQ$ *is a morphism of* $\mathcal{A}$ *such that* $A \in \mathcal{A}_\alpha$ *and* $Q \in \mathcal{P}$ *then there is* $P \in \mathcal{P}'$ *and morphisms* $r: A \to JP$ *and* $t: P \to Q$, *such that* $f = J(t)r$.

Proof. The objects of the inserter $\mathcal{P}$ may be considered as ordered pairs $R = (C,c)$ where $C = JR$ and $\mu_R = c: FC \to GC$. Say $B = JQ$ and $b = \mu_Q: FB \to GB$. Now B is the α-filtered colimit of $T: A_\alpha/B \to A$, implying $GL = \operatorname{colim} GT$ is an α-filtered colimit. Thus $bF(f): FA \to GB$ factorizes as $G(f_1)\xi_1$ where A_1 is α-presentable and $\xi_1: FA \to GA_1$ and $f_1: A_1 \to B$ are morphisms. Moreover, we may assume that there is s_1 such that $f_1 s_1 = f$. Continuing this process, and setting $f = f_o$ and $A = A_o$, we obtain α-presentable objects A_i, and morphisms $\xi_{i+1}: FA_i \to GA_{i+1}$, $s_{i+1}: A_i \to A_{i+1}$ and $f_i: A_i \to B$, such that $f_i = f_{i+1}s_{i+1}$ and $G(f_{i+1})\xi_{i+1} = bF(f_i)$. Set $D = \operatorname{colim}_{i<\omega} A_i$ and

$\xi = \operatorname{colim}_{i<\omega} \xi_i \colon FD \to GD$. This gives an object $P = (D,\xi)$ of P. Since each A_i is α-presentable so is the α-colimit D; that is $P \in P'$. The morphism $s\colon D \to B$, given by the morphisms $f_i\colon A_i \to B$, is of the form $J(t)$ for a unique $t\colon P \to Q$, and $r\colon A \to JP = D$ is given by the morphisms $s_i\colon A_i \to A_{i+1}$. □

Corollary 3.12. *The subcategory* P' *is dense in* P.

Proof. For $Q \in P$ with $JQ = B$ the functor $P'/Q \to A_\alpha/B$ induced by J is cofinal by Lemma 3.11. Since J preserves and creates colimits Q is the α-filtered colimit of $P'/Q \to P$. So, using the results of Chapter 0 and Lemma 3.10, the subcategory P' is dense. □

Using the same proof as that given for Proposition 3.9 yields

Proposition 3.13. *The 2-category* α-*Ladj* *admits inserters and the inclusion* α-*Ladj* $\to$ CAT *preserves them.* □

When $\alpha = \aleph_0$. Lemma 3.6 and Lemma 3.10 still hold, but Lemma 3.7 and Lemma 3.11, and their corollaries, do not. As a counterexample consider the inserter of $F = 1\colon \mathit{Set} \to \mathit{Set}$ and $G = - \times 2\colon \mathit{Set} \to \mathit{Set}$, the functor which assigns to each set B the disjoint union $B \coprod B$ of two copies of B. Thus objects of the inserter P are pairs (A,f) where $A \in \mathit{Set}$ and $f\colon A \to A \coprod A$. A morphism $\bar{h}\colon (A,f) \to (B,g)$ is a morphism $h\colon A \to B$ such that $gh = (h \coprod h)f$.

From Proposition 3.13 the category P is locally $\aleph_1$-presentable. It is, in fact, locally finitely presentable. Associated with any object

(A,f) is the endomorphism $f^{\dagger}: A \to A$ formed by composing $f: A \to A \coprod A$ and the codiagonal $\Delta: A \coprod A \to A$. Any endomorphism $t: C \to C$ and any object $c \in C$ generate a set $C(c,t) = \{c, t(c), t^2(c), \ldots\}$ of images of c under iterates of t. An object (A,f) of P is finitely presentable if and only if $A = \bigcup_{i=1}^{n} A(a_i, f^{\dagger})$ where $\{a_1, \ldots, a_n\}$ is a finite subset of A. The objects of the form $(\mathbb{N}, t)$, the map $t^{\dagger}: \mathbb{N} \to \mathbb{N}$ being the successor-function, are thus finitely presentable. Let N be the full subcategory of these objects. For any object $(A,f) \in P$ and any element $a \in A$ there is a unique object $(\mathbb{N},t) \in N$ and a unique morphism $\bar{h}: (\mathbb{N},t) \to (A,f)$ such that $h(0) = a$. So N is a strong generator of finitely presentable objects.

The functor $J: P \to \mathit{Set}$ does not preserve finitely presentable objects; the category P, with the associated 1-cell and 2-cell, is not the inserter in $\aleph_0$-*Ladj*.

3.4 The retract-type completeness of *Ladj*

Using Theorem 1.25 we may combine the results of the previous sections.

Proposition 3.14. *Let α be an uncountable cardinal. Then the 2-category α-Ladj admits all indexed limits of retract type of size α. The inclusion into CAT preserves these limits.* □

The union, in CAT, of the sub-2-categories α-*Ladj* is *Ladj*. So

Theorem 3.15. *The 2-category Ladj is retract-type complete and the inclusion α-Ladj $\to$ CAT preserves indexed limits of retract type.* □

3.5 Some applications

In Chapter 5 we shall consider a symmetric monoidal closed structure on *Ladj*. The internal hom is given by $Ladj(A,B)$; the category $Ladj(A,B)$ is locally presentable if A and B are.

If K and B are α-cocomplete categories, with K small, then $\alpha\text{-}Cocomp(K,B)$ is the full subcategory of $[K,B]$ whose objects are those functors preserving α-colimits.

Proposition 3.16. (Kelly [17], Theorem 5.56). *If B is cocomplete and A is locally α-presentable then*

$$Ladj(A,B) \simeq \alpha\text{-}Cocomp(A_\alpha,B). \qquad \square$$

Thus to show that $Ladj(A,B)$ is locally presentable when A and B are it is sufficient to prove that $\alpha\text{-}Cocomp(K,B)$ is locally presentable for any small α-cocomplete category K and any locally-presentable category B. The class of (isomorphism classes of) functors $M_i\colon K_i \to K$, where K_i is an α-category, is in fact a set. For each such $M_i\colon K_i \to K$ let S_i and $T_i\colon [K,B] \to B$ be the obvious functors such that $S_i(F) = \operatorname{colim} FM_i$ and $T_i(F) = F \operatorname{colim} M_i$, and let $\eta_i\colon S_i \to T_i$ be the natural transformation such that $(\eta_i)_F\colon \operatorname{colim} FM_i \to F \operatorname{colim} M_i$ is the canonical comparison morphism. The functors S_i and T_i are cocontinuous, and hence in *Ladj*. The category $\alpha\text{-}Cocomp(K,B)$ is the joint inverter of the 2-cells η_i. So, by Theorem 3.15,

Proposition 3.17. *If K is a small α-cocomplete category and B is locally presentable then $\alpha\text{-}Cocomp(K,B)$ is locally presentable.* $\square$

Corollary 3.18. *If A and B are locally presentable then $Ladj(A,B)$ is also locally presentable.* □

Lawvere, in this study of duality, has raised the following question: Let (A,J) be a topology and E the category of sheaves on (A,J). So the inclusion $I: E \to [A^{op},Set]$ has a left-exact left adjoint $L \dashv I$ and the reflection is $\eta: 1 \to IL: [A^{op},Set] \to [A^{op},Set]$. The category F is the full subcategory of $[A,Set]$ consisting of those functors $F: A \to Set$ such that $F \star \eta_G: F \star G \to F \star ILG$ is an isomorphism for all $G: A^{op} \to Set$. (Recall that $F \star G$ is the indexed colimit; see Kelly [17], p.73). Obviously F is cocomplete and the inclusion is cocontinuous. Is F a coreflective subcategory?

The answer is affirmative.

Preopostion 3.19. *In the situation above F is locally presentable, and hence a coreflective subcategory of $[A,Set]$.*

Proof. It is well-known that a Grothendieck topos is locally presentable; a bound for the presentation rank may be given in terms of the size of a suitable topology. Because I is a right adjoint, there is a regular cardinal α such that E is locally α-presentable and I has rank α.

Indexed colimits are cocontinuous in each variable, and IL has rank α. Thus, since every object in $[A^{op},Set]$ is the α-filtered colimit of α-presentable objects, the morphism $F \star \eta_G$ is invertible for each $G \in [A^{op},Set]$ if and only if $F \star \eta_H$ is invertible for each α-presentable object $H: A^{op} \to Set$. For each such object H let $S_H = - \star H: [A,Set] \to Set$ and $T_H = - \star ILH: [A,Set] \to Set$, and let $\tau_H = - \star \eta_H: S_H \to T_H$. The functors S_H and T_H are cocontinuous and hence in *Ladj*. The category F is the joint inverter of the set of 2-cells $\{\tau_H\}$ and hence the inclusion $F \to [A, Set]$ is in *Ladj*. □

The deeper question of characterizing such subcategories F is, to the author's knowledge, still open.

Chapter 4. Purity and large limits

We have already seen that *Ladj* admits all small indexed limits of retract type, and that these limits are formed as in *CAT*. The same is true of certain important large limits. Our proof that these large limits, when formed in *CAT*, are in fact locally presentable categories uses the concept of *purity*.

Such results were , in effect, given in an unpublished manuscript by Ulmer [27], but we use a different notion of purity due to Fakir [8] to give simpler proofs more in accord with the classical notions of purity. These proofs suggest a more general theory of pure monomorphisms.

One application, also given by Ulmer, is to the cocontinuous analogue of the results in Freyd and Kelly [9].

4.1. Basic and pure monomorphisms

Throughout this chapter α is a fixed regular cardinal and A is a locally α-presentable category. Thus the functor category $[2,A]$, whose objects may be construed as morphisms of A, is also locally α-presentable; its α-presentable objects are the morphisms $f: A \to B$ of A for which both A and B are α-presentable.

The category *Mono*(A) is the full subcategory of $[2,A]$ whose objects are the monomorphisms of A.

Proposition 4.1. *The inclusion* T: *Mono*(A) $\to$ $[2,A]$ *has a left adjoint* S: $[2,A] \to$ *Mono*(A). *Moreover* T *has rank* α, *and so* *Mono*(A) *is locally* α-*presentable*.

Proof. A morphism $f: A \to B$ of $\mathcal{A}$ admits a factorization $f = f'e$, where e is a strong epimorphism and f' is a monomorphism. This factorization gives the reflection $S: [2,\mathcal{A}] \to Mono(\mathcal{A})$ such that $S(f) = f'$.

Since α-filtered colimits in $\mathcal{A}$ commute with finite limits, it is obvious that $Mono(\mathcal{A})$ admits α-filtered colimits and that T preserves them. Hence $Mono(\mathcal{A})$ is a full reflective subcategory of a locally α-presentable category, the inclusion having rank α. So $Mono(\mathcal{A})$ is itself locally α-presentable. □

We can readily identity the α-presentable objects of $Mono(\mathcal{A})$. A monomorphism $m: A \to B$ of $\mathcal{A}$ is *basic* if A is α-generated and B is α-presentable.

Proposition 4.2. *A monomorphism* $m: A \to B$ *of* $\mathcal{A}$ *is an* α*-presentable object of* $Mono(\mathcal{A})$ *if and only if it is basic.*

Proof. Let $f: C \to B$ be an α-presentable object of $[2,\mathcal{A}]$. Then $S(f): A \to B$ is a monomorphism such that B is α-presentable and A is the quotient of an α-presentable object, that is, A is α-generated.

Now let $m: A \to B$ be a basic monomorphism. Then there is an α-presentable object C and a strong epimorphism $e: C \to A$. Thus $m = S(me)$ is the image, under S, of an α-presentable object of $[2,\mathcal{A}]$.

So the basic monomorphisms, constituting the image of the α-presentable objects of $[2,\mathcal{A}]$ under S, form a strong generator of $Mono(\mathcal{A})$ consisting of α-presentable objects. To prove that every α-presentable object of $Mono(\mathcal{A})$ is a basic monomorphism of $\mathcal{A}$ it is sufficient to establish that the basic monomorphisms are closed under α-colimits in $\mathit{Mono}(A)$.

Let K be a category of size α, and let $L: K \to \mathit{Mono}(A)$ be a functor such that $L(K)$ is a basic monomorphism for each object K of K. Since the colimit $t: X \to Y$ of the composite $TL: K \to [2,A]$ is formed pointwise, and since both the α-generated objects and the α-presentable objects of A are closed under α-colimits, X is an α-generated object of A and Y is an α-presentable object. The α-generated objects of A are also closed under taking quotient objects. So $\operatorname{colim} L = S(t)$ is a basic monomorphism. □

Since every object in a locally α-presentable object is an α-filtered colimit of α-presentable objects we have

Corollary 4.3. *Every monomorphism in A is an α-filtered colimit of basic monomorphisms.* □

These colimits may be considered in $\mathit{Mono}(A)$ or $[2,A]$, for the functor T preserves and reflects α-filtered colimits.

An alternative proof of the corollary may be founded in Fakir [8], from whence he establishes Proposition 4.2.

When α is $\aleph_0$ and A is the category Grp of groups, the basic monomorphisms are readily identified by the well-known theorem of Higman, Neumann and Neumann; a finitely-generated group is the domain of a basic monomorphism if and only if it is recursively presentable.

A *pure monomorphism*, called "un morphisme α-algébriquement clos" by Fakir [8], is a monomorphism $n: C \to D$ of $\mathcal{A}$ such that for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{m} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{n} & D \end{array} ,$$

where m is a basic monomorphism, there is a morphism $h: B \to C$ for which $f = hm$. When α is $\aleph_0$ and $\mathcal{A}$ is the category of algebras for a one-sorted finitary algebraic theory, then this notion of purity coincides with the classical notion given in equational terms by Cohn [4] (see Fakir [8] for details). To prove another useful characterization of pure monomorphisms we first establish

Lemma 4.4. (Fakir [8]).

(1) *Every coretraction in* $\mathcal{A}$ *is a pure monomorphism.*

(2) *An* α*-filtered colimit of pure monomorphisms is itself a pure monomorphism.*

Proof. (1) Let $n: A \to B$ and $p: B \to A$ be morphisms in $\mathcal{A}$ such that $pn = 1$, and let

$$\begin{array}{ccc} C & \xrightarrow{m} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[n]{} & B \end{array}$$

be a commutative diagram. Then $f = pnf = (pg)m$.

(2) Let the monomorphism $n: A \to B$ be the α-filtered colimit of pure monomorphisms $n_i: A_i \to B_i$, with coprojections given by the commutative diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\;n_i\;} & B_i \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle b_i} \\ A & \xrightarrow[\;n\;]{} & B \end{array} .$$

Let

$$\begin{array}{ccc} C & \xrightarrow{\;m\;} & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[\;n\;]{} & B \end{array}$$

be a commutative diagram, where m is a basic monomorphism. Since m is an α-presentable object of $Mono(A)$, and since n is an α-filtered colimit, there is an i and a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\;m\;} & D \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ A_i & \xrightarrow[\;n_i\;]{} & B_i \end{array}$$

such that $a_i f' = f$ and $b_i g' = g$. Now n_i is pure, and so there is a morphism $h': D \to A_i$ such that $h'm = f'$. Hence $f = (a_i h')m$. □

Proposition 4.5. (Fakir [8]). *A monomorphism* $f: A \to B$ *of* A *is pure if and only if it is an* α*-filtered colimit of coretractions.*

Proof. From Lemma 4.4 every α-filtered colimit of coretractions in A is pure.

Let $f: A \to B$ be a pure monomorphism. Then, from Corollary 4.3, f is the α-filtered colimit of basic monomorphisms $n_i: A_i \to B_i$, with coprojections

$$\begin{array}{ccc} A_i & \xrightarrow{\;n_i\;} & B_i \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle b_i} \\ A & \xrightarrow[\;f\;]{} & B \end{array}$$

Since f is pure there are morphisms $h_i: B_i \to A$ such that $h_i n_i = a_i$. For each i, let

$$\begin{array}{ccc} A_i & \xrightarrow{\;n_i\;} & B_i \\ {\scriptstyle a_i}\downarrow & & \downarrow{\scriptstyle g_i} \\ A & \xrightarrow[\;f_i\;]{} & D_i \end{array}$$

be a pushout. So there is, for each i, a unique morphism $t_i: D_i \to A$ such that $t_i f_i = 1$ and $t_i g_i = h_i$. Taking the colimit of the pushout diagrams above, and recalling that colimits commute with colimits, we see that f is the α-filtered colimit of the coretractions f_i.

Corollary 4.6. *A functor* $T: A \to B$ *with rank* α, *between locally* α-*presentable categories* A *and* B, *preserves pure monomorphisms.*

Proof. Every functor preserves coretractions. So T preserves α-filtered colimits of coretractions. □

Most notions of purity are used with two principal results in mind. First, any subobject is contained in a pure subobject which is, in some sense, not too much bigger (for instance, with regard to the size of the underlying set or with regard to the presentation rank). Second, for a certain class of properties (for instance, being a divisible module) if B has these properties and if $m: A \to B$ is a pure subobject then A has these properties. Our intentions are no different.

First, we need a technical result. As throughout this chapter, $\mathcal{A}$ is a locally α-presentable category. For each object A of $\mathcal{A}$ let

$$|A| = \Sigma_G |\mathcal{A}(G,A)| ,$$

where the sum is taken over representatives G of the isomorphism classes of α-presentable objects in $\mathcal{A}$. Let τ^+ denote the successor cardinal of the cardinal τ; infinite successor cardinals are always regular.

Lemma 4.7. (Gabriel and Ulmer [10], Theorem 9.3). *Let $\beta \geq \alpha$ be a regular cardinal. An object* A *of the locally α-presentable category $\mathcal{A}$ is β-generated if and only if there is a strong epimorphism $g: \Sigma G_i \to A$, where each G_i is α-presentable and there are fewer than β summands.* □

Proposition 4.8. (Gabriel and Ulmer [10]). *Let $\gamma \geq \alpha$ be a cardinal such that $(2^\gamma)^+ > |B|$ for each α-generated object* B. *Set $\delta = (2^\gamma)^+$. Then, for each object* A *of* $\mathcal{A}$, *the following are equivalent:*

(1) $|A| < \delta$.

(2) A *is δ-presentable*

(3) A *is δ-generated.*

Proof. (1) ⇒ (2). Assume (1). Consider the objects $h: G_h \to A$ of $\mathcal{A}_\alpha/A$. There are $|A|$ (isomorphism classes of) such objects. Between any two such objects there are fewer than δ morphisms, since

$\delta > |G_h| = \Sigma_G |A(G,G_h)|$ for each object $h: G_h \to A$ of A_α/A. Thus the colimit A of the canonical functor $A_\alpha/A \to A$ is a δ-colimit of α-presentable objects and thus δ-presentable.

(2) $\Rightarrow$ (3). This implication is trivial.

(3) $\Rightarrow$ (1). Assume (3). Then, by Proposition 4.7, there is a strong epimorphism $q: \Sigma_{k\in K} G_k \to A$, where $|K| < \delta$ and each G_k is α-presentable. For each subset $J \subseteq K$, with $|J| < \alpha$, the composite morphism $G_J = \Sigma_{j\in J} G_j \to \Sigma_{k\in K} G_k \xrightarrow{q} A$ factorizes as $m_J \ell_J$, where $e_J: G_J \to A_J$ is a strong epimorphism and $m_J: A_J \to A$ is a monomorphism. The set of such subsets J, ordered by inclusion, is α-filtered. Considering colimits over this pre-ordered set, we see that the strong epimorphism q is the composite $(\text{colim } m_J)(\text{colim } e_J)$. But $\text{colim } e_J: G = \text{colim } G_J \to \text{colim } A_J$ is a strong epimorphism and $\text{colim } m_J: \text{colim } A_J \to A$, being the α-filtered colimit of monomorphisms, is a monomorphism. Hence $A \cong \text{colim } A_J$. Note that each object A_J is α-generated, and that there are at most $(2^\gamma)^\alpha = 2^\gamma$ such objects. So

$$\begin{aligned} |A| &= \Sigma_G |A(G,A)| \\ &= \Sigma_G |A(G,\text{colim } A_J)| \\ &= \Sigma_G |\text{colim } A(G,A_J)| , \end{aligned}$$

since each G is α-presentable,

$$\begin{aligned} &\leq \Sigma_G \Sigma_J |A(G,A_J)| \\ &= \Sigma_J \Sigma_G |A(G,A_J)| \\ &= \Sigma_J |A_J| \\ &= (2^\gamma)(2^\gamma) \\ &< \delta \quad , \end{aligned}$$

since $|A_J| < (2^\gamma)^+ = \delta$ for each of the α-generated objects A_J. □

Since the sets of α-presentable objects and α-generated objects of $\mathcal{A}$ are small, the set of (isomorphism classes of) basic monomorphisms is small.

Proposition 4.9. *Let the locally α-presentable category $\mathcal{A}$ be given. For every regular cardinal β there is a regular cardinal β' such that, whenever $m: A \to B$ is a subobject of B in $\mathcal{A}$, with A a β-presentable object, there is a pure subobject $m': A' \to B$, containing m, with A' a β'-presentable object of $\mathcal{A}$. Moreover, there are arbitrarily large regular cardinals δ for which we may take $\delta' = \delta$ and for which every δ-generated object is δ-presentable.*

Proof. For any subobject $m: A \to B$, consider those pairs r,s of morphisms

$$\begin{array}{ccc} E_{rs} & \xrightarrow{\ s\ } & F_{rs} \\ {\scriptstyle r}\downarrow & & \\ A & & \end{array}$$

with s basic, for which mr factorizes through s, say as $h_{rs}s$. Let the colimit of this small diagram, as r and s vary, be given by

$$\begin{array}{ccc} E_{rs} & \xrightarrow{\ s\ } & F_{rs} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle j_{rs}} \\ A & \xrightarrow[\ i_{rs}\ ]{} & X \end{array}\quad .$$

Then there is a morphism $h: X \to B$ with $hi_{rs} = m$ and $hj_{rs} = h_{rs}$. Now $h = \bar{m}e$, where $e: X \to \bar{A}$ is a strong epimorphism and $\bar{m}: \bar{A} \to B$ is a monomorphism. The composite morphisms ei_{rs} are equal, and we denote their common value by $m : A \to \bar{A}$. Thus, for every pair r,s as above, we have a morphism $t: F_{rs} \to \bar{A}$ such that $ts = m\,r$.

Observe that $\bar{A}$ is a strong-epimorphic quotient of $A + \Sigma_{r,s}F_{rs}$, and that the number of summands in the coproduct $\Sigma_{r,s}F_{rs}$ is bounded when the presentation rank β of A is known. Thus the presentation rank of $\bar{A}$ does not exceed some $\bar{\beta}$ depending only on β.

Put $A_o = A_1$, $A_1 = \bar{A}$, $m_o = m$, $m_1 = \bar{m}$ and $m^* = m_{0,1}$. Proceeding by transfinite induction we construct subobjects $m_i: A_i \to B$, indexed by the ordinals $i \le \alpha$, and morphisms $m_{i,j}: A_i \to A_j$, indexed by the pairs of ordinals $i \le j \le \alpha$, such that $m_j m_{i,j} = m_i$ for $i \le j \le \alpha$; here $A_{\gamma+1} = \bar{A}_\gamma$ and $m_{\gamma+1} = \bar{m}_\gamma$, and $A_\gamma = \cup_{\delta<\gamma} A_\delta$ when is a limit ordinal.

Set $A' = A_\alpha$ and $m' = m_\alpha$. It is clear that the presentation rank of A' does not exceed some β' depending only on β. We claim that $m_\alpha: A_\alpha \to B$ is pure. Note that $m_\alpha: A_\alpha \to B$ is not merely the union of the subobjects $m_\delta: A_\delta \to B$, $\delta < \alpha$, but is also the α-filtered colimit of the diagram (A_i, m_{ij}); for α-filtered colimit commute with finite limits, and hence preserve monomorphisms. Let

$$\begin{array}{ccc} E & \xrightarrow{s} & F \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle f} \\ A_\alpha & \xrightarrow[m_\alpha]{} & B \end{array}$$

be a commutative diagram where s is a basic monomorphism. Since E is α-generated the morphism r factorizes as $r = m_{\rho,\alpha}t$ for some $\rho < \alpha$ and some morphism $t: E \to A_\rho$. From the construction of $A_{\rho+1}$ there is a morphism $q: F \to A_{\rho+1}$ such that $qs = m_{\rho,\rho+1}t$, and hence $r = m_{\rho,\alpha}t = m_{\rho+1,\alpha}m_{\rho,\rho+1}t = (m_{\rho+1,\alpha}q)s$. So m_α is pure as required.

Let $\bar{\alpha}$ be a regular cardinal such that every α-generated object is α-presentable. Replacing α by $\bar{\alpha}$, so that A is temporarily considered as a locally $\bar{\alpha}$-presentable category, there is an arbitrarily large regular cardinal $\delta > \alpha$ such that the three conditions of Proposition 4.8 (with $\bar{\alpha}$ replacing α) are equivalent. Thus, for any δ-presentable object A and any $\bar{\alpha}$-presentable object E, we have $|A(E,A)| < \delta$. In addition, let δ be sufficiently large so that there are fewer than δ (isomorphism classes of) basic monomorphisms. Thus, if A is δ-presentable, the pairs r,s above are fewer than δ in number. So the colimit defining X is a δ-colimit. Hence X is δ-presentable and its strong-epimorphic quotient $\bar{A}$ is δ-generated, and hence is itself δ-presentable. For each limit ordinal $i \leq \alpha$, the object A_i is a strong-eipmorphic quotient of the δ-colimit $\Sigma_{j<i} A_j$. Thus each A_i - in particular $A' = A_\alpha$ - is δ-presentable. □

Recall that an object A of the locally α-presentable category A is the α-filtered colimit of its α-generated subobjects. Recall also that any cocontinuous functor between locally-presentable categories has a right adjoint.

Theorem 4.10. *Let A be a locally α-presentable category and B a full subcategory closed under colimits and pure subobjects. Then B is* *locally presentable, and hence a coreflective subcategory.*

Proof. Let $\delta \geq \alpha$ be a regular cardinal such that we may take $\delta' = \delta$ in Proposition 4.9, and such that every δ-generated object of A is δ-presentable. Since the full subcategories of δ-presentable and δ-generated objects coincide, any object A of A is the δ-filtered colimit of the canonical functor S_A: $Mon(A_\delta/A) \to A$, where $Mon(A_\delta/A)$ is the full subcategory of A_δ/A whose objects $C \to A$ are monomorphisms in A. Since we may take $\delta' = \delta$, Proposition 4.9 ensures that the inclusion T_A: $PureMon(A_\delta/A) \to Mon(A_\delta/A)$, of the full subcategory whose objects are the pure monomorphisms, is a final functor. Thus $A = \operatorname{colim} S_A = \operatorname{colim}(S_A T_A)$. If A is in B then, since B is closed under pure subobjects, the functor $S_A T_A$ takes its values in B. Hence the objects in $B \cap B_\delta$ give a strong generator for B.

Now $B \cap A_\delta \subseteq B_\delta$, for colimits in B are formed as in A. Hence B is locally δ-presentable, and the inclusion $B \to A$ has a right adjoint.

In fact the inclusion preserves δ-presentable objects. □

If so desired, an explicit bound, in terms of A, may be given for the presentation rank of such subcategories B.

Remark 4.11. Recall that we have been considering in this Section 4.1 a locally α-presentable category A. Such an A is locally β-presentable for each regular cardinal $\beta \geq \alpha$. Note, however, that the notion of basic monomorphism, and hence that of pure monomorphism, depends upon α. Since every α-basic monomorphism is β-basic, so every β-pure monomorphism is α-pure; but a monomorphism that is β-pure for all regular cardinals $\beta \geq \alpha$ must, since it is β-basic for some such β, be a coretraction.

4.2 Some large limits in *Ladj*

The results in Chapters 2 and 3 were concerned only with small limits in *Loc* on *Ladj*. Using Theorem 4.10 we may establish the existence of certain large limits.

In *CAT*, the joint inverter of a family of natural transformations $\sigma_i \colon F_i \to G_i \colon A \to B_i$ is the full subcategory of A consisting of those objects A such that each $(\sigma_i)_A$ is invertible. We have seen that if the family is a small set, and if each σ_i is in *Ladj*, then the joint inverter is in *Ladj*. This is true for a large set also - provided each σ_i is in α-*Ladj*, for some fixed regular cardinal α, and the components of the natural transformations are strong epimorphisms. First recall that:

Lemma 4.12. *A cocontinuous functor between locally-presentable categories preserves strong epimorphisms.*

Proof. Right adjoint functors preserve monomorphisms, and so any left adjoint functor preserves strong epimorphisms. But any cocontinuous functor between locally-presentable categories is a left adjoint. □

We can now prove

Theorem 4.13. *Let* $\sigma_i \colon F_i \to G_i \colon A \to B_i$ *be a family of natural transformations in* *Ladj* *such that the locally-presentable categories*

A, B_i are all locally α-presentable for some regular cardinal α, and such that each component $(\sigma_i)_A : F_iA \to G_iA$ is a strong epimorphism. Then Ladj admits the joint inverter of this family, which is formed as in CAT.

Proof. Let $J: P \to A$ be the joint inverter in *CAT* of the family of natural transformations. Then, by Theorem 2.7, P is cocomplete and J is cocontinuous. We claim that P is closed under pure (that is, α-pure) subobjects.

If $m: A \to B$ is a pure monomorphism in A, then, by Corollary 4.6, $F_i(m)$ and $G_i(m)$ are monomorphisms. If, moreover, B is an object of the subcategory P, then each component $(\sigma_i)_B$ is an isomorphism. Thus $(\sigma_i)_A$ is a monomorphism, since $G_i(m)(\sigma_i)_A = (\sigma_i)_B G_i(m)$. But $(\sigma_i)_A$ is also a strong epimorphism, and hence an isomorphism, so that $A \in P$. Thus, by Theorem 4.10, the category P is locally presentable.

If $T: C \to P$ is a functor such that JT is in *Ladj* then T is cocontinuous, and so in *Ladj*. Thus the locally-presentable category P, with the functor $J: P \to A$, is the joint inverter in *Ladj*. □

Remark 4.14. (1) Clearly the hypothesis of Theorem 4.13 may be weakened to the requirement that all but a small set of the natural transformations are, componentwise, strong epimorphisms.

(2) Note that, even if the functors F_i, G_i are in α-*Ladj*, the colimit P does not, in general, lie in α-*Ladj*.

A similar proposition holds for joint equifiers.

Proposition 4.15. *Let $\sigma_i, \rho_i : F_i \to G_i : A \to B_i$ be a family of parallel 2-cells in $Ladj$ such that the locally-presentable categories A, B_i are all locally α-presentable for some regular cardinal α. Then the 2-category $Ladj$ admits the joint equifier of this family, and it is preserved by the inclusion $Ladj \to CAT$.*

Proof. For each $A \in A$ let $(\tau_i)_A : G_i A \to H_i A$ be the coequalizer of $(\sigma_i)_A$ and $(\rho_i)_A$. Then there is a unique way of extending H_i to a functor $H_i : A \to B_i$ so that τ_i is a natural transformation. Since F_i and G_i are cocontinuous so is H_i.

The joint equifier of the pairs (σ_i, ρ_i) is given by the joint inverter of the natural transformations τ_i. Hence we may apply Theorem 4.13 to obtain the desired result. □

Alternatively, a proof of this proposition could be given by verifying directly that the subcategory of A which is the joint equifier in CAT is closed under pure subobjects.

4.3 Categories of cocontinuous functors

The main application of the results so far established in this chapter is to categories of cocontinuous functors. As mentioned above, although Proposition 4.16 and Theorem 4.18 have already been proved by Ulmer [27], our proofs use a different, and simpler, notion of purity.

Let A, B and C be locally-presentable categories, and let $T : B \times A \to C$ be a functor which is cocontinuous in each variable - that is, one for which each $T(B,-)$ and $T(-,A)$ is a left adjoint.

For a family Σ of morphisms $\sigma: X_\sigma \to Y_\sigma$ in $\mathcal{B}$, let A_Σ be the full subcategory of A whose objects A are those for which each $T(\sigma,A)$ is an isomorphism. Then

Proposition 4.16. *If the set of isomorphism classes of the codomains* Y_σ *is small then* A_Σ *is locally presentable.*

Proof. Let $e_\sigma: X_\sigma \to Z_\sigma$ and $m_\sigma: Z_\sigma \to Y_\sigma$ give the factorization of $\sigma: X_\sigma \to Y_\sigma$ into a strong epimorphism and a monomorphism. Since $\mathcal{B}$ is wellpowered the family of monomorphisms m_σ is a small set. Because $T(-,A)$ is a left adjoint, each $T(e_\sigma,A)$ is a strong epimorphism. Thus $T(\sigma,A)$ is an isomorphism if and only if $T(e_\sigma,A)$ and $T(m_\sigma,A)$ are. By Theorem 4.13 and Remark 4.14(1), the category A_Σ is locally presentable. □

Remark 4.17. The proposition above still holds under conditions on $\mathcal{B}$ weaker than being a locally-presentable category. For, if $\mathcal{B}$ is complete and wellpowered it certainly admits strong epimorphism-monomorphism factorizations. Moreover, $T(-,A)$ preserves strong epimorphisms whenever it is a left adjoint.

A *cocone* with vertex X in a category $\mathcal{C}$ is a natural transformation $\gamma: S \to \Delta X: \mathcal{D} \to \mathcal{C}$. For a family Γ of small cocones $\gamma: S_\gamma \to \Delta X_\gamma: \mathcal{D}_\gamma \to \mathcal{C}$ in a small category $\mathcal{C}$, the category $[\mathcal{C},A]_\Gamma$ is the full subcategory of the functor category $[\mathcal{C},A]$ whose objects G are the functors for which each $G\gamma$ is a colimit cocone.

The cocontinuous analogue of Theorem 5.21 of Freyd and Kelly [9], for locally-presentable categories, is

Theorem 4.18. *Let* A *be a locally-presentable category and* Γ *a (not necessarily small) family of small cocones in a small category* C. *Then* $[C,A]_\Gamma$ *is locally presentable, and hence a coreflective subcategory of* $[C,A]$.

Proof. For a functor $S\colon \mathcal{D} \to \mathcal{C}$, let $\hat{S}\colon \mathcal{C}^{op} \to [\mathcal{D},\mathcal{S}et]$ denote the functor defined by $\hat{S}C = \mathcal{C}(C,S-)$. A cocone $\gamma\colon S \to \Delta X\colon \mathcal{D} \to \mathcal{C}$ and an object C of $\mathcal{C}$ determine a natural transformation $\mathcal{C}(C,\gamma)\colon \mathcal{C}(C,S-) \to \Delta\mathcal{C}(C,X)\colon \mathcal{D} \to \mathcal{S}et$. The induced morphisms $\mathrm{colim}_D \mathcal{C}(C,SD) \to \mathcal{C}(C,X)$ are the components of a natural transformation

$$\bar{\gamma}\colon (\mathrm{colim}_D \circ \hat{S}) \to \mathcal{C}(-,X)\colon \mathcal{C}^{op} \to \mathcal{S}et.$$

Now, the cocone $G\gamma\colon GS \to \Delta GX\colon \mathcal{D} \to A$, given by composition with the functor $G\colon C \to A$, is a colimit cocone if and only if the induced morphism $\mathrm{colim}\ GS \to GX$ is invertible. But by Kelly [17] (3.10) and (4.1), and by the cocontinuity of the indexed colimit in the first variable, we have $GX \cong C(-,X) * G$ and $\mathrm{colim}\ GS \cong \mathrm{colim}(\hat{S} * G) \cong (\mathrm{colim}\ \hat{S}) * G$. Thus we see that $G\gamma$ is a colimit cocone if and only if $\bar{\gamma} * G\colon (\mathrm{colim}\ \hat{S}) * G \to C(-,X) * G$ is invertible.

Taking $T\colon [C^{op},\mathcal{S}et] \times [C,A] \to A$ to be the indexed colimit $T(F,G) = F * G$ and the family Σ to be the natural transformation $\bar{\gamma}\colon \mathrm{colim}\ \hat{S}_\gamma \to C(-,X_\gamma)$, the hypotheses of Proposition 4.16 are satisfied. Hence $[C,A]_\Gamma$ is locally presentable.

Obviously the inclusion $[C,A]_\Gamma \to [C,A]$ is cocontinuous, and so has a right adjoint. □

Chapter 5. Symmetric monoidal closed structures

In a 2-category equivalence of objects is more common than isomorphism. We say that a 2-category M, or more generally a bicategory, is *symmetric monoidal closed* if there is given a tensor product $- \otimes -: M \times M \to M$, which is associative, unitary, and commutative to within *equivalences* (and no longer isomorphisms), subject to suitable coherence axioms; and if each $- \otimes M: M \to M$ admits a right biadjoint $\mathrm{Hom}(M,-)$. We do not intend to examine here what the appropriate coherence conditions may be, for we are considering particular and naturally-occurring cases where whatever conditions are appropriate are surely satisfied.

The 2-category α-*Th* has a symmetric monoidal closed structure in this sense (see Kelly [17] and [18]). This structure transfers to one on α-*Ladj*, which in turn gives rise to such a structure on *Ladj*. In this chapter we explore these various symmetric monoidal closed structures, and also the biclosed structure of *Ladj*.

5.1 Some symmetric monoidal closed 2-categories

The 2-category α-*Cont* of all α-complete categories, α-continuous functors and natural transformations contains the full sub-2-category α-*Th* of all *small* α-complete categories. Kelly [18] describes an obvious symmetric monoidal closed structure on α-*Th*; the category $\alpha\text{-}Th(A \otimes B, C)$ may be seen as the full subcategory of $CAT(A \times B, C)$ whose objects $F: A \times B \to C$ are those functors α-continuous in each variable separately. Denoting this tensor product by $A \underset{\alpha}{\otimes} B$ we have $\alpha\text{-}Cont(A \underset{\alpha}{\otimes} B, C) = \alpha\text{-}Cont(A, \alpha\text{-}Cont(B,C))$, where C is any α-complete category.

Associated with the α-theory A is the locally α-presentable category $\mathcal{A} = \alpha\text{-}Cont(A, Set) = A\text{-}Alg$. Under the biequivalence $\alpha\text{-}Th \sim (\alpha\text{-}Ladj)^{co}$, the 2-category $(\alpha\text{-}Ladj)^{co}$, and thus $\alpha\text{-}Ladj$, inherits a symmetric monoidal closed structure. Using Theorem 9.9 of Kelly [18], we have

$$\begin{aligned}(A \underset{\alpha}{\otimes} B)\text{-}Alg &= \alpha\text{-}Cont(A \underset{\alpha}{\otimes} B, Set)\\ &\simeq \alpha\text{-}Cont(A, \alpha\text{-}Cont(B, Set))\\ &= \alpha\text{-}Cont(A, \mathcal{B})\\ &\simeq Ladj(\mathcal{A}, \mathcal{B}^{op})^{op}\ ;\end{aligned}$$

so that the tensor product, in $\alpha\text{-}Ladj$, may be taken to be

$$\mathcal{A} \underset{\alpha}{\otimes} \mathcal{B} = Ladj(\mathcal{A}, \mathcal{B}^{op})^{op}\ , \tag{5.1}$$

which is independent of α. The unit of the tensor product is Set, and the internal hom in $\alpha\text{-}Ladj$ is

$$Hom(\mathcal{A}, \mathcal{B}) = \alpha\text{-}Th(\mathcal{A}_\alpha^{op}, \mathcal{B}_\alpha^{op})\text{-}Alg.$$

Let α, β be a pair of regular cardinals such that $\alpha \leq \beta$. Then every β-complete category is trivially α-complete, and every locally α-presentable category is locally β-presentable. Thus there is a functor $T\colon \beta\text{-}Th \to \alpha\text{-}Th$, and a pseudo-functor $S\colon \alpha\text{-}Th \to \beta\text{-}Th$, given by the composite of the biequivalence $\alpha\text{-}Th \sim (\alpha\text{-}Ladj)^{co}$, the inclusion $(\alpha\text{-}Ladj)^{co} \to (\beta\text{-}Ladj)^{co}$, and the biequivalence $(\beta\text{-}Ladj)^{co} \sim \beta\text{-}Th$. For an α-theory A, it follows from Kelly [17] that SA is the β-theory generated by the sketch $(A;\Phi)$, where Φ consists of all α-limit cones in A.

Proposition 5.2. *The pseudo-functors* S *and* T *give a biadjunction*

$$S \dashv T\colon \beta\text{-}Th \to \alpha\text{-}Th\ .$$

Proof. If A is a small α-complete category and B is a small β-complete category, then, by the comments above, $\beta\text{-}Th(SA,B) \simeq \alpha\text{-}Th(A,B) \simeq \alpha\text{-}Th(A,TB)$. □

Since, by (5.1), the inclusion $\alpha\text{-}Ladj \to \beta\text{-}Ladj$ preserves the tensor product, so does S; so that T has a monoidal structure (see Kelly [30]).

Moreover (5.1) defines a tensor product on the union *Ladj* of the 2-categories α-*Ladj*, giving a symmetric monoidal structure on *Ladj*, with *Set* as the unit of the tensor product.

Proposition 5.3. *The symmetric monoidal structure on* Ladj, *given by* (5.1), *has an internal hom given by*

$$\mathrm{Hom}(A,B) = Ladj(A,B).$$

Proof. Let the regular cardinal α be large enough so that the locally--presentable categories *A*, *B* and *C* are locally α-presentable. Then we have

$$\begin{aligned} Ladj(A,Ladj(B,C)) &\simeq \alpha\text{-}Cocont(A\ ,Ladj(B,C)) \\ &\simeq \alpha\text{-}Cocont(A_\alpha,\alpha\text{-}Cocont(B_\alpha,C)) \\ &\simeq Comod(A_\alpha \times B_\alpha,C) \\ &\simeq \alpha\text{-}Cont(A_\alpha^{op} \otimes_\alpha B_\alpha^{op},C^{op})^{op} \\ &\simeq Ladj(A \otimes B,C)\ , \end{aligned}$$

by Theorem 9.9 of Kelly [18]; here $Comod(A^{op} \times B^{op},C)$ is the full subcategory of $CAT(A^{op} \times B^{op},C)$ consisting of those functors which are α-cocontinuous in each variable separately. □

5.2 The biclosed structure of *Ladj*

Recall that a bicategory M is biclosed if, for each 1-cell $f: A \to B$ and each object C, the functors $M(f,1): M(B,C) \to M(A,C)$ and $M(1,f): M(C,A) \to M(C,B)$ given by composition with f have right adjoints.

Proposition 5.4. *The 2-category Ladj is biclosed.*

Proof. Let A, B and C be locally-presentable categories and $T: A \to B$ a left adjoint functor. The left adjoint functors between locally-presentable categories are precisely the cocontinuous functors. Hence colimits in $Ladj(A,B)$ are formed pointwise in B, and the functors $Ladj(T,1): Ladj(B,C) \to Ladj(A,C)$ and $Ladj(1,T): Ladj(C,A) \to Ladj(C,B)$ are cocontinuous functors between locally-presentable categories. Hence they have right adjoints. □

Chapter 6. Locally-presentable enriched categories

Most of the important results concerning locally-presentable categories can, with only slight alteration, be stated and proved in an enriched context - provided the base-category is suitable. Kelly has initiated such a program in [18]. This chapter furthers that program by giving results analogous to those in Chapters 2 and 3.

Throughout this chapter V is a complete and cocomplete symmetric monoidal closed category. For convenience we use the terms "category", "functor" and "monad" for "V-category", "V-functor" and "V-monad", and distinguish a V-category A from its underlying ordinary category A_o. It is assumed that the reader is acquainted with the basic notions of enriched category theory; we refer them to Kelly [17] for any unfamiliar terms or notation.

6.1 Limits of V-categories

The 2-category $V\text{-}CAT$ consists of V-categories, V-functors and V-natural transformations. Of course its hom-categories are not, in general, locally small. Yet is admits all small indexed limits. The 2-functor $(\ )_o : V\text{-}CAT \to CAT$, assigning to each V-category its underlying category, preserves these limits; indeed there is a left adjoint $L: CAT \to V\text{-}CAT$, the free V-category LK on the ordinary category having the same objects as K and having $LK(K,K') = K(K,K').I$, the coproduct of $K(K,K')$ copies of I.

We describe the basic indexed limits of retract type in $V\text{-}CAT$.

The product ΠA_i of a set of categories - that is, of V-categories - has as objects the set $\Pi ob(A_i)$, and the typical hom-object is $(\Pi A_i)(A,B) = \Pi A_i(A_i,B_i)$, where A_i and B_i are the i-components of A and B respectively.

The cotensor product $\{K,A\}$ of a small ordinary category K and a V-category A is the enriched functor category $[LK,A]$. Given any V-category B, there is a suitable extension V' of V such that $[B,A]$ is a V'-category (see Kelly [17], Section 3.11). Then

$$\begin{aligned} V\text{-}CAT(B,[LK,A]) &\cong V\text{-}CAT(B \otimes LK,A) \\ &\cong V'\text{-}CAT(B \otimes LK,A) \\ &\cong V'\text{-}CAT(LK,[B,A]) \\ &\cong CAT(K,V'\text{-}CAT(B,A)) \\ &\cong CAT(K,V\text{-}CAT(B,A)) \ . \end{aligned}$$

Under the isomorphism $\Psi\colon V\text{-}CAT(B,[LK,A]) \xrightarrow{\cong} CAT(K,V\text{-}CAT(B,A))$ we have

$$(\Psi S)K = E_K S\colon B \to A \qquad (6.1)$$

where $E_K\colon [LK,A] \to A$ is the V-functor given by evaluation at K.

The equifier $J\colon P \to A$ of a parallel pair of natural transformations $\rho,\sigma\colon F \to G\colon A \to B$ is the full subcategory of A whose objects A satisfy $\rho_A = \sigma_A$.

The objects of the inserter $Q = Ins(F/G)$ for $F,G\colon A \to B$ are pairs (A,a) where A is an object of A and $a\colon FA \to GA$ is a morphism in B_o, and the hom-object $Q((A,a),(B,b))$ is the equalizer of the two morphisms $\mathcal{B}(1,b)F_{A,B}$ and $\mathcal{B}(a,1)G_{A,B}$: $\mathcal{A}(A,B) \to \mathcal{B}(FA,GB)$. Here the morphism $F_{A,B}$: $\mathcal{A}(A,B) \to \mathcal{B}(FA,FB)$, which is often abbreviated to F, is part of the data for the functor F, and the morphism $\mathcal{B}(f,g)$: $\mathcal{B}(C,D) \to \mathcal{B}(X,Y)$, for $f: X \to C$ and $g: D \to Y$ in $\mathcal{B}_o$, is that described, for instance, in Kelly [17], p.37.

For the splitting $R: \mathcal{A} \to \mathcal{B}$ and $J: \mathcal{B} \to \mathcal{A}$ of the idempotent endofunctor $F: \mathcal{A} \to \mathcal{A}$, the objects of $\mathcal{B}$ are the objects A of $\mathcal{A}$ such that $FA = A$ and the hom-object $\mathcal{B}(A,B)$ is given by the splitting of the idempotent $F_{A,B}$: $\mathcal{A}(A,B) \to \mathcal{A}(A,B)$.

Let $\mathcal{J} = \{J_i: L_i \to V\}$ be a class of indexing types. As in Chapter 2, where $V = \mathit{Set}$, the 2-category $\mathcal{J}$-*Comp* consists of all (small) categories admitting J_i-indexed limits for all $J_i \in \mathcal{J}$, those functors which preserve these limits, and natural transformations between them. Modifying the proof of Theorem 2.6 gives

Proposition 6.2. *The 2-category $\mathcal{J}$-Comp admits all indexed limits of retract type and the inclusion into V-CAT preserves them.* □

6.2 Some basic facts about locally-presentable enriched categories

To develop results analogous to those for locally α-presentable ordinary categories we must impose certain restrictions on the base-category V, namely that V_o be locally α-presentable and that $(V_o)_\alpha$ be closed under the monoidal structure, in the sense that $X \otimes Y \in V_{o\alpha}$ when $X,Y \in V_{o\alpha}$ and that the unit I for the tensor product lies in $V_{o\alpha}$. Such a V is said to be *locally α-presentable as a closed category* (see Kelly [18]). Obviously such a closed category V is then locally

β-presentable as a closed category for each regular cardinal $\beta \geq \alpha$. In the sequel the base-category V will always be locally α-presentable as a closed category. In most important cases V is locally finitely presentable as a closed category.

A category A *admits* β*-filtered colimits* if it admits the V-colimit of each functor $S: K \to A_0$ (see Kelly [17], pp.94-96), where the small ordinary category K is β-filtered. An object A in such a category is β*-presentable* if the functor $A(A,-): A \to V$ preserves β-filtered colimtits. A *locally* β*-presentable category* A, where $\beta \geq \alpha$ is a regular cardinal, is one that is cocomplete and has a small strong generator consisting of β-presentable objects. Since V is locally β-presentable as a closed category we may assume, when considering 2-categories of locally β-presentable categories, that $\beta = \alpha$.

As shown in Corollary 7.3 of Kelly [18], the full subcategory A_α consisting of the α-presentable objects of the locally α-presentable category A is, in fact, a dense generator.

A V-category M is an α*-category* if it has fewer than α objects and if each hom-object $M(M,M')$ is in V_α. A functor $J: M \to V$ is an α*-functor* if M is an α-category and J factorizes through the inclusion $V_\alpha \to V$; an α*-limit* is a limit indexed by an α-functor.

Proposition 6.3. (Kelly [18], Proposition 4.9). *In a locally* α*-presentable category* α*-limits commute with* α*-filtered colimits.* □

Proposition 6.4. (Kelly [18], Theorem 7.2). *If* A *is a locally* α*-presentable category then the subcategory* A_α *is closed under* α*-colimits.* □

Kelly also gives, in his Proposition 7.5, a very useful characterization of locally α-presentable categories in terms of their underlying categories:

Proposition 6.5. *Let* A *be a cocomplete category. If* A_o *is locally* α*-presentable and if* $A_{o\alpha} \subseteq A_{\alpha o}$ *then* A *is locally* α*-presentable.* □

When A is locally α-presentable it is true that $A_{o\alpha} = A_{\alpha o}$. The result above is useful in dealing with V-α-*Ladj*, the 2-category of locally α-presentable V-categories corresponding to α-*Ladj*. In the case of V-α-*Loc* we need instead some properties of monads on locally α-presentable categories.

Recall, from Section 1.3, how the object of algebras for a monad may be constructed from inserters and equifiers. To show the completeness of the category of algebras for a monad on a complete category, we need modify but slightly the proof of Theorem 2.6.

Proposition 6.6. *Let* $F,G\colon A \to B$ *be a pair of 1-cells in* V-CAT, *and let* $L\colon K \to V$ *be an indexing type such that* A *admits* L*-indexed limits and* G *preserves them. Then the inserter* $P = \mathrm{Ins}(F/G)$ *admits* L*-indexed limits and the associated functor* $J\colon P \to A$ *preserves them.*

Proof. Let $T\colon K \to P$ be a functor. The natural transformation $\mu\colon FJ \to GJ$ associated with the inserter induces a morphism $\{L,\mu T\}\colon \{L,FJT\} \to \{L,GJT\}$. There are also canonical morphisms $t\colon F\{L,JT\} \to \{L,FJT\}$ and $s\colon G\{L,JT\} \to \{L,GJT\}$, the latter being an isomorphism since G preserves L-indexed limits. Set $m = s^{-1}\{L,\mu T\}t\colon F\{L,JT\} \to G\{L,JT\}$. We claim that the object $(\{L,JT\},m)$ of the inserter P is the L-indexed limit of T.

For $(P,p) \in \mathcal{P}$ the hom-object $\mathcal{P}((P,p),(\{L,JT\},m))$ is the equalizer of $\mathcal{B}(1,m)F$ and $\mathcal{B}(p,1)G\colon \mathcal{A}(P,\{L,JT\}) \to \mathcal{B}(FP,G\{L,JT\})$; thus it is also the equalizer of $\mathcal{B}(1,s)\mathcal{B}(1,m)F = \mathcal{B}(1,\{L,\mu T\})\mathcal{B}(1,t)F$ and $\mathcal{B}(1,s)\mathcal{B}(p,1)G = \mathcal{B}(p,s)G$. Hence, composing with the natural isomorphisms $[K,V](L,\mathcal{A}(P,JT-)) = \mathcal{A}(P,\{L,JT\})$ and $\mathcal{B}(FP,\{L,GJT\}) = K,V\,(L,\mathcal{B}(FP,GJT-))$, we see that $\mathcal{P}((P,p),(\{L,JT\},m)) = [K,V](L,\mathcal{P}((P,p),T-))$, the isomorphism being natural in (P,p). Thus $(\{L,JT\},m)$ is indeed the L-indexed limit of T, and it is preserved by J. □

Proposition 6.7. *Let* $\sigma,\rho\colon F \to G\colon A \to \mathcal{B}$ *be a pair of 2-cells in* V-CAT, *and let* $L\colon K \to V$ *be an indexing type such that* A *admits* L-*indexed limits and* G *preserves them. Then the equifier* $P = \text{Equif}(F,G)$ *admits* L-*indexed limits and the associated functor* $J\colon P \to A$ *preserves them.*

Proof. We consider the equifier $J\colon P \to A$ as a full subcategory of A. For a functor $T\colon K \to P$ let $t\colon F\{L,JT\} \to \{L,FJT\}$ and $s\colon G\{L,JT\} \to \{L,GJT\}$ again be the canonical morphisms. It is sufficient to prove that $P = \{L,JT\}$ is in the full subcateogry P - that is, $\sigma_P = \rho_P$. Since $\sigma J = \rho J\colon FJ \to GJ$, the induced morphisms $\{L,\sigma JT\}$ and $\{L,\rho JT\}\colon \{L,FJT\} \to \{L,GJT\}$ are equal. Thus $s\sigma_P = \{L,\sigma JT\}t = \{L,\rho JT\}t = s\rho_P$. But s is an isomorphism, and hence $\sigma_P = \rho_P$ as required.

For a monad (T,η,μ) on a category A the Eilenberg-Moore category is denoted by A^T. The forgetful functor $U\colon A^T \to A$ and the functor $F\colon A \to A^T$, assigning to each object of A the corresponding free algebras, give an adjunction $F \dashv U\colon A^T \to A$.

Theorem 6.18. *Let* (T,η,μ) *be a monad on the category* A.

(1) *If* A *admits* J*-indexed limits, for an indexing type* $J\colon K \to V$, *then* A^T *admits* J*-indexed limits and* U *preserves them.*

(2) *If* A *admits* J*-indexed colimits, for an indexing type* $J\colon K^{op} \to V$, *and if the functor* T *preserves them, then* A^T *admits* J*-indexed colimits and* U *preserves them.*

Proof. (1) The result follows using Propositions 6.6 and 6.7 and the construction in Section 1.3 of objects of algebras by means of inserters and equifiers.

(2) In this case, we use the duals of Propositions 6.6 and 6.7. □

When A is locally α-presentable and the endofunctor T has rank α more may be said about A^T.

Theorem 6.9. *Let* (T,η,μ) *be a monad on the locally* α*-presentable category* A *such that* T *has rank* α. *Then* A^T *is itself locally* α*-presentable and* $U\colon A^T \to A$ *is a continuous functor with rank* α.

Proof. By Theorem 6.8 the category A^T is complete and admits α-filtered colimits; the functor $U\colon A^T \to A$ preserves all limits and all α-filtered colimits.

Let K be the full subcategory of A^T whose objects are the free algebras on the α-presentable objects of A. The free algebra FA on the α-presentable object A of A is itself α-presentable in A^T, for $A^T(FA,-) \cong A(A,-)U$ has rank α. We shall use Theorem 5.19 of Kelly [17] to prove that the inclusion $J\colon K \to A^T$ is dense.

An object A of $\mathcal{A}$ is the colimit of the canonical functor $T_A\colon \mathcal{A}_\alpha/A \to \mathcal{A}$. Thus the free algebra FA on A is the colimit of the functor FT_A. Moreover, since this colimit is α-filtered, it is J-absolute (see Kelly [17], p.170). For any algebra (B,b) the diagram

$$FTB \underset{\mu_B}{\overset{Fb}{\rightrightarrows}} FB \xrightarrow{\ b\ } (B,b)$$

exhibits (B,b) as a coequalizer. Since this colimit is U-split, it is preserved by each $\mathcal{A}^T(FA,-) \cong \mathcal{A}(A,-)U$, and hence is J-absolute. Thus, by Theorem 5.19 of Kelly [17], $\mathcal{K}$ is a dense generator consisting of α-presentable objects.

It remains to prove that $\mathcal{A}^T$ is cocomplete. Let $G\colon \mathcal{A}_\alpha \to \mathcal{K}$ be the restriction of F, and let $H = G^{op}\colon \mathcal{A}_\alpha^{op} \to \mathcal{K}^{op}$. Then $[H,1]\tilde{J} = \tilde{I}U\colon \mathcal{A}^T \to [\mathcal{A}_\alpha^{op},\mathcal{V}]$, where $I\colon \mathcal{A}_\alpha \to \mathcal{A}$ is the inclusion. Now H is surjective on objects, and every natural transformation $s\colon S \to S'\colon \mathcal{A}_\alpha^{op} \to \mathcal{V}$ is determined by the morphisms $s_A\colon SA \to S'A$. Thus $[H,1]\colon [\mathcal{K}^{op},\mathcal{V}] \to [\mathcal{A}_\alpha^{op},\mathcal{V}]$ is conservative. Since $[H,1]$ also preserves all colimits, it reflects all colimits, so that the adjunction $\mathrm{Lan}_H \dashv [H,1]\colon [\mathcal{K}^{op},\mathcal{V}] \to [\mathcal{A}_\alpha^{op},\mathcal{V}]$ is monadic. The functor $\tilde{I}U$ has a left adjoint since U and $\tilde{I}$ each have a left adjoint. Thus Dubuc's adjoint triangle theorem [5] is applicable, and $\tilde{J}$ has a left adjoint – that is, $\mathcal{A}^T$ is a reflective subcategory of $[\mathcal{K}^{op},\mathcal{V}]$. Hence $\mathcal{A}^T$ is cocomplete as required. $\square$

6.3 The retract-type completeness of V-*Loc*

As for V = *Set*, the objects of the 2-category V-α-*Loc* are the locally α-presentable categories, the 1-cells are the continuous functors with rank α - all such have left adjoints by Kelly [18] - and the 2-cells are the natural transformations. The union, taken over all regular cardinals $\beta \geq \alpha$, of the 2-categories V-β-*Loc* is V-*Loc*; its objects are locally--presentable categories, its 1-cells are functors which have a left adjoint, and its 2-cells are natural transformations. To prove the retract-type completeness of V-*Loc* we first establish the retract-type completeness of V-α-*Loc*.

Theorem 6.10. *The 2-category V-α-Loc admits*

(1) *products*

(2) *cotensor products*

(3) *inserters*

(4) *equifiers*

and

(5) *splittings of idempotents.*

Hence V-α-Loc admits all indexed limits of retract type. Moreover the inclusion of V-α-Loc in V-CAT preserves these limits.

Proof. (1) *Products.* The proof given in Proposition 2.11 holds for the enriched case. The product, in V-*CAT*, of locally α-presentable categories is complete and cocomplete, the projections $P_j : \prod A_i \to A_j$ are continuous and cocontinuous, and a strong generator formed of α-presentable objects is given by the objects of the form $Q_j(A)$, where $Q_j \dashv P_j$ and $A \in A_{j\alpha}$.

For any set of 1-cells $B \to A_i$ in V-α-Loc the associated functor is continuous and has rank α.

(2) *Cotensor products.* For any small ordinary category K and any locally α-presentable category A the enriched functor category $[LK,A]$ is locally α-presentable; for it is complete and cocomplete, since A is, and the objects of the form $(LK)(K,-) \otimes A$, where $K \in LK$ and $A \in A_\alpha$, give a strong generator consisting of α-presentable objects. The evaluation functors $E_K: [LK,A] \to A$ preserve, and jointly detect, limits and colimits. In Section 6.1 we established the isomorphism $\Psi: V\text{-}CAT(B,[LK,A]) \xrightarrow{\cong} CAT(K, V\text{-}CAT(B,A))$. In the case where B is locally α-presentable we see, from (6.1), that the functor $S: B \to [LK,A]$ is continuous and has rank α if and only if each $(\Psi S)K$ is continuous and has rank α - that is, ΨS takes its values in V-α-$Loc(B,A)$. Hence Ψ restricts to an isomorphism V-α-$Loc(B,[LK,A]) \cong CAT(K, V$-$\alpha$-$Loc(B,A))$.

(3) *Inserters.* Let $F,G: A \to B$ be a parallel pair of 1-cells in V-α-Loc whose inserter, in V-CAT, is $J: P \to A$. By Proposition 6.2 and the corresponding assertion for a class of indexing types for colimits, the category P admits all limits and α-filtered colimits, and the functor J preserves them. Since J_o has a left adjoint, and since J preserves cotensor products, J has a left adjoint (see Kelly [29]). Now, since J_o is monadic, so is J by Theorem 2.2.1 of Dubuc [6]. Thus, applying Theorem 6.9, the category P is locally α-presentable. If $R: C \to P$ is a functor such that $JR: C \to A$ is in V-α-Loc then R is continuous with rank α. Hence $J: P \to A$, with the associated 2-cell, is in fact the inserter in V-α-Loc.

(4) *Equifiers.* The proof is almost identical to that for inserters.

(5) *Splittings of idempotents.* Again we apply Proposition 6.2 and Theorem 6.9. For an idempotent $F: A \to A$ the monad on A arises from the adjunction $I_* \dashv I: B \to A$ given in Proposition 2.2. □

Theorem 6.11. *The 2-category V-Loc admits all indexed limits of retract type and they are preserved by the inclusion V-Loc → V-CAT.*

Proof. The result follows immediately from Theorem 6.10. □

6.4 The retract-type completeness of *V-Ladj*

The objects of the 2-category *V-α-Ladj* are the locally α-presentable categories, the 1-cells are cocontinuous functors which preserve α-presentable objects (that is, functors whose right adjoints have rank α) and the 2-cells are natural transformations. The 2-category *V-Ladj* is the union of all such 2-categories. To establish the retract-type completeness of *V-Ladj* we consider indexed limits of retract type in *V-α-Ladj* which are formed as in *V-CAT*. As with the *Set*-based case, the size of α places restrictions on the size of these limits.

Proposition 6.12. *The 2-category V-α-Ladj admits splitting of idempotents.*

Proof. If $F: A \to A$ is an idempotent endofunctor in *V-α-Ladj*, then it has a splitting $R: A \to B$, $I: B \to A$ such that B is cocomplete and R and I are cocontinuous. Moreover, since R_o and I_o give a splitting of F_o, the ordinary category B_o is locally α-presentable. The same proof as in Corollary 3.2 shows that the objects RA, where $A \in A_\alpha$, are α-presentable in B. Hence $B_{o\alpha} \subseteq B_{\alpha o}$, and so B is locally α-presentable, with R and I preserving α-presentable objects. □

A proof that products with fewer than α components exist in V-α-*Ladj* lifts directly from that given in Proposition 3.3 for the particular case $V = Set$.

Proposition 6.13. *The 2-category V-α-Ladj admits all products with fewer than α components and these products are preserved by the inclusion into V-CAT.* $\square$

Before examining the cotensor product we shall make some observations about evaluation functors. If M is an object of the small category M and if B is a complete and cocomplete category then the evaluation functor $E_M: [M,B] \to B$ has both adjoints $S_M \dashv E_M \dashv T_M$ where $S_M(B) = M(-,M) \otimes B: M \to B$ and $T_M(B) = M(M,-) \pitchfork B: M \to B$. As usual $V \pitchfork B$ denotes the cotensor product, and $V \otimes B$ the tensor product, of $V \in V$ and $B \in B$.

Proposition 6.14. *Let B be a locally α-presentable category and M an α-category. A functor $F: M \to B$ which factorizes through the inclusion $B_\alpha \to B$ is an α-presentable object of $[M,B]$.*

Proof. Since $E_M: [M,B] \to B$ and $B(FN,-): B \to V$, where M and N are objects of M, each have rank α, so does the representable functor $[M,B](M(M,-) \otimes FN,-) \cong B(FN,-)E_M$ - that is, $M(M,-) \otimes FN$ is an α-presentable object of $[M,B]$. Since $M(N,M)$ is α-presentable in V, so also $M(N,M) \otimes M(M,-) \otimes FN$ is α-presentable in $[M,B]$. Now, the functor F is the coend $\int^M M(M,-) \otimes FM$. Thus the corresponding coequalizer diagram (see (2.2) of Kelly [17])

$$\Sigma_{M,N} M(N,M) \otimes M(M,-) \otimes FN \rightrightarrows \Sigma_M M(M,-) \otimes FM \to \int^M M(M,-) \otimes FM$$

exhibits F as an iterated α-colimit of α-presentable objects, since the coproducts have fewer than α summands. Hence F itself is an α-presentable object of $[M,B]$. □

If K is an ordinary α-category then LK is an α-category. Using the proposition above, and a modification of the proof of Proposition 3.5, we have:

Proposition 6.15. *If K is an ordinary α-category and B is a locally α-presentable category, then the cotensor product exists in V-α-Ladj and is $\{K,B\} = [L(K),B]$. Hence cotensor products with α-categories are preserved by the inclusion V-α-Ladj $\to$ V-CAT.* □

For the existence of inserters and equifiers in V-α-$Ladj$ we impose, as before, the restriction that α be uncountable.

Theorem 6.16. *Let α be an uncountable regular cardinal. Then V-α-Ladj admits*

(1) *inserters*

and

(2) *equifiers.*

Moreover the inclusion into V-CAT preserves them.

Proof. (1) *Inserters.* For a parallel pair of 1-cells $F,G: A \to B$ in V-α-$Ladj$ let $J: P \to A$, with associated 2-cell $\mu: FJ \to GJ$, be the inserter in V-CAT. Since A and B are cocomplete and F and G are cocontinuous the category P is cocomplete and J is cocontinuous. From Proposition 3.13 we know that the underlying category P_o is locally α-presentable; the α-presentable objects (P,p) are those objects such that $J(P,p) = P \in A_{o\alpha} = A_{\alpha o}$. Now, the hom-object $P((P,p),(B,b))$ is the equalizer of a pair of morphisms $f,g\colon A(P,B) \to B(FP,GB)$. Since $A(P,-)$, $B(FP,-)$, and G have rank α, and since α-filtered colimits commute with equalizers in the locally α-presentable category V, we readily see that $P((P,p),-)\colon P \to V$ has rank α. Thus $P_{o\alpha} \subseteq P_{o\alpha}$, and P is locally α-presentable by Proposition 6.5. The functor $J\colon P \to A$ preserves α-presentable objects since $P_{o\alpha} = P_{\alpha o}$.

Any functor $T\colon C \to P$ with $JT\colon C \to A$ in V-α-*Ladj* must be cocontinuous and preserve α-presentable objects. Hence $J\colon P \to A$, with the 2-cell μ, is the inserter in V-α-*Ladj*.

(2) *Equifiers.* Let $\rho,\sigma\colon F \to G\colon A \to B$ be a parallel pair of 2-cells in V-α-*Ladj* whose equifier, in V-*CAT*, is the full subcategory $J\colon P \to A$. The category P is cocomplete and J is cocontinuous. The α-presentable objects of the locally α-presentable category P_o are $P_{o\alpha} = A_{o\alpha} \cap P = A_{\alpha o} \cap P$. So, if $P \in P_{o\alpha}$ then $P(P,-) = A(P,-)J$ has rank α – that is, $P_{o\alpha} \subseteq P_{\alpha o}$. By Proposition 6.5, P is locally α-presentable. Moreover J preserves α-presentable objects.

Again, any functor $T\colon C \to P$ with $JT\colon C \to A$ in V-α-*Ladj* must be cocontinuous and preserve α-presentable objects. Hence $J\colon P \to A$ is the equifier in V-α-*Ladj*. □

Combining the propositions of this section yields

Theorem 6.17. (1) *If* α *is an uncountable regular cardinal then* *V-*α*-Ladj admits all indexed limits of retract type of size less than* α.

(2) *The 2-category V-Ladj admits all indexed limits of retract type.*

Moreover, the inclusion into V-CAT *preserves these limits.* □

BIBLIOGRAPHY

[1] J. Adámek and V. Trnková, *Varietors and machines in a category*, Algebra Universalis, 13(1981), 89-132.

[2] M. Barr, *★-Autonomous Categories*, Lecture Notes in Math. 752, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[3] J. Bénabou, *Introduction to bicategories*, Lecture Notes in Math. 47, Springer-Verlag, Berlin-Heidelberg-New York, 1967, 1-77.

[4] P.M. Cohn, *On the free product of associative rings*, Math. Z. 71(1959), 380-398.

[5] E.J. Dubuc, *Adjoint triangles*, Lecture Notes in Math. 61, Springer-Verlag, Berlin-Heidelberg-New York,1968, 69-91.

[6] E.J. Dubuc, *Kan Extensions in Enriched Category Theory*, Lecture Notes in Math. 145, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[7] S. Eilenberg and G.M. Kelly, *Closed categories*, in Proc. Conf. Categorical Algebra, La Jolla 1965, 421-562 (Springer-Verlag, Berlin-Heidelberg-New York, 1966).

[8] S. Fakir, *Objets algébriquement clos et injectifs dans les catégories localement présentables*, Bull. Soc. Math. France, Mémoire 42, 1975.

[9] P.J. Freyd and G.M. Kelly, *Categories of continuous functions* I, J. Pure Appl. Algebra 2(1972), 169-191.

[10] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math. 221, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[11] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[12] J.W. Gray, *Formal Category Theory: Adjointness for 2-categories*, Lecture Notes in Math. 391, Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[13] J.W. Gray, *Closed categories, lax limits and homotopy limits*, J. Pure Appl. Algebra, 19(1980), 127-158.

[14] J.R. Isbell, *Structure of categories*, Bull. Amer. Math. Soc. 72(1966), 619-655.

[15] G.M. Kelly, *On clubs and doctrines*, Lecture Notes in Math. 420, Springer-Verlag, Berlin-Heidelberg-New York, 1974, 181-256.

[16] G.M. Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. 22(1980), 1-83.

[17] G.M. Kelly, *The Basic Concepts of Enriched Category Theory*, Cambridge University Press, Cambridge, 1982.

[18] G.M. Kelly, *Structures defined by finite limits in the enriched context* I, Cahiers Topologie Géom. Différentielle, 23(1982), 3-42.

[19] G.M. Kelly and R.H. Street, *Review of the elements of 2-categories*, Lecture Notes in Math. 420, Springer-Verlag, Berlin-Heidelberg-New York, 1974, 75-103.

[20] J. Lambek, *Subequalizers*, Canad. Math. Bull. 13(1970), 337-349.

[21] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[22] R.H. Street, *Two constructions on lax functors*, Cahiers Topologie Géom. Différentielle, 13(1972), 217-264.

[23] R.H. Street, *Limits indexed by category-valued 2-functors*, J. Pure Appl. Algebra, 8(1976), 149-181.

[24] R.H. Street, *Fibrations in bicategories*, Cahiers Topologie Géom. Différentielle, 21(1980), 111-160.

[25] F. Ulmer, *Locally α-presentable and locally α-generated categories*, Lecture Notes in Math. 195, Springer-Verlag, Berlin-Heidelberg-New York, 1971, 230-247.

[26] F. Ulmer, *On categories of cocontinuous functors, applications*, 1975 (unpublished).

[27] F. Ulmer, *Locally presentable categories* II, 1975 (unpublished).

[28] F. Ulmer, *Bialgebras in locally presentable categories*, 1977 (unpublished).

[29] G.M. Kelly, *Adjunctions for enriched categories*, Lecture Notes in Math. 106, Springer-Verlag, Berlin-Heidelberg-New York, 1969, 166-177.

[30] G.M. Kelly, *Doctrinal adjunction*, Lecture Notes in Math., Springer--Verlag, Berlin-Heidelberg-New York, 1974, 257-280.