Notes on Differential Equations

by Bob Terrell
bterrell@math.cornell.edu

These are introductory notes on ordinary and partial differential equations. Assumed background is calculus and a little physics. Linear algebra is not assumed, and is introduced here in four of the lectures. Those four lectures have been used in the Engineering Mathematics course at Cornell University for several years. The notes as a whole have not been used as a text, to my knowledge, but have been freely available for many years from my web page. The notes as written are quite close to lectures I have delivered while attempting to explain various textbooks to my students.

The notes contain about one third of the material of the typical differential equations book, and are therefore concerned with only the most important ideas. The style of the notes is approximately that of the author's lectures, in that most ideas are taught by example, and that I view the ideas as being at least as important as the calculations. For fuller coverage, see any of the excellent books:

1. Agnew, Ralph Palmer, *Differential Equations*, McGraw–Hill, 1960

2. Hubbard, John H., and West, Beverly H.,*Differential Equations, a Dynamical Systems Approach, Parts 1 and 2,* Springer, 1995 and 1996

3. Churchill, Ruel V., *Fourier Series and Boundary Value Problems,* McGraw Hill, 1941

Some of the exercises have the format *What's rong with this?* ... ?!? Most of these are errors taken from test papers of students in this class, so it could be quite beneficial to study them.

Bob Terrell
January 1997
updated September 2005

Today:

A survey, some examples involving money and other things, and nice pictures called slope fields. Reading a differential equation.

Welcome to the world of differential equations! Although you have somehow survived life for 18 or more years without hearing much about them, differential equations happen to describe many of the processes in the world around you. Today we are going to survey what some of them look like.

A differential equation is an equation which contains an unknown function and one of its unknown derivatives. Here is a differential equation.

$$\frac{dy}{dt} = .028y$$

It doesn't look too exciting does it? Really it is, though. It might for example represent your bank account, where the balance is y at a time t years after you open the account, and the account is earning 2.8% interest. Regardless of the specific interpretation, let's see what the equation says. Since we see the term dy/dt we can tell that y is a function of t, and that the rate of change is a multiple, namely .028, of the value of y itself. We definitely should always write $y(t)$ instead of just y, and we will sometimes, but it is traditional to be sloppy.

For example, if y happens to be 2000 at a particular time t, the rate of change of y is then $.028(2000) = 56$, and the units of this rate in the bank account case are dollars/year. Thus y is increasing, whenever y is positive. What do you estimate the balance will be, roughly, a year from now, if it is 2000 and is growing at 56 dollars/year? This is not supposed to be a hard question. (By the way, when I ask a question, don't cheat yourself by ignoring it. Think about it, and future things will be easier. I promise.) Later when y is, say, 5875.33, its rate of change will be $.028(5875.33) = 164.509$ which is much faster. We'll sometimes refer to $y' = .028y$ as the banker's equation.

Do you begin to see how you can get useful information from a differential equation fairly easily, by just reading it carefully? One of the most important skills to learn about differential equations is how to read them. For example in the equation

$$y' = .028y - 10$$

there is a new negative influence on the rate of change, due to the –10. This –10 could represent withdrawals from the account.

For Practice: What must be the units of the –10?

Whether the resulting value of y' is actually negative depends on the current value of y. That is an example of "reading" a differential equation. As a result of this reading skill, you can perhaps recognize that the banker's equation is very idealized: It does not account for deposits or changes in interest rate. It didn't account for withdrawals until we appended the –10, but even that is an unrealistic continuous rate of withdrawal. You may wish to think about how the equation might be modified to include those things more realistically.

It is also significant that you can make graphs of the solutions sometimes. In the bank account problem we have already noticed that the larger y is, the greater the rate of increase. This can be displayed by sketching a "slope field" as in Figure 1.1. Slope fields are done as follows. First, the general form of a differential equation is

$$y' = f(y, t)$$

where $y(t)$ is the unknown function, and f is given. To make a slope field for this equation, choose some points (y, t) and evaluate f there. According to the differential equation, these numbers must be equal to the derivative y', which is the slope for the graph of the solution.

These resulting values of y' are then plotted using small line segments to indicate the slopes. For example, at the point $t = 6, y = 20$, the equation $y' = .028y$ says that the slope must be $.028(20) = .56$. So we go to this point on the graph and place a mark having this slope. Solution curves then must be tangent to the slope marks. This can be done by hand or computer, *without* solving the differential equation.
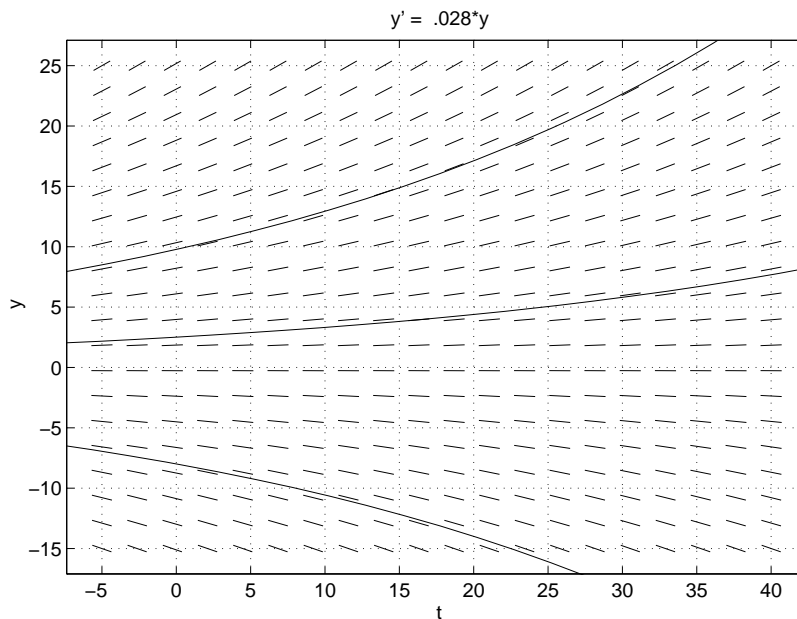
Fig 1.1 *The slope field for* $y' = .028y$ *as made by the* `dfield` *utility in Matlab. In Lecture 3 we will discuss where to get* `dfield`*. Some solutions are also shown.*

Note that we have included the cases $y = 0$ and $y < 0$ in the slope field even though they might not apply to your bank account. (Let us hope.)

For Practice: You should try making a slope field for $y' = y + t$.

There is also a way to explicitly solve the banker's equation. Assume we are looking for a positive solution. Then it is alright to divide the equation by y, getting

$$\frac{1}{y}\frac{dy}{dt} = .028$$

Then integrate, using the chain rule:

$$\ln(y) = .028t + c$$

where c is some constant. Then

$$y(t) = e^{.028t+c} = c_1 e^{.028t}$$

Here we have used a property of the exponential function, that $e^{a+b} = e^a e^b$, and set $c_1 = e^c$. The potential answer which we have found must now be checked by substituting it into the differential equation to make sure it really works. You should do this. Now. You will notice that the constant c_1 can in fact be any constant, in spite of the fact that our derivation of it seemed to suggest that it be positive.

6

This is always the way with differential equations, that it doesn't matter how you solve them, what is important is that you check to see whether you are right. Some kinds of answers in math and science can't be checked very well, but these can, so do it. Even *guessing* answers is a highly respected method! if you check them.

Your healthy scepticism may be complaining to you at this point about any bank account which grows exponentially. If so, good. It is clearly impossible for anything to grow exponentially forever. Perhaps it is reasonable for a limited time. The hope of applied mathematics is that our models will be idealistic enough to solve while being realistic enough to be worthwhile.

The last point we want to make about this example concerns the constant c_1 further. What is the amount of money you originally deposited, $y(0)$? Do you see that it is the same as c_1? That is because $y(0) = c_1 e^0$ and $e^0 = 1$. If your original deposit was 300 dollars then $c_1 = 300$. This value $y(0)$ is called an "initial condition", and serves to pick the solution we are interested in out from among all those which might be drawn in Figure 1.1.

sample

Be sure you can do the following kind

$$\begin{cases} x' = -3x \\ x(0) = 5 \end{cases}$$

Like before, we get a solution $x(t) = ce^{-3t}$. Then $x(0) = 5 = ce^0 = c$ so $c = 5$ and the answer is

$$x(t) = 5e^{-3t}$$

Check it to be absolutely sure.

A Gallery of Differential Equations ───────────────

Now that we have seen one example, here is a list and preview of things to come. Not all differential equations are about money.

1. $dy/dt = .028(600 - y)$ This equation is a model for the heating of a pizza in a 600 degree oven. The physical law involved is called, believe it or not, "Newton's Law of Cooling". We'll do it in the fourth lecture.

2. Newton had other laws as well, one of them being the "$F = ma$" law of inertia. You might have seen this in a physics class, but not realised that it is a differential equation. That is because it concerns the unknown position of a mass, and the second derivative a, of that position. In fact, the original differential equation, the very first one over 300 years ago, was made by Newton for the case in which F was the gravity force between the earth and moon, and m was the moon's mass. Before that, nobody knew what a differential equation was, and nobody knew that gravity had anything to do with the motion of the moon. You have probably heard a garbled version of the story in which Newton saw or was hit by a falling apple. What is frequently omitted is that he looked up to see whence it had fallen, and saw the moon up there, behind the branches. Thus it occured to him that there might be a connection.

There is a joke by the famous physicist Richard Feynmann, who said that once people thought there were angels up there pushing against the side of the moon to keep it moving around the earth. This is of course ridiculous, he said. After Newton everyone knows that the angels are in fact on the far side of the moon pushing it toward us!

Anyway Newton's law often leads to second order differential equations, the simplest of which may look like $d^2x/dt^2 = -x$. By the way, you should be able to guess some solutions to this equation. What are they?

3. There are also things called Partial Differential Equations. This does not mean that somebody forgot to write out the whole thing! It means that the unknown functions depend

on more than one variable, so that partial derivatives show up in the equation. One example is $u_t = u_{xx}$ where the subscripts denote partial derivatives. This is called the heat equation, and $u(x,t)$ is the temperature at position x and time t, when heat is allowed to conduct only along the x axis, as through a wall or along a metal bar. It concerns a different aspect of heat than does Newton's law of cooling, and we will discuss it more later.

4. $u_{tt} = u_{xx}$ looks sort of like the heat equation, but is very different because of the second time derivative. This is the wave equation, which is about electromagnetic waves (radio), music, and water waves, in decreasing order of accuracy. The equation for vibrations of a drum head is the two dimensional wave equation, $u_{tt} = u_{xx} + u_{yy}$.

5. There are others which we won't study but some of the ideas we use can be applied to them. The reason for mentioning these here is to convince you that the earlier statement about "many processes" in the world around you is correct. If you bang your fist on the table top, and the table top is somewhat rigid, not like a drum head, then the sound which comes out is caused by vibrations of the wood, and these are described by solutions of $u_{tt} = -(u_{xxxx} + 2u_{xxyy} + u_{yyyy})$. Here $u(x,y,t)$ is the vertical deflection of the table top while it bends and vibrates. Can you imagine that happening at a small scale? At a much smaller scale, the behavior of electrons in an atom is described by Schrödinger's equation $iu_t = -(u_{xx} + u_{yy} + u_{zz}) + V(x,y,z)u$. In this case, $u(x,y,z,t)$ is related to the probability that the electron is at (x,y,z) at time t. At the other end of the scale there is an equation we won't write down, but which was worked out by Einstein. (Not $E = mc^2$, but a differential equation.) That must be why Einstein is so famous. He wrote down a differential equation for the whole universe.

Problems ———————————————————————————————— **1**

1. Make slope fields for $x' = x$, $x' = -x$, $x' = x + t$, $x' = -x + \cos t$, $x' = t$.

2. Sketch some solution curves onto the given slope field in Figure 1.2 below.

3. What general fact do you know from calculus about the graph of a function y if $y' > 0$? Apply this fact to any solution of $y' = y - y^3$: consider the intervals $(-\infty, -1), (-1, 0)$, $(0, 1)$, and $(1, \infty)$. For each interval, state whether y is increasing or decreasing.

4. (continuing 3.) If $y' = y - y^3$ and $y(0) = \frac{1}{2}$, what do you think will be the $\lim_{t \to \infty} y(t)$? Make a slope field if you're not sure.

5. Reconsider the banker's equation $y' = .028y$. If the interest rate is 3% at the beginning of the year and expected to rise linearly to 4% over the next two years, what would you replace .028 with in the equation? You are not asked to solve the equation.

6. In $y' = .028y - 10$, suppose the withdrawals are changed from $10 per year continuously, to $200 every other week. What could you replace the -10 by? This is a difficult rhetorical question only– something to think about. Do you think it would be alright to use a smooth function of the form $a \cos bt$ to approximate the withdrawals?
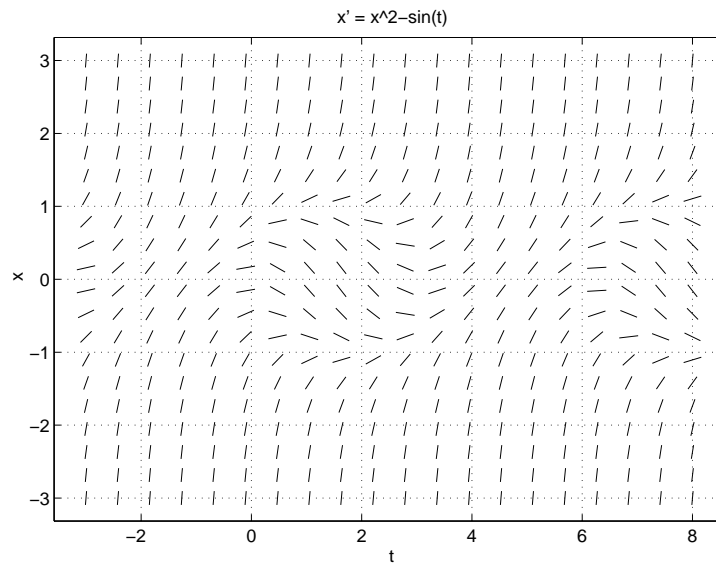
Figure 1.2 Slope field for Problem 2. (figure made by `dfield`*)*

Today:

The logistic equation and other separable examples.

We saw last time that the banker's equation $\frac{dy}{dt} = .028y$ has exponentially growing solutions. It also has a completely different interpretation from the bank account idea. Suppose that you have a population containing about $y(t)$ individuals. The word "about" is used because if $y = 32.51$ then we will have to interpret how many individuals that is. Also the units could be, say, thousands of individuals, rather than just plain individuals. The population could be anything from people on earth, to deer in a certain forest, to bacteria in a certain Petrie dish. We can read this differential equation to say that the rate of change of the population is proportional to the number present. That perhaps captures some element of truth, yet we see right away that no population can grow exponentially forever, since sooner or later there will be a limit imposed by space, or food, or energy, or something.

The Logistic Equation ————————————————————————

Here is a modification to the banker's equation that overcomes the previous objection.

$$\frac{dy}{dt} = .028y(1 - y)$$

We will use units here such that y is in thousands. In order to understand why this avoids the exponential growth problem we must read the differential equation carefully. Remember that I said this is an important skill.

Here we go. You may rewrite the right-hand side as $.028(y - y^2)$. You know that when y is small, y^2 is very small. Consequently the rate of change is still about $.028y$ when y is small, and you will get exponential growth, approximately. After this goes on for a while, it is plausible that the y^2 term will become important. In fact as y increases toward 1 (one thousand), the rate of change approaches 0!

Therefore if we make a slope field for this equation we see something like Fig 2.1.

Fig. 2.1 The Logistic Equation. Note that solutions starting near 0 have about the same shape as the exponentials in Fig. 1.1, until they get near 1. (figure made using `dfield`*)*

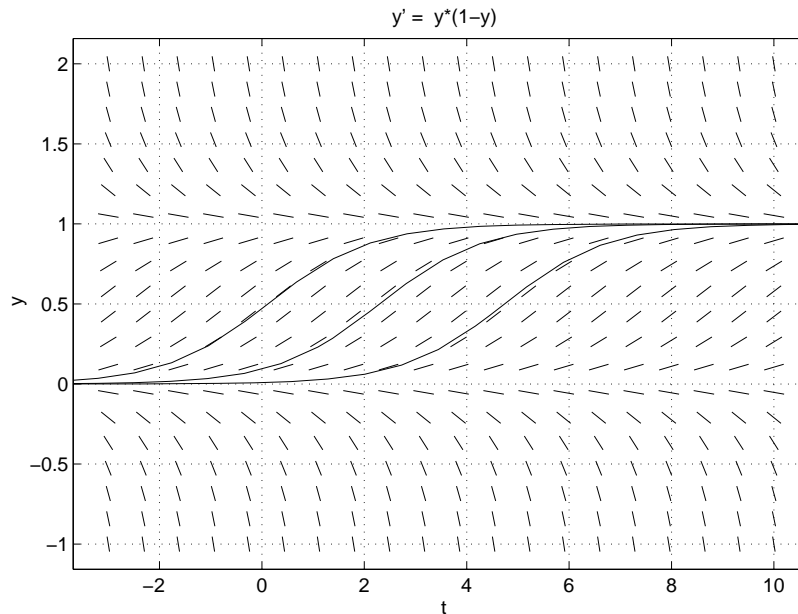The solutions which begin with initial conditions between 0 and 1 evidently grow toward 1 as a limit. This in fact can be verified by finding an explicit formula for the solution. Proceeding much as we did for the bank account problem, first write

$$\frac{1}{y - y^2} \, dy = .028 \, dt$$

To make this easier to integrate, we'll use a trick which was discovered by a student in *this* class, and multiply first by y^{-2}/y^{-2}. Then integrate

$$\int \frac{y^{-2}}{y^{-1} - 1} \, dy = \int .028 dt$$

$$-\ln(y^{-1} - 1) = .028t + c$$

The integral can be done without the trick, using partial fractions, but that is much longer. Now solve for $y(t)$

$$y^{-1} - 1 = e^{-(.028t + c)} = c_1 e^{-.028t}$$

$$y(t) = \frac{1}{1 + c_1 e^{-.028t}}$$

Note that these manipulations would be somewhat scary if we had not specified that we are interested in y values between 0 and 1. For example, the ln of a negative number is not defined. However, we emphasize that the main point is to *check* any formulas found by such manipulations. So let's check it:

$$y' = \frac{.028 C_1 e^{-.028t}}{(1 + c_1 e^{-.028t})^2}$$

We must compare this expression to

$$.028y(1 - y) = .028y - .028y^2$$

$$= \frac{.028}{1 + c_1 e^{-.028t}} - \frac{.028}{(1 + c_1 e^{-.028t})^2}$$

$$= \frac{.028(1 + c_1 e^{-.028t}) - .028}{(1 + c_1 e^{-.028t})^2}$$

You can see that this matches y'. Note that the value of c_1 is not restricted to be positive, as the derivation above may have suggested. We have seen this kind of thing before, so checking is very important. The only restriction here occurs when the denominator of y is 0, which can occur if c_1 is negative. If you stare long enough at y you will see that this does not happen if the initial condition is between 0 and 1, and that it restricts the domain of definition of y if the initial conditions are outside of this interval. All this fits very well with the slope field above. In fact, there is only one solution to the equation which is not contained in our formula. Can you see what it is? You won't have to look very far.

Separation of Variables ───

Equations of the form

$$\frac{dx}{dt} = f(x)g(t)$$

include the banker's and logistic equations and some other useful equations. You may attack these by "separating variables": Write $\frac{dx}{f(x)} = g(t) \, dt$, then try to integrate and solve for $x(t)$. Remember that this is what we did with the banker's and logistic equations.

sample

$$\frac{dx}{dt} = tx, \frac{dx}{x} = t \, dt, \ln |x| = \frac{t^2}{2} + c, x(t) = c_1 e^{(t^2/2)}, \text{ check} : x' = \frac{2t}{2} c_1 e^{(t^2/2)} = tx$$

11

The skeptical reader will wonder why I said "try" to integrate and solve for x. The next two examples illustrate that there is no guarantee that either of these steps may be completed.

sample

$\frac{dx}{dt} = \frac{1}{1+3x^2}, (1+3x^2)\,dx = dt, x+x^3 = t+c$. Here you can do the integrals, but you can't very easily solve for x.

sample

$\frac{dx}{dt} = e^{-t^2}, dx = e^{-t^2}\,dt$. Here you can't do the integral.

The Leaking Bucket ────────────────────────────────

(This example comes from the Hubbard and West book. See preface.)

We're about to write out a differential equation to model the depth of water in a leaking bucket, based on plausible assumptions. We'll then try to solve for the water depth at various times. In particular we will try to reconstruct the history of an empty bucket. This problem is solvable by separation of variables, but it has far greater significance that the problem has *more than one* solution. Physically if you see an empty bucket with a hole at the bottom, you just can't tell when it last held water.

Here is a crude derivation of our equation which is not intended to be physically rigorous. We assume that the water has depth $y(t)$ and that the speed of a molecule pouring out at the bottom is determined by conservation of energy. Namely, as it fell from the top surface it gained potential energy proportional to y and this became kinetic energy at the bottom, proportional to $(y')^2$. Actually y' is the speed of the top surface of the water, not the bottom, but these are proportional in a cylindrical tank. Also we could assume a fraction of the energy is lost in friction. In any event we get $(y')^2$ proportional to y, or

$$y' = -a\sqrt{y}$$

The constant is negative because the water depth is decreasing. We'll set $a = 1$ for convenience.
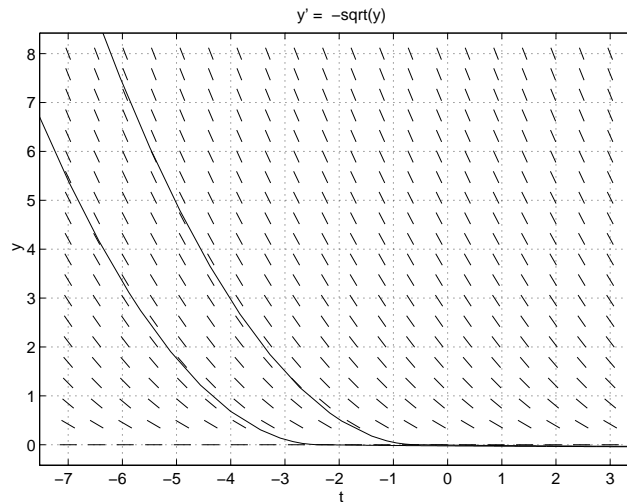
The slope field looks like the figure.

Fig. 2.2 Depth of water in the leaking bucket (figure made by `dfield`)

12

Separation of variables gives, if $y > 0$,

$$-y^{-1/2}\, dy = dt$$

$$2y^{1/2} = -t + c$$

so here we must have $t < c$, otherwise the square root would be negative.

$$y = \left(\frac{-t+c}{2}\right)^2$$

Note that if $t = c$ then $y = 0$ so the derivation is no good there. Comparing our formula to the slope field we see what must happen:

$$y(t) = \begin{cases} \left(\frac{-t+c}{2}\right)^2, & \text{if } t < c; \\ 0, & \text{if } t \geq c. \end{cases}$$

The final point here is that if our initial condition is, say, $y(0) = 0$, then *any* choice of $c < 0$ gives an acceptable solution. This means that if the bucket is empty now, it could have become empty at any time in the past. Again, we consider this kind of behavior to be unusual, in the sense that we don't usually notice or want examples of nonunique behavior. Ironically in this problem, nonuniqueness fits reality perfectly.

sample

$$\begin{cases} \frac{dx}{dt} = \frac{1}{2}x^3 \\ x(0) = -3 \end{cases}$$

$x^{-3}\, dx = \frac{1}{2}\, dt$, $-\frac{1}{2}x^{-2} = \frac{1}{2}t + c$, $x^{-2} = -t - 2c$, (so $t < -2c$). Then at $t = 0$ we get $(-3)^{-2} = -0 - 2c$, $2c = -1/9$, $x^{-2} = -t + 1/9$, $x = \pm(-t + 1/9)^{-1/2}$. Now recheck the initial conditions to determine the sign: $-3 = \pm(0 + 1/9)^{-1/2}$, therefore use the minus sign:

$$x = -(-t + 1/9)^{-1/2} \text{ for } (t < 1/9).$$

Check it:

$$\begin{cases} -3 = -(1/9)^{-1/2} & \text{ok} \\ x' = -(-\frac{1}{2})(-t + \frac{1}{9})^{-\frac{3}{2}} = \frac{1}{2}x^3 & \text{ok} \end{cases}$$

Problems ——————————————————————————————————————— **2**

1. Repeat the previous example, $x' = \frac{1}{2}x^3$, for the case $x(0) = 3$.
2. Solve $x' = -\frac{1}{2}x^3$ with initial condition $x(0) = -3$.
3. Solve $x' = -\frac{1}{2}x^3$ with initial condition $x(0) = 3$.
4. Sketch slope fields for problems 1 and 3.
5. Solve $y' = y - y^3$.
6. Solve $x' + t^2 x = 0$.
7. Solve $x' + t^3 x = 0$.
8. Solve $x' = t^4 x$.
9. Solve $x' = .028(1 + \cos t)x$.
10. Solve $x' = a(t)x$.
11. *What's rong with this?* $dx/dt = x^2$, $dx/x^2 = dt$, $\ln(x^2) = t$, $x^2 = e^t + c$. ?!? There are at least two errors.

13

Today:

> *We learn that some equations have unique solutions, some have too many, and some have none. Also an introduction to some of the available software.*

Last time we saw an initial value problem, the leakey bucket problem, for which several solutions existed. This phenomenon is not generally desirable, because if you are running an experiment you would like to think that the same results will follow from the same initial conditions each time you repeat the experiment.

On the other hand, it can also happen that an innocent-looking differential equation has no solution at all, or a solution which is not defined for all time. As one example, consider

$$\left(\frac{dx}{dt}\right)^2 + x^2 + 1 = 0$$

This equation has no real solution. See why? For another example consider

$$\frac{dx}{dt} = x^2$$

By separation of variables we can find some solutions, such as $x(t) = \frac{1}{3-t}$, but this function is not defined when $t = 3$. In fact we should point out that this formula $\frac{1}{3-t}$ has to be considered to define two functions, not one, the domains being $(-\infty, 3)$ and $(3, \infty)$ respectively. The reason for this distinction is simply that the solution of a differential equation has by definition to be differentiable, and therefore continuous.

We therefore are interested in the following general statement about what sorts of equations have solutions, and when they are unique, and how long, in time, these solutions last. It is called the "Fundamental Existence and Uniqueness Theorem for Ordinary Differential Equations", which means, among other things, that it is considered to be important.

The statement of this theorem includes the notion of partial derivative, which some may not have seen. So we mention here that the partial derivative of a function of several variables is defined to mean that the derivative is constructed by holding all other variables constant. For example, if $f(t, x) = x^2 t - \cos(t)$ then $\frac{\partial f}{\partial x} = 2xt$ and $\frac{\partial f}{\partial t} = x^2 + \sin(t)$.

Existence and Uniqueness Theorem Consider an initial value problem of the form

$$\frac{dx}{dt} = f(x, t)$$

$$x(t_0) = x_0$$

where f, t_0, and x_0 are given. Suppose it is true that f and $\frac{\partial f}{\partial x}$ are continuous functions of t and x in at least some small rectangle containing the initial condition (x_0, t_0). Then the conclusion is that there is a solution to the problem, it is defined at least for a small amount of time both before and after t_0, and there is only one such solution. There is no guarantee about how long a time interval the solution is defined for.

Now let's see what went wrong with some of our previous examples. For the leakey bucket equation, we have $f(t, x) = -\sqrt{x}$ and $\frac{\partial f}{\partial x} = -1/(2\sqrt{x})$. These are just fine and dandy as long as $x_0 > 0$, but when $x_0 = 0$ you can't draw a rectangle around the initial conditions in the domain of f, and even worse, the partial derivative is not defined because you would have to divide by 0. The conclusion is that the Theorem does not apply.

Note carefully what that conclusion was! It was not that existence or uniqueness are impossible, but only that the theorem doesn't have anything to say about it. When this

happens you have to make a more detailed analysis. A theorem is really like the guarantee on your new car. It is guaranteed for 10,000 miles, but hopefully it does not just collapse and fall to pieces all over the road as soon as you reach the magic number. Similarly some of the conclusions of a theorem can still be true in a particular example even if the hypotheses do not hold. To see it you can't apply the theorem but you can make a detailed analysis of your example. So here in the case of the leakey bucket we do have existence anyway.

For the first example above, $(\frac{dx}{dt})^2 + x^2 + 1 = 0$, the equation is not at all in the form required for the theorem. You could perhaps rearrange it as $\frac{dx}{dt} = \sqrt{-1 - x^2}$ but this hurts my eyes–the right side is not even defined within the realm of real numbers in which we are working. So the theorem again says nothing about it.

For the other example above, $\frac{dx}{dt} = x^2$, the form is alright and the right side is $f(t, x) = x^2$. This is continuous, and $\frac{\partial f}{\partial x} = 2x$, which is also continuous. The theorem applies here, but look at the conclusion. There is a solution defined for some interval of t around the initial time, but there is no statement about how long that interval is. So our original puzzlement over the short duration of the solution remains. If you were a scientist working on something which might blow up, you would be glad to be able to predict when or if the explosion will occur. But this requires a more detailed analysis in each case–there is no general theorem about it.

A very important point about uniqueness is the implication it has for our pictures of slope fields and solution curves. Two solution curves cannot cross or touch, when uniqueness applies.

Software

In spite of examples we have seen so far, it turns out that it is not possible to write down solution formulas for most differential equations. This means that we have to draw slope fields or go to the computer for approximate solutions. We soon will study how approximate solutions can be computed. Meanwhile we are going to introduce you to some of the available tools.

There are several software packages available to help your study of differential equations. These tend to fall into two categories. On one side are programs which are visual and easy to use, with the focus being on using the computer to show you what some of the possibilities are. You point and click and try things, and see a lot without working too hard. On the other side are programs in which you have to do some active programing yourself. While they often have a visual component, you have to get your hands dirty to get anything out of these. The result is that you get a more concrete understanding of various fundamentals. Both kinds have advantages. It seems that the best situation is not to choose one exclusively, but to have and use both kinds.

In the easy visual category there are java applets available at

`http://www.math.cornell.edu/~ bterrell/de`

and

`http://math.rice.edu/dfield/dfpp.html`

(i.e., you can now use the world wide web to solve differential equations. This is probably not what they had in mind;-) Probably the earliest userfriendly DE software was MacMath, by John Hubbard. MacMath was the inspiration for `de`.

The other approach is to do some programing in any of several available languages. These include `matlab`, its free counterpart `octave`, and the freeware program `xpp`.

There are also java applets on partial differential equations. These are called Heat Equation 1D, Heat Equation 2D, Wave Equation 1D, and Wave Equation 2D, and are available from

`http://www.math.cornell.edu/~ bterrell` or

`http:  or ftp://archives.math.utk.edu`

in the directory `/software/mac/diffEquations/PDE_Progs`.
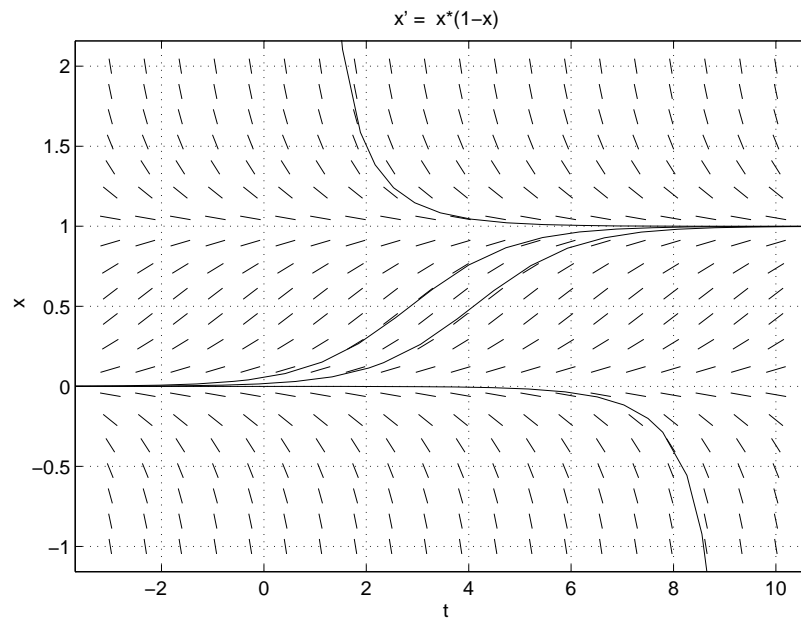
Fig 3.1 This is the slope field for the Logistic Equation as made by `dfield` *in Matlab. Question: Do these curves really touch? Read the Theorem again if you're not sure.*

Problems ─── **3**

1. Find out how to download and use some of the programs mentioned in this lecture, try a few simple things, and read some of the online help which they contain.
2. Answer the question in the caption for Figure 3.1.

Today:

Applications and a solution method for first order linear equations.

Today we consider equations like $\frac{dx}{dt} + ax = b$ or

$$x' + ax = b$$

These are called first order linear equations. "Linear" is here a more-or-less archaic use of the word and means that x and x' only occur to the first power. We will say later what the word "linear" means in a more modern sense. The coefficients a and b can be constants or functions of t. You should check that when a and b are constant, the solutions always look like $x(t) = c_1 e^{-at} + c_2$, where c_1 is an arbitrary constant but c_2 has to be something specific. When I say "check", it doesn't mean derive it, but just substitute into the equation and verify. Make sure you see how similar this is to the case when $b = 0$.

For Practice: solve $x' + 2x = 3$.

Among these equations is the very first one we ever did, $y' = .028y$, which we solved by separation of variables. Some, but not all of these equations can be solved the same way. The simplest one of them is $x' = 0$ which is so easy as not to be useful. The next simplest is $x' = b$ and this one is very important because it holds a key to solving all the others. We call this one the "easy kind" of differential equation because it does not involve x at all and can be solved just by integrating:

$$x' = b$$

$$x = \int b\, dt$$

Wouldn't it be nice if all differential equations could be solved so easily?

Maybe some can. Somebody once noticed that all these equations can be put into the form of the easy kind, if you will only multiply the equation by a cleverly chosen "integrating factor". For example, consider again

$$x' + 2x = 3$$

Multiply by e^{2t} and you will get

$$e^{2t} x' + 2e^{2t} x = 3e^{2t}$$

Then, and this is the cool part, recognise the left hand side as a total derivative, like you have in the easy kind:

$$(e^{2t} x)' = 3e^{2t}$$

This uses the product rule for derivatives. Now all you have to do is integrate and you are done.

$$e^{2t} x = \int 3e^{2t}\, dt = \frac{3}{2} e^{2t} + C$$

so

$$x(t) = \frac{3}{2} + Ce^{-2t}$$

What do you think the integrating factor should be for $x' - 1.3x = \cos(t)$? In this case, which will be a homework problem, you still have an integral to do which is a little harder, and you probably have to do integration by parts. There is a general formula for

17

the integrating factor which is not too important, but here it is. For the equation above, $x' + ax = b$, an integrating factor is $e^{\int a\,dt}$. This always works, at least within the limits of your ability to do the integrals. As for where this formula comes from, we have devised a homework problem so that you can answer that yourself. As to the deeper question of how anybody ever thought up such a scheme as integrating factors in the first place, that is much harder to answer.

Now let's look at some applications of these equations.

Newton's Law of Cooling

Newton's law of cooling is the statement that the exponential growth equation $x' = kx$ applies sometimes to the temperature of an object, provided that x is taken to mean the difference in temperature between the object and its surroundings.

For example suppose we have placed a 100 degree pizza in a 600 degree oven. We let $x(t)$ be the pizza temperature at time t, minus 600. This makes x negative, while x' is certainly positive because the pizza is heating up. Therefore the constant k which occurs must be a negative number. The solution to the equation does not require an integrating factor; it is $x(t) = Ce^{kt}$, and $C = 100 - 600$ as we have done previously. Therefore the pizza temperature is $600 + x(t) = 600 - 500e^{kt}$. We don't have any way to get k using the information given. It would suffice though, to be told that after the pizza has been in the oven for 15 minutes, its temperature is 583 degrees. This says that $583 = 600 - 500e^{15k}$. So we can solve for k and then answer any questions about temperature at other times. Notice that this law of cooling for the pizza can be written as $p' - kp = 600k$ where $p(t)$ is the pizza temperature, which is a first order linear equation. A different environment arrives if we move the pizza to the 80 degree kitchen. A plot of the temperature history under such conditions is in Fig 4.1. It is not claimed that the solution is differentiable at $t = 6$, when the environment changes.
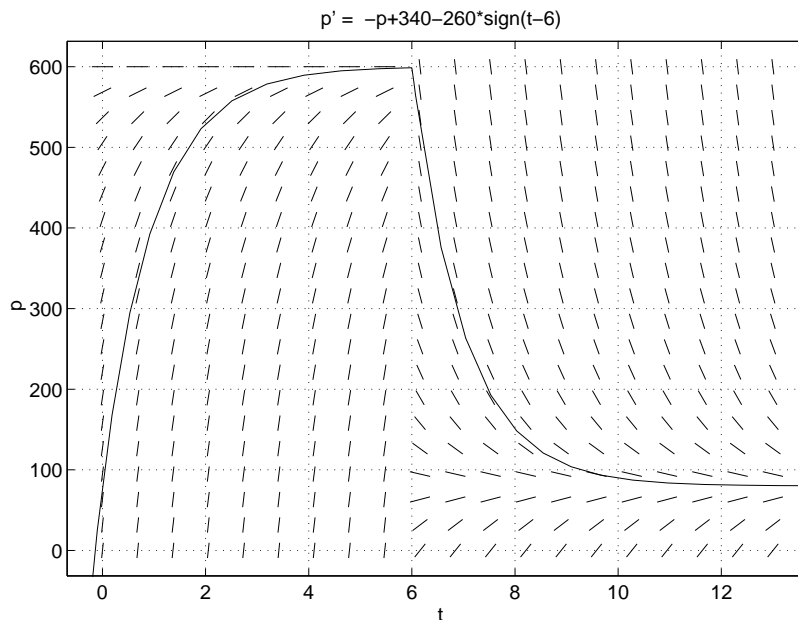

$$p' = -p + 340 - 260*\text{sign}(t-6)$$

Fig. 4.1 The pizza heats and then cools. Note the trick used to change the environment from 600 degrees to 80 degrees at time 6: the equation was written as $p' = -p + a - b * sign(t - 6)$, and a and b selected to achieve the 600 and 80. (figure by `dfield`)

There is another use of Newton's law of cooling which is not so pleasant to discuss, but it is real. The object whose temperature we are interested in is sometimes of interest because the police want to know the last time it was alive. The body temperature of a murder victim begins at 98.6 degrees F, or 37 C, and decreases at a rate dependent on the environment. A 50 degree road is one environment, and a 35 degree morgue table is another environment. Part of the history is known, namely that which occurs once the body has been found. Measurements of time and temperature can be used to find k. Then that part of the history from the unknown time of death to the known time of discovery has to be reconstructed using Newton's law of cooling. This involves functions not unlike those in Fig 4.1. It should be clear that this involves estimates and inaccuracies, but is useful, nonetheless.

Investments

Moving to a more uplifting subject, our bank account equation $y' = .028y$ can be made more realistic and interesting. Suppose we make withdrawals at a rate of $3500 per year. This can be included in the equation as a negative influence on the rate of change.

$$y' = .028y - 3500$$

Again we see a first order linear equation. But the equation is good for more than an idealised bank account. Suppose you buy a car at 2.8% financing, paying $3500 per year. Now loosen up your point of view and imagine what the bank sees. From the point of view of the bank, they just invested a certain amount in *you*, at 2.8% interest, and the balance decreases by "withdrawals" of about $3500 per year.

So the same equation describes two apparently different kinds of investments. In fact it is probably more realistic as a model of the loan than as a bank account. By changing the –3500 to +3500 you can model still other kinds of investments.

sample

A car is bought using the loan as described above. If the loan is to be paid off in 6 years, what price can we afford? The price is $y(0)$.

$$\begin{cases} y' = .028y - 3500 \\ y(6) = 0 \end{cases}$$

$e^{-.028t}y' - .028e^{-.028t}y = -3500e^{-.028t}$, $(e^{-.028t}y)' = -3500e^{-.028t}$, $e^{-.028t}y = (3500/.028)e^{-.028t} + c$ $y = (3500/.028) + ce^{.028t} = 125000 + ce^{.028t}$, $y(6) = 0 = 125000 + ce^{.028(6)}$ implies $c \doteq -105669$, $y(0) \doteq 125000 - 105669 = 19331$

Problems _____ 4

1. Solve $x' - 1.3x = \cos t$.

2. This problem shows where the formula $e^{\int a\,dt}$ for the integrating factor comes from. Suppose we have the idea to multiply $x' + ax = b$ by a factor f, so that the result of the multiplication is $(fx)' = fb$. Show that for this plan to work, you will need to require that $f' = af$. Deduce that $e^{\int a\,dt}$ will be a suitable choice for f.

3. The temperature of an apple pie is recorded as a function of time. It begins in the oven at 450 degrees, and is moved to an 80 degree kitchen. Later it is moved to a 40 degree refrigerator, and finally back to the 80 degree kitchen. Make a sketch somewhat like Figure 4.1, which shows qualitatively the temperature history of the pie.

4. A body is found at midnight, on a night when the air temperature is 16 degrees C. Its temperature is 32 degrees, and after another hour, its temperature has gone down to 30.5 degrees. Estimate the time of death.

5. Sara's employer contributes $3000 per year to a retirement fund, which earns 3% interest. Set up and solve an initial value problem to model the balance in her fund, if it began with $0 when she was hired. How much money will she have after 20 years?

6. Show that the change of variables $x = 1/y$ converts the logistic equation $y' = .028(y - y^2)$ of Lecture 2 to the first order equation $x' = -.028(x - 1)$, and figure out a philosophy for why this might hold.

Today:

A numerical method for solving differential equations either by hand or on the computer, several ways to run it, and how your calculator works.

Today we return to one of the first questions we asked. "If your bank balance $y(t)$ is \$2000 now, and $dy/dt = .028y$ so that its rate of change is \$56 per year now, about how much will you have in one year?" Hopefully you guess that \$2056 is a reasonable first approximation, and then realize that as soon as the balance grows even a little, the rate of change goes up too. The answer is therefore somewhat more than \$2056.

The reasoning which lead you to \$2056 can be formalised as follows. We consider

$$\frac{dx}{dt} = f(x, t)$$

$$x(t_0) = x_0$$

Choose a "stepsize" h and look at the points $t_1 = t_0 + h$, $t_2 = t_0 + 2h$, etc. We plan to calculate values x_n which are intended to approximate the true values of the solution $x(t_n)$ at those time points. The method relies on knowing the definition of the derivative

$$x'(t) = \lim_{h \to 0} \frac{x(t + h) - x(t)}{h}$$

We make the approximation

$$x'(t_n) \doteq \frac{x_{n+1} - x_n}{h}$$

Then the differential equation is approximated by the difference equation

$$\frac{x_{n+1} - x_n}{h} = f(x_n, t_n)$$

example   With $h = 1$ the bank account equation becomes

$$\frac{y_{n+1} - y_n}{h} = .028y_n$$

or

$$y_{n+1} = y_n + .028hy_n$$

This leads to $y_1 = y_0 + .028hy_0 = 1.028y_0 = 2056$ as we did in the first place. For a better approximation we may take $h = .2$, but then 5 steps are required to reach the one-year mark. We calculate successively

$$y_1 = y_0 + .028hy_0 = 1.0056y_0 = 2011.200000$$

$$y_2 = y_1 + .028hy_1 = 1.0056y_1 = 2022.462720$$

$$y_3 = y_2 + .028hy_2 = 1.0056y_2 = 2033.788511$$

$$y_4 = y_3 + .028hy_3 = 1.0056y_3 = 2045.177726$$

$$y_5 = y_4 + .028hy_4 = 1.0056y_4 = 2056.630721$$

Look, you get more money if you calculate more accurately! Here the bank has in effect calculated interest 5 times during the year. "Continuously Compounded" interest means taking h so close to 0 that you are in the limiting situation of calculus.

Now at this point we happen to know the answer to this particular problem. It is $y(1) = 2000e^{.028(1)} = 2056.791369....$ Continuous compounding gets you the most money.

Usually we do not have such formulas for solutions, and then we have to use this or some other numerical method.

This method is called Euler's Method, in honour of Leonard Euler, a mathematician of the 18th century. He was great, really. The collected works of most famous scientists typically fill a few books. To see all of his in the library you will have to look over several *shelves*. This is more impressive still when you find out that though he was Swiss, he worked in Russia, wrote in Latin, and was in fact blind. Do you know what "*e*" stands for, as in 2.718...? "exponential" maybe? It stands for Euler. He also invented some things which go by other people's names. So show some respect, and pronounce his name correctly, "oiler".

example   Now we'll do one for which the answer is not as easily known ahead of time. Since the calculations can be tedious, this is a good time to use the computer. We will set up the problem by hand, and solve it two ways on the machine. Assume that $p(t)$ is the proportion of a population which carries but is not affected by a certain disease virus, initially 8%. The rate of change is influenced by two factors. First, each year about 5% of the carriers get sick, so are no longer counted in p. Second, the number of new carriers each year is about .02 of the population but varies a lot seasonally. The differential equation is

$$\begin{cases} p' = -.05p + .02(1 + \sin(2\pi t)) \\ p(0) = .08 \end{cases}$$

Euler's approximation is to choose a stepsize h, put $t_n = nh$, $p_0 = .08$, and approximate $p(t_n)$ by p_n where

$$p_{n+1} = p_n + h(-.05p_n + .02(1 + \sin(t_n)))$$

To compute in **octave** or **matlab**, use any text editor to make a function file to calculate the right hand side:

```
%file myfunc.m
function rate = myfunc(t,p)
rate = -.05*p + .02*(1+sin(2*pi*t));
%end file myfunc.m
```

Then make another file to drive the computation:

```
%file oiler.m
h = .01;
p(1) = .08;
for n = 1:5999
  p(n+1) = p(n) + h*myfunc(n*h,p(n));
end
plot(0:h:60-h,p)
%end file oiler.m
```

Then at the **octave** prompt, give the command **oiler**. You should get a graph like the following. Note that we used 6000 steps of size .01, so we have computed for $0 \leq t \leq 60$.
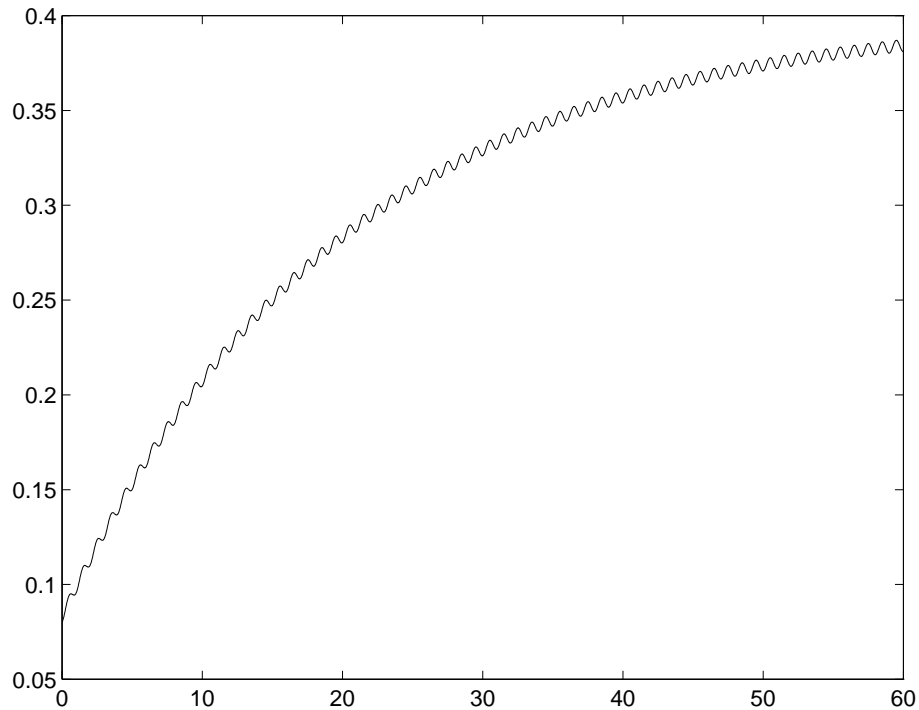
Fig. 5.1 You can see the seasonal variation plainly, and there appears to be a trend to level off. This is a dangerous disease, apparently. (figure made by the `plot` *commands shown in the text)*

For additional practise you should vary h and the equation to see what happens. There are also more sophisticated methods than Euler's. One of them is built into `matlab` under the name `ode23` and you are encouraged to use it even though we aren't studying the internals of how it works. You should type `help ode23` in `matlab` to get information on it. To solve the carrier problem using `ode23`, you may proceed as follows. Use the same `myfunc.m` file as before. Then in Matlab give the commands

```
[t,p] = ode23('myfunc',0,60,.08);
plot(t,p)
```

Your results should be about the same as before, but generally `ode23` will give more accuracy than Euler's method. There is also an `ode45`. We won't study error estimates in this course. However we will show one more example to convince you that these computations come close to things you already know. Look again at the simple equation $x' = x$, with $x(0) = 1$. You know the solution to this by now, right? Euler's method with step h gives

$$x_{n+1} = x_n + hx_n$$

This implies that

$$x_1 = (1 + h)x_0 = 1 + h$$

$$x_2 = (1 + h)x_1 = (1 + h)^2$$

$$\ldots$$

$$x_n = (1 + h)^n$$

Thus to get an approximation for $x(1) = e$ in n steps, we put $h = 1/n$ and receive

23

$$e \doteq (1 + \frac{1}{n})^n$$

Let's see if this looks right. With $n = 2$ we get $(3/2)^2 = 9/4 = 2.25$. With $n = 6$ and some arithmetic we get $(7/6)^6 \doteq 2.521626$, and so forth. The point is not the accuracy right now, but just the fact that these calculations can be done without a scientific calculator. You can even go to the grocery store, find one of those calculators that only does +−*/, and use it to compute important things.

Did you ever wonder how your scientific calculator works? Sometimes people think all the answers are stored in there somewhere. But really it uses ideas and methods like the ones here to calculate everything based only on +−*/. Isn't that nice?

sample

We'll estimate some cube roots by starting with a differential equation for $\sqrt[3]{t}$. $x(t) = t^{1/3}$, $x(1) = 1$, $x'(t) = \frac{1}{3}t^{-2/3}$. These give the differential equation

$$x' = \frac{1}{3x^2}$$

Then Euler's method says $x_{n+1} = x_n + \frac{h}{3x_n^2}$, and we will use $x_0 = 1, h = .1$:

$$x_1 = 1 + \frac{.1}{3} = 1.033333...$$

Therefore $\sqrt[3]{1.1} \doteq 1.0333$.

$$x_2 = x_1 + \frac{h}{3x_1^2} \doteq 1.0645$$

Therefore $\sqrt[3]{1.2} \doteq 1.0645$ etc. For better accuracy, h can be decreased.

Problems ─── **5**

1. What does Euler's method give for $\sqrt{2}$, if you approximate it by setting $x(t) = \sqrt{t}$ and solving

$$\begin{cases} x' = \frac{1}{2x} \\ x(1) = 1 \end{cases}$$

Use 1, 2, and 4 steps, i.e., $h = 1, .5, .25$ respectively.
2. Solve

$$\begin{cases} y' = (\cos y)^2 \\ y(0) = 0 \end{cases}$$

for $0 \le t \le 3$ by modifying the `oiler.m` and `myfunc.m` files given in the text.
3. Solve the differential equation in problem 2 by separation of variables.
4. Redo problem 2 using `ode45` and your modified `myfunc.m`, as in the text.
5. Run `dfield` on the equation of problem 2. Estimate the value of $y(3)$ from the graph.
6. Compare your answers to problems 2 and 3. Is it true that you just computed $\tan^{-1}(3)$ using only +−*/ and cosine? Figure out a way to compute $\tan^{-1}(3)$ using only +−*/.
7. Solve the carrier equation $p' = -.05p + .02(1 + \sin(2\pi t))$ using the integrating factor method of Lecture 4. The integral is pretty hard, but you can do it. Predict from your solution, the proportion of the population at which the number of carriers "levels off" after a long time, remembering from Fig. 5.1 that there will probably continue to be fluctuations about this value. Does your number seem to agree with the picture? Note that your number is probably more accurate than the computer's.

8. Newton's law of cooling $u' = -a(u - u_s)$ is sometimes replaced by the Stephan-Boltzmann law $u' = -a(u^4 - u_s^4)$, where u_s is the surrounding temperature. Which of these laws predicts faster cooling when $u(0)$ is near u_s and $u_s = 100$? $u_s = 0.1$?

Today:

  2nd order equations and some applications. The characteristic equation. Conservation laws.

 The prototype for today's subject is $x'' = -x$. You know the solutions to this already, though you may not realize it. Think about the functions and derivatives you know from calculus. In fact, here is a good method for any differential equation, not just this one. Make a list of the functions you know, starting with the very simplest. Your list might be

0

1

c

t

t^n

e^t

$\cos(t)$

$\dots$

 Now run down the list trying things in the differential equation. In $x'' = -x$ try 0. Well! what do you know? It works. The next few don't work. Then $\cos(t)$ works. Also $\sin(t)$ works. Frequently, as here, you don't need to use a very long list before finding something. As it happens, $\cos(t)$ and $\sin(t)$ are not the only solutions to $x'' = -x$. You wouldn't think of it right away, but $2\cos(t) - 5\sin(t)$ also works, and in fact any $c_1\cos(t) + c_2\sin(t)$ is a solution.

For Practice: find similar solutions to $x'' = -9x$.

 The equations $x'' = cx$ happen to occur frequently enough that you should know all their solutions. Try to guess all solutions of $x'' = 0$ and $x'' = 25x$ for practise. Besides these, we will consider equations of the forms

$$ax'' + bx' + cx = 0$$

$$ax'' + bx' + cx = d\cos(ft)$$

These are called second order linear equations, and the first one is called homogeneous. The strict meaning of the word 'linear' is being stretched somewhat here, but the terminology is traditional. Really an operation L which may be applied to functions, such as $L(x) = ax'' + bx' + cx$, is linear if it is true that $L(x + y) = L(x) + L(y)$ and $L(kx) = kL(x)$. This does hold here, and the consequence is that if x and y are both solutions to $L(x) = 0$, then linear combinations like $2x + 5y$ are also solutions. That is true for the homogeneous equation because we have $L(2x + 5y) = L(2x) + L(5y) = 2L(x) + 5L(y) = 0$. However, the nonhomogeneous second equation does not have this property. You can see this yourself. Suppose x and y are solutions to the same equation, say $x'' + 3x = 2$ and $y'' + 3y = 2$. Then we want to see whether the sum $s = x + y$ is also a solution. We have $s'' + 3s = x'' + y'' + 3x + 3y = 2 + 2 = 4$. This is a _different_ equation, since we got $s'' + 3s = 4$, not $s'' + 3s = 2$. So it is a bit of a stretch to call the nonhomogeneous equation 'linear', but that is the standard terminology. You should also be sure you understand that for an equation like $x'' + x^2 = 0$, even with 0 on the right side, the sum of solutions is not usually a solution.

 Next we look at a method for solving the homogeneous second order linear equations. Some of the nonhomogeneous ones will be done in Lecture 8. The motivation for this method is the observation that exponential functions have appeared several times in the equations which we have been able to solve. Let us try looking down our list of functions until we come to the ones like e^{rt}. Trying $x = e^{rt}$ in

$$ax'' + bx' + cx = 0$$

we find $ar^2 e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt}$ This will be zero only if

$$ar^2 + br + c = 0$$

since the exponential is never 0. This is called the characteristic equation. For example, the characteristic equation of $x'' + 4x' - 3x = 0$ is $r^2 + 4r - 3 = 0$. See the similarity? We converted a differential equation to an algebraic equation that looks abstractly similar. Since we ended up with quadratic equations, we will get two roots usually, hence two solutions to our differential equation.

example $x'' + 4x' - 3x = 0$ has characteristic equation $r^2 + 4r - 3 = 0$. Using the quadratic formula, the roots are $r = -2 \pm \sqrt{7}$. So we have found two solutions $x = e^{(-2-\sqrt{7})t}$ and $x = e^{(-2+\sqrt{7})t}$. By linearity we can form linear combinations

$$x(t) = c_1 e^{(-2-\sqrt{7})t} + c_2 e^{(-2+\sqrt{7})t}$$

example You already know $x'' = -x$ very well, right? But our method gives the characteristic equation $r^2 + 1 = 0$. This does not have real solutions. In lecture 7 we'll see how the use of complex numbers at this stage will unify the sines and cosines we know, with the exponential functions which our characteristic method produces.

example The weirdest thing which can happen is the case in which the quadratic equation has a repeated root. For example, $(r+5)^2 = r^2 + 10r + 25 = 0$ corresponds to $x'' + 10x' + 25x = 0$. The only root is –5, so we only get one solution $x(t) = e^{-5t}$, and multiples of that. If these were the only solutions, we could solve for say, the initial conditions $x(0) = 3$ and $x'(0) = -15$, but could not solve for $x(0) = 3$ and $x'(0) = -14$. As it turns out, we have not made a general enough guess, and have to go further down our list of functions, so to speak, until we get to things like te^{rt}. These work. So we end up with solutions $x(t) = (c_1 + c_2 t)e^{-5t}$ for this problem. This is a special case, and not too important, but it takes more work than the others for some reason.

sample

$$\begin{cases} x'' = -5x \\ x(0) = 1 \\ x'(0) = 2 \end{cases}$$

general solution is $x(t) = c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)$. Use initial conditions $x(0) = c_1 = 1, x'(0) = \sqrt{5}c_2 = 2$, $c_2 = \frac{2}{\sqrt{5}}$. So

$$x(t) = \cos(\sqrt{5}t) + \frac{2}{\sqrt{5}} \sin(\sqrt{5}t)$$

sample

$$\begin{cases} y'' = 4y \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$$

characteristic equation is $r^2 = 4$, $r + \pm 2$, $y = c_1 e^{2t} + c_2 e^{-2t}$, $y(0) = c_1 + c_2 = 1$ and $y'(0) = 2c_1 - 2c_2 = 2$. Then eliminate c_2: $2y(0) + y'(0) = 4c_1 = 4, c_1 = 1$, $c_2 = 1 - c_1 = 0$, $y(t) = e^{2t}$, you check it.

conservation laws

Sometimes a second order equation can be integrated once to yield a first order equation. Good things happen when this is possible, since we have many tools to use on first order equations, and because usually the resulting equation can be interpreted as conservation of energy or some similar thing.

For example, let's pretend that we don't know how to solve the equation $x'' = -x$. You can try to integrate this equation with respect to t. Look what happens:

$$\int x'' \, dt = -\int x \, dt$$

You can do the left side, getting x', but what happens on the right? You can't do it can you? The right side doesn't integrate to xt or something like that does it? I'm asking all these questions because this is a common place to make mistakes. Really, you just can't do the integral on the right, because the integrand $x(t)$ is not yet known. But there is a way to fix this problem. It is very clever because it seems at first to complicate the problem rather than simplifying it. Multiply the equation $x'' = -x$ by x' and see what happens. You get

$$x'x'' = -xx'$$

Now stare at that for a while, and see if you can integrate it now.

$$\int x'x'' \, dt = -\int xx' \, dt$$

You may notice, if you look at this long enough, that according to the chain rule you *can* now integrate both sides. That is because xx' is the derivative of $\frac{1}{2}(x)^2$, and $x'x''$ is the derivative of $\frac{1}{2}(x')^2$. So integrating, you get

$$\frac{1}{2}(x')^2 = c - \frac{1}{2}(x)^2$$

Now there are several things to observe about this. First we don't have a second order equation any more, but a first order instead. Second we can make a picture for this equation in the (x, x') plane, and it just says that solutions to our problem go around circles in this plane, since we just got the equation of a circle whose size depends on the constant c. Third, there is a physical interpretation for the first order equation, which is conservation of energy. Not everyone who takes this course has had physics yet, so the next bit of this discussion is optional, and may not make too much sense if you have not seen an introduction to mechanics. Conservation of energy means the following. x is the position and x' the velocity of a vibrating object. Peek ahead to Figure 8.1 if you like, to see what it is. The energy of this object is in two forms, which are kinetic energy and potential energy. The kinetic energy "$\frac{1}{2}mv^2$" is the $(x')^2$ term. The potential energy "$\frac{1}{2}kx^2$" is the x^2 term. So what is c? It is the total energy of the oscillator, and the going around in circles is physically the fact that when something swings back and forth, the energy is periodically transferred from to potential to kinetic and back. The child on a swing has a lot of kinetic energy at the bottom of the path moving fast, and all that becomes potential energy by the top of the swing when he is instantaneously not moving, and the cycle repeats.

This kind of calculation, finding the integral of the equation or some multiple of the equation, is called finding a conservation law. It cannot always be used, but is a powerful thing to know about.

Problems ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯ **6**

1. Suppose $x_1'' + x_1{}^2 = 0$ and $x_2'' + x_2{}^2 = 0$, and set $s = x_1 + x_2$. Calculate $s'' + s^2$ to see why the sum of solutions is not usually a solution.

2. Suppose x_1 and x_2 both solve
$$\begin{cases} x'' + x = 0 \\ x'(0) = 2 \end{cases}$$

and set $s = x_1 + x_2$. Find out why s is *not* a solution.

3. Suppose x_1 and x_2 both solve

$$\begin{cases} x'' + x = \cos 8t \\ x'(0) = 0 \end{cases}$$

and set $s = x_1 + x_2$. Is s a solution?

4. Solve
$$\begin{cases} 4y'' - 3y' - y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

5. Solve the same equation as in problem 4, but with initial conditions

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

6. Solve the same equation as in problem 4, but with initial conditions

$$\begin{cases} y(0) = 2 \\ y'(0) = 3 \end{cases}$$

7. Can you see that the answer to problem 6 should be 2 times the answer to problem 4 plus 3 times the answer to problem 5?

8. Find a conservation law for the equation $x'' + x^3 = 0$.

9. Do you think there are any conservation laws for $x'' + x' + x = 0$?

10. *What's rong with this?* $y'' + y^2 = 0$, $r^2 + 1 = 0$, $r = -1$, $y = e^{-t} + c$. ?!? There are at least 3 errors.

Today:

> *A review of complex numbers and a motivation for the definition of e^{a+bi}.*
> *More second order equations.*

Complex numbers are expressions of the form $a + bi$ where a and b are real numbers. You add, subtract, and multiply them just the way you think you do, except that $i^2 = -1$. So for example,

$$(2.5 + 3i)^2 = 6.25 + 2(2.5)(3i) + 9i^2 = -3.25 + 15i$$

If you plot these points on a plane, plotting the point (x, y) for the complex number $x + yi$, you will see that the angle from the positive x axis to $2.5 + 3i$ gets doubled when you square, and the length gets squared. Addition and multiplication are in fact both very geometric, as you can see from the figures.
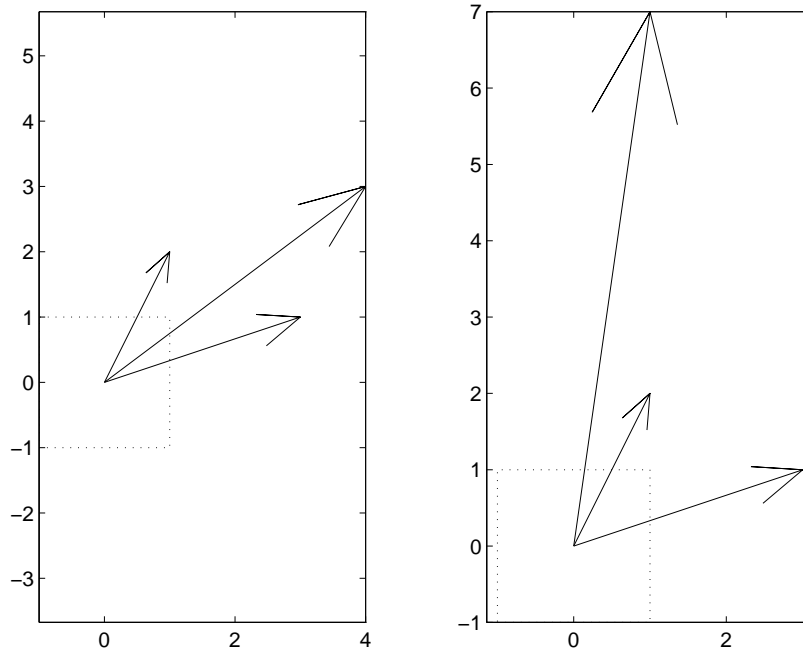
Fig 7.1 *Addition of complex numbers is the same as for vectors. Multiplication adds the angles while multiplying the lengths. The left picture illustrates the sum of $3 + i$ and $1 + 2i$, while the right is for the product. (figure made by the* `compass` *command in* `matlab`*)*

For Practice: Use the geometric interpretation of multiplication to figure out a square root of i.

Division of complex numbers is best accomplished by using this formula for reciprocals:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2}$$

For Practice: Verify that this reciprocal formula is correct, i.e. that using it you get $\frac{1}{a+bi}(a + bi) = 1$.

You are now equiped to do arithmetic, such as solving $2 - (2 - 5i)z = 3/i$ for z. To do algebra correctly we should point out that two complex numbers are equal by definition when the real and imaginery parts are equal. If $z = a + bi$ and a and b are real, then we write $\text{re}(z) = a$ and $\text{im}(z) = b$. (not bi.) *Warning*: If you know that $a + bi = c + di$ you cannot conclude that $a = c$ and $b = d$. For example, $1 + 0i = 0 + (-i)i$. However, if you know in addition that a, b, c, and d are real, then you can conclude that $a = c$ and $b = d$.

The quadratic formula for solving $ax^2 + bx + c = 0$ is still good if everything in sight is complex, but taking square roots can be tedious. That is why we practised it above.

There is another important construction which is motivated by algebra and differential equations, and very useful. That is raising e to a complex power. We set $e^{(s+ti)} = e^s e^{ti}$ by analogy with known properties of real exponentials, but this still requires a definition of e^{ti}. We claim that the only reasonable choice is $\cos(t) + i\sin(t)$. The reason is as follows. The whole process of solving second order equations by the characteristic equation method depends on the formula

$$\frac{de^{rt}}{dt} = re^{rt}$$

Let's require that this hold also when $r = i$. Writing $e^{it} = f(t) + ig(t)$ this requires

$$f' + ig' = i(f + ig) = -g + if$$

so $f' = -g$ and $g' = f$. These should be solved using the initial conditions $e^0 = 1 = f(0) + ig(0)$. These give $f'(0) = 0$ and $f'' = -f$. The only solution is $f(t) = \cos(t)$ and $g(t) = \sin(t)$. Therefore our definition becomes

$$e^{(s+ti)} = e^s(\cos(t) + i\sin(t))$$

Now it turns out that we actually get $\frac{de^{rt}}{dt} = re^{rt}$ for *all* complex r. You should verify this for practise, using our definition. This is a case of pulling oneself up by one's own bootstraps: we solve a simple differential equation to make a definition of e to a complex power, so that we can use that to solve more complicated differential equations.

Now we can do more examples.

example $x'' + 2x' + 3x = 0$ The characteristic equation, which works by all that has been done above, is $r^2 + 2r + 3 = 0$. The solutions are $r = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$. So there are solutions to the differential equation of the form

$$x(t) = c_1 e^{(-1+i\sqrt{2})t} + c_2 e^{(-1-i\sqrt{2})t}$$

Given some initial conditions, we can find the constants. For instance if $x(0) = 5$ and $x'(0) = 0$ then we must solve

$$\begin{cases} x(0) = 5 = c_1 + c_2 \\ x'(0) = 0 = (-1 + i\sqrt{2})c_1 + (-1 - i\sqrt{2})c_2 \end{cases}$$

The sum of these gives $5 = i\sqrt{2}(c_1 - c_2)$. So

$$c_1 = c_2 + \frac{5}{i\sqrt{2}} = c_2 - \frac{5i}{\sqrt{2}}$$

The first equation then gives

$$5 = 2c_2 - \frac{5i}{\sqrt{2}}$$

Finally $c_2 = (5/2)(1 - 1/\sqrt{2})$ and $c_1 = 5 - c_2 = (5/2)(-1 - 1/\sqrt{2})$. So

$$x(t) = (5/2)(-1 - 1/\sqrt{2})e^{(-1+i\sqrt{2})t} + (5/2)(1 - 1/\sqrt{2})e^{(-1-i\sqrt{2})t}$$

31

You may notice that the answer, besides being messy, looks very complex. Now the last time you went into the lab, all the instruments were probably reading out *real* numbers, weren't they? So it is common to rewrite our solution in a way which shows that it really is real after all. To do this, the following new idea is required: If $x(t)$ is a complex solution to a second order linear differential equation with real coefficients then the real and imaginery parts of x are also solutions! For example, e^{3it} is a solution to $x'' = -9x$. The real and imaginery parts are respectively $\cos(3t)$ and $\sin(3t)$, and these are certainly solutions also. To see why this works in general, suppose that $x = u + iv$ solves $ax'' + bx' + cx = 0$. This says that $a(u'' + iv'') + b(u' + iv') + c(u + iv) = (au'' + bu' + cu) + i(av'' + bv' + cv) = 0$ Assuming that a, b, c, u, and v are all real, you can conclude that $au'' + bu' + cu$ and $av'' + bv' + cv$ are also 0. But note that if some of the coefficients had not been real, or if the equation had been something like $x'' + x^2 = 0$ then the argument would not have worked. (Equations with explicit complex coefficients sometimes occur in physics, in spite of what we all noticed above about lab instruments. Schrödinger's Equation is an example.)

So let's rework the previous example slightly. After finding the values of r above, we wrote down among other things a complex solution $x(t) = e^{(-1+i\sqrt{2})t} = e^{-t}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t))$ Using the real and imaginery parts, we form solutions

$$x(t) = a_1 e^{-t}\cos(\sqrt{2}t) + a_2 e^{-t}\sin(\sqrt{2}t)$$

If we apply the same initial conditions again, we need $x(0) = 5 = a_1$ and $x'(0) = 0 = -a_1 + \sqrt{2}a_2$. Solving these we end with

$$x(t) = 5e^{-t}\cos(\sqrt{2}t) - (5/\sqrt{2})e^{-t}\sin(\sqrt{2}t)$$

It is important to realise that this is equal to the complex-looking solution we found previously.

sample

Solve for a:
$$\frac{3}{i} - (2+i)a = 7$$

$$a = \frac{7 - \frac{3}{i}}{-(2+i)} = -\frac{7+3i}{2+i} = -(7+3i)\frac{2-i}{2^2+1^2} = \frac{-14-6i+7i+3i^2}{5} = \frac{-17+i}{5}$$

sample

Solve
$$y'' + 2y' + 2y = 0$$

The characteristic equation is $r^2 + 2r + 2 = 0$, so by the quadratic formula $r = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$. One solution is $e^{(-1+i)t} = e^{-t}(\cos(t) + i\sin(t))$. Taking the real and imaginery parts, the solution is

$$y(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t)$$

Don't forget to check it.

sample

$$x'' + 2x' + x = 0$$

$r^2 + 2r + 1 = (r+1)^2 = 0, r = -1, -1, x = (c_1 + c_2 t)e^{-t}$, Check: $x' = c_2 e^{-t} - (c_1 + c_2 t)e^{-t} = c_2 e^{-t} - x$. $x'' = -c_2 e^{-t} - x' = -c_2 e^{-t} - (c_2 e^{-t} - x) = -2c_2 e^{-t} + x$, $x'' + 2x' + x = -2c_2 e^{-t} + x + 2(c_2 e^{-t} - x) + x = 0$

sample

$$x'' + 2x' - x = 0$$

$$r^2 + 2r - 1 = 0, r = -1 \pm \sqrt{1+1}, \ x(t) = c_1 e^{(-1-\sqrt{2})t} + c_2 e^{(-1+\sqrt{2})t}$$

Problems ────────────────────────────────── **7**

1. Solve $r^2 - 6r + 10 = 0$, $r^3 - 6r^2 + 10r = 0$, and $r^2 - 6r - 10 = 0$.

2. Solve $y'' + 3y' + 4y = 0$.

3. Find a, if $a + 1/a = 0$; if $a + 1/a = i$.

4. Sketch the graphs of the functions $e^{-t}\cos(t)$, $e^{-t}\cos(3t)$, and $e^{-4t}\sin(3t)$.

5. Verify that $d(e^{rt})/dt = re^{rt}$ for $r = a + bi$, using the definition $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$.

6. For each t, $e^{(1+i)t}$ is a complex number, which is a point in the plane. So as t varies, a curve is traced in the plane. Sketch it.

7. A polynomial $r^2 + br + c = 0$ has roots $r = -2 \pm i$. Find b and c.

8. A characteristic equation $r^2 + br + c = 0$ has roots $r = -2 \pm i$. What was the differential equation?

9. *What's rong with this?* $x'' + 4x = 0$, $r^2 + 4 = 0$, $r = \pm 2i$, $x = c_1(e^{2it} + e^{-2it})$. ?!?

Today:

Forced equations, and how to solve them using the undetermined coefficients method. Natural frequency.

Consider the equations

$$x'' + 2x = 0$$

$$y'' + 2y = 9 \sin 3t$$

The first one is called the homogeneous form of the second one, or the second is called a forced form of the first, and there are variations on this terminology. Mechanically what they mean is as follows. Since we know the solutions to the first one (don't we?) are

$$x(t) = c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t$$

it seems reasonable that this first equation is about something vibrating or oscillating. It can be interpreted as a case of Newton's $F = ma$ law, if you write it as $-2x = 1x''$. Here x is the position of a unit mass, x'' is its acceleration, and there is a force $-2x$ which seems to oppose the displacement x. We call this a "spring–mass" system. It can be drawn as in Figure 8.1, where x is measured up. The $-2x$ is interpreted as a spring force because it is in the direction opposite x: if you pull the spring 1.54 units up, then $x = 1.54$ and the force is –3.08, or 3.08 units downward. This is also a system without friction as we see from the fact that there are no other forces except for the spring force, and that the oscillation continues undiminished forever. Note that the "natural frequency" of this system is $\sqrt{2}/2\pi$ cycles/second, in the sense that the period of x is $2\pi/\sqrt{2}$: $x(t + 2\pi/\sqrt{2}) = x(t)$.

The so–called forced equation involves an additional force, as you can see if you write it as $-2y - 9 \sin 3t = 1y''$. The picture in this case is like the right side of Figure 8.1.
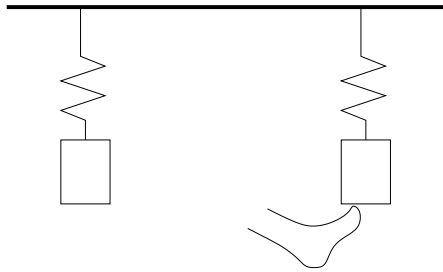

Fig. 8.1 Unforced and forced spring–mass systems.

Now we turn to solution methods for the forced equation. We are guided by the physics. That is, ask yourself what will happen if you start with a system which wants to vibrate at a frequency of $\sqrt{2}/2\pi$, and somebody reaches in and shakes it at a frequency of $3/2\pi$. What could happen? If you think about it, it is plausible that the motion might be rather complicated, but that a part of the motion could be at each of these frequencies.

Let's try that. Assume

$$y(t) = x(t) + A \sin(3t)$$

where x is the solution given above for the unforced equation. Then

$$y'' + 2y = x'' - 3^2 A \sin(3t) + 2(x(t) + A \sin(3t)) = -8A \sin(3t)$$

We want this to equal $9 \sin(3t)$. Notice how the terms involving x dropped out. We need $-8A = 9$ so $A = -9/8$. Our solution becomes

$$y(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) - \frac{9}{8}\sin(3t)$$

In particular, our physics–style thinking did lead to a solution. Some people who take this course have had physics, and some have not had physics. In any case we hope you can see that a little physics (or banking, or later biological) knowledge goes a long way in helping to set up and solve differential equations.

Notice also the coefficients of the terms in this solution. There are terms at the natural frequency $\sqrt{2}/2\pi$ and a term at the forced frequency $3/2\pi$. The coefficient of the forced–frequency term is fixed at –9/8, while the natural–frequency terms have arbitrary coefficients. If you have initial conditions to apply, now is the time.

example   To solve

$$\begin{cases} y'' + 2y = 9\sin(3t) \\ y(0) = 4 \\ y'(0) = 5 \end{cases}$$

we use the solution found above

$$y(t) = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) - \frac{9}{8}\sin(3t)$$

We get $y(0) = 4 = c_1$, $y'(0) = 5 = c_2\sqrt{2} - 3(\frac{9}{8})$. The answer becomes

$$y(t) = 4\cos(\sqrt{2}t) + \left(\frac{5 + \frac{27}{8}}{\sqrt{2}}\right)\sin(\sqrt{2}t) - \frac{9}{8}\sin(3t)$$

This is graphed in the figure 8.2.
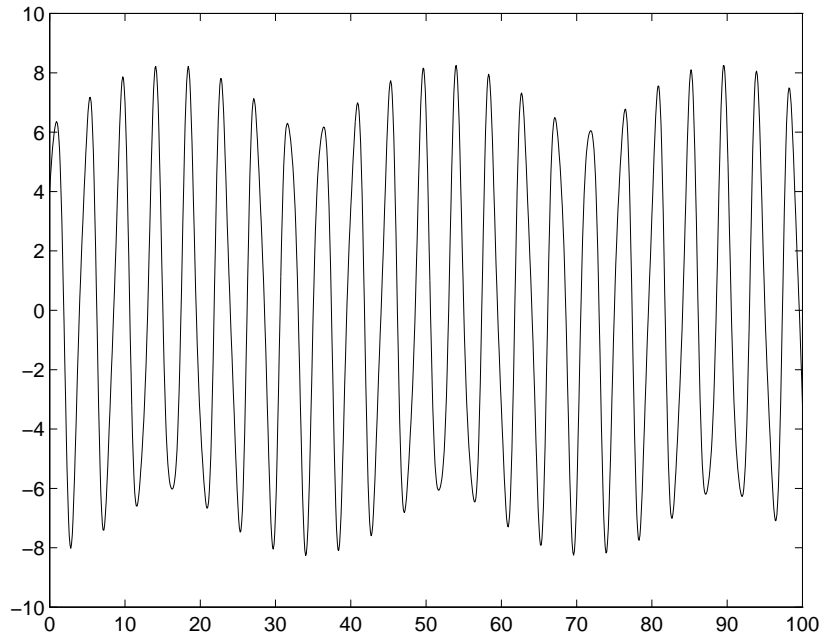
Fig. 8.2. *Can you see two frequencies at work here? They are not the obvious ones! Do problems 1–3 to see what they are. (figure made by the* `plot` *command in* `matlab`*)*

There is a terminology that traditionally goes with this solution. One calls $-\frac{9}{8}\sin(3t)$ a "particular" solution to $y'' + 2y = 9\sin(3t)$. It is also true that $29.3\cos(\sqrt{2}t) - \frac{9}{8}\sin(3t)$ is a particular solution, as are many other things. One says that the solution is the sum of a particular solution and a solution of the homogeneous equation.

Other equations can be solved by the same method. For example, it is possible to solve $y'' + y' + y = 3 + e^t$ by guessing a particular solution of the form $A + Be^t$. This will work. However this type of forcing is really not very important for us. Sines and cosines are the important things because they are the most practical; if you are sitting in class and happen to notice a fan vibration in the ceiling of the room, that is an example which can be modeled by one of these equations with sinusoidal forcing. The spring and mass correspond to some of the ceiling materials, and the sine or cosine forcing term represents the fan motor running at a constant speed.

example

$$y'' + 3y' + 2y = \sin(t)$$

Try $y = A\sin(t) + B\cos(t)$, since sine alone will not work, considering the $3y'$ term. Then $y'' + 3y' + 2y = (-A + 2A - 3B)\sin(t) + (-B + 2B + 3A)\cos(t)$. We need

$$\begin{cases} A - 3B = 1 \\ B + 3A = 0 \end{cases}$$

Substituting the second of these into the first gives $A - 3(-3A) = 10A = 1$. Then $A = .1$ and $B = -3A = -.3$. Our particular solution is $.1\sin(t) - .3\cos(t)$. For the homogeneous equation the characteristic equation is

$$r^2 + 3r + 2 = (r+1)(r+2) = 0$$

so $r = -1$ or $r = -2$. The answer is then

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} + .1\sin(t) - .3\cos(t)$$

This is shown in figure 8.3, for the case $c_1 = 1$, $c_2 = -1$.

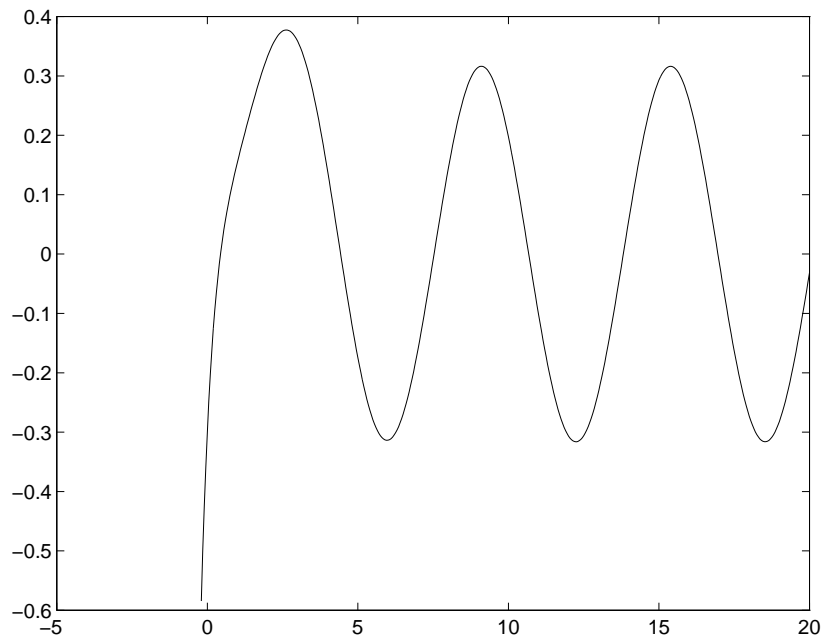Fig. 8.3. A solution to $y'' + 3y' + 2y = \sin t$. Only the forced solution remains after some time; the e^{-t} and e^{-2t} terms are "transient". (figure made by `plot` in `matlab`)

There is one special case which requires more work to solve than the others. This is the case which occurs when the forcing is at exactly the natural frequency, and is called "resonance". Like:

$$x'' + 4x = \cos(2t)$$

Think about what happens then. If your little brother is on the swing going back and forth every 3 seconds, and you "help" by pushing once every 3 seconds, pretty soon he will be going higher than he wants to go! Mathematically, if you try a particular solution $A\cos(2t)$ it won't work because you get 0 when you substitute it into the left side of the equation.

Taking these things together, it is reasonable to try something like $At\cos(2t)+Bt\sin(2t)$. These actually work, giving solutions which grow with time.

For Practice: Verify that the solution to $x''+4x = \cos(2t)$ is $x(t) = \frac{1}{4}t\sin(2t)+c_1\cos(2t)+c_2\sin(2t)$

Problems ────────────────────────────────── **8**

1. Use one of the addition formulas $\sin(a+b) = \sin(a)\cos(b)+ \cos(a)\sin(b)$ or $\cos(a+b) = \cos(a)\cos(b)- \sin(a)\sin(b)$ to show that any function of the form $A\cos(ft) + B\sin(ft)$ can be written in the form $C\sin(ft + \phi)$, for some numbers C and ϕ. Note that here we were combining sinusoids having the same frequency $f/2\pi$.

2. Similarly to problem 1, show how a combination of two sinusoids of *different* frequencies can be written as a sum of products of sinusoids: derive for example the identity $\sin(8t) + 2.3\sin(7t) = 3.3\sin(7.5t)\cos(.5t) - 1.3\cos(7.5t)\sin(.5t)$

3. Combine the ideas of problems 1 and 2 to explain the two obvious frequencies in Figure 8.2, in particular that they are *not* $\sqrt{2}/2\pi$ and $3/2\pi$.

4. Solve $x'' + x' + (1/4)x = 7\sin 6t$.

5. Solve $x'' + 4x = \sin 3t$, and $x'' + 4x = \sin 2t$, and discuss the difference.

6. Show that the addition formulas for the sine and cosine are equivalent to the complex exponential rule $e^{a+b} = e^a e^b$.

7. *What's rong with this?* $x'' + 4x = 0$, $x = \cos(2t) + \sin(2t) + C$?!?

37

Today:

Why would anybody want to study more than one equation at a time? Find out. Phase plane.

A chemical engineering problem ――――――――――――――――――――――――――――

Jane's Candy Factory contains many things of importance to chemical engineers. One of the processing lines contains two tanks of sugar solution. Pure water and sugar continually enter the first one where they are mixed, and an equal volume of the solution flows to the second, where more sugar is added. The solution is taken out at the same rate from the second tank. The flow rates are as shown in the table.

	tank 1	*tank 2*
sugar input	10 lb/hr	6 lb/hr
water input	25 gal/hr	0
solution out	25 gal/hr	25 gal/hr
tank contents	100 gal	125 gal
weight of sugar	x lb	y lb

Jane's control system tracks the weight of sugar present in the tanks in the variables x and y. We can figure out the rates of change easily. The basic principle here is conservation of mass.

$$x' \text{ [lb/hr]} = \text{ the rate in} - \text{ the rate out} = 10 \text{ [lb/hr]} - \frac{x \text{ [lb]}}{100 \text{ [gal]}}(25 \text{ [gal/hr]})$$

See, you just keep an eye on the units, and everything works out. Similarly

$$y' = 6 + \frac{x}{100}(25) - \frac{y}{125}(25)$$

Jane would like to predict the length of time required to start up this system, i.e., beginning with pure water in brand new tanks, how long before a "steady state" condition is reached (if ever)? This is an initial value problem:

$$\begin{cases} x' = 10 - .25x \\ y' = 6 + .25x - .2y \\ x(0) = 0, y(0) = 0 \end{cases}$$

Let's solve it the quickest way we can, and then we'll do more of an overview of systems in general. Look–the x equation is first order linear. Jane's Candy Factory is in luck! We know the solution is of the form $x = ae^{-.25t} + b$. Using the equation and the initial condition yields $x(t) = 40 - 40e^{-.25t}$. Then the y equation reads $y' = 6 + .25(40 - 40e^{-.25t}) - .2y$ which is *also* first order linear. The integrating factor is $e^{.2t}$, which brings the y equation to $(e^{.2t}y)' = 16e^{.2t} - 10e^{-.05t}$. Then $e^{.2t}y = 80e^{.2t} - 200e^{-.05t} + c$. The initial condition $y(0) = 0$ gives finally $c = 120$ and our solution is

$$\begin{cases} x(t) = 40 - 40e^{-.25t} \\ y(t) = 80 - 200e^{-.25t} + 120e^{-.2t} \end{cases}$$

The figure shows a plot of these solutions. It was made using the matlab commands

```
t = 0:.2:30; % this was a guess, that 30 hours would be enough
x = 40 - 40*exp(-.25*t);
y = 80 - 200*exp(-.25*t) + 120*exp(-.2*t);
plot(t,x); hold; plot(t,y)
```
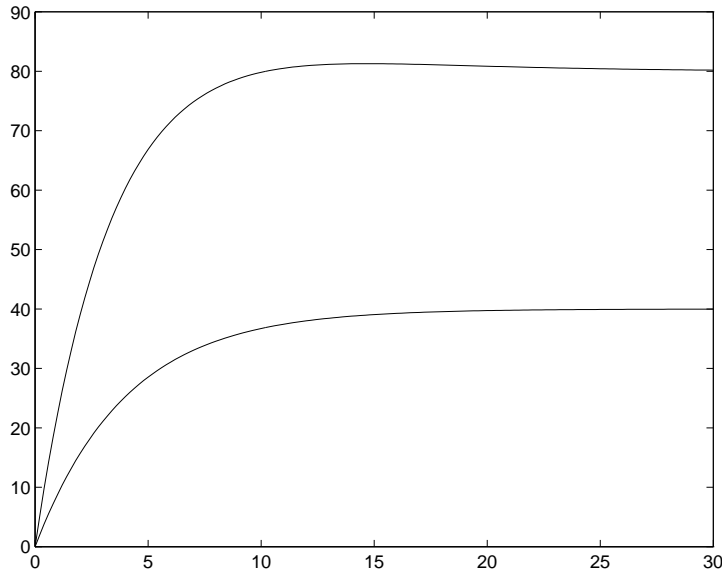
Fig. 9.1 For Jane's Candy Factory. See whether you can determine which curve is for the first tank. (figure made by `plot` commands as given in the text)

Finally we can answer Jane's question about how long the plant is going to take to start up. As $t \to \infty$, $x(t)$ approaches 40 and $y(t)$ approaches 80. Using the graph, we will be within 10% of these values after about 8 hours, and well over half there in only 3. That kind of information could be useful, if you run a candy factory, or an oil refinery . . . .

phase plane

There is an easier way to produce a plot for Jane's system, although reading it requires a bit of practice. The `pplane` utility makes the following picture.

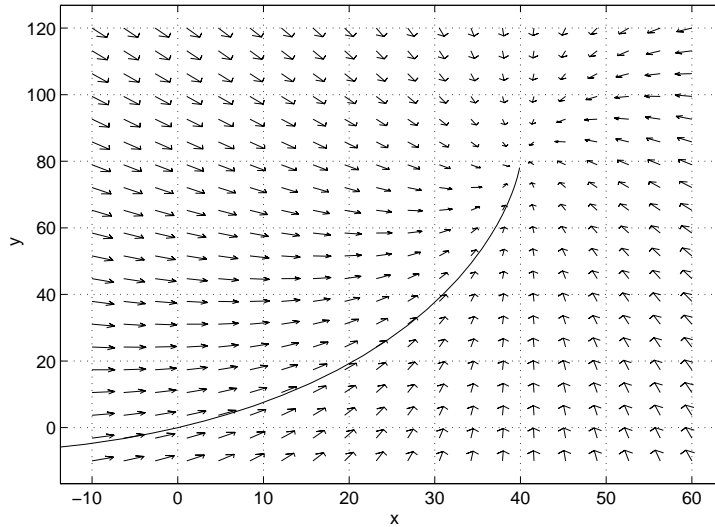Fig. 9.2 Jane's Candy Factory, t hidden. The limit point (40,80) corresponds to the limiting values in Fig. 9.1. (figure made by `pplane`)

In order to interpret this figure, one has to realise that the time is not shown, so that the relation between x and y may be displayed. This is called the "phase plane" for the system. Here we explain the idea of a phase plane. The vectors shown are made by using

the expressions for x' and y' as components. This can be done for any system of the form

$$\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

It means that we are thinking of the functions f and g as components of a vector field. We also think of $(x(t), y(t))$ as parametric equations for a curve in the (x, y) plane. Then from this point of view, look at what the differential equations say. The right–hand sides are the vector field, while the left–hand sides give the velocity. Consequently, the differential equation system just says that the velocity of a solution curve must agree with the given vector field. This is the *entire content* of the diffential equation. You may observe in Fig. 9.2 that the solution curves are tangent to the vectors. You do observe that, right?

For Practice: Choose a point in the (x, y) plane, and work out the right hand sides of Jane's system for these coordinates. Then draw a vector with these components starting from your point (x, y). See if it is the same as the one `pplane` drew.

converting a system to a higher order equation

It is occasionally of interest to convert a system of differential equations into a single higher order equation. (This cannot always be done.) This can be carried out for Jane's system as follows. The y equation gives $.25x = y' - 6 + .2y$, or $x = 4y' - 24 + .8y$. Substituting this into the x equation gives $x' = 4y'' - 0 + .8y' = 10 - .25(4y' - 24 + .8y) = 16 - y' - .2y$. Rearranging, we find that

$$4y'' + 1.8y' + .2y = 16$$

The initial conditions become $y(0) = 0$, and $y'(0) = 6 + .25x(0) - .2y(0) = 6$. Ordinarily as we'll see, it is more useful to convert higher order equations to lower. That always can be done.

Problems ———————————————————————————— 9

1. Find a second order equation for x_1, if

$$\begin{cases} x_1' = x_1 - x_2 \\ x_2' = x_1 + 2x_2 \end{cases}$$

2. Find a second order equation for x_2, for the same system as in problem 1. Is it the same as the equation for x_1?

3. Make a sketch of the phase plane for the system of problem 1. You should do this first by hand, and then let the computer do the work for you by running `pplane` on the system.

4. Here you will find another system for the second order equation of problem 1. Set $z_1 = y$, $z_2 = y' - y$, and find the equations for the derivatives of z_1 and z_2. Did you get the same system as in problem 1?

5. Make a sketch of the phase plane for the system

$$\begin{cases} x' = 1 \\ y' = y - x \end{cases}$$

and make a sketch of the slope field for the equation

$$y' = y - t$$

How do these two sketches compare, and why?

6. Set up a system for a modified version of Jane's Candy Factory, in which there are now third and fourth tanks in series with the first two, with parameters as shown.

	tank 3	tank 4
sugar input	0 lb/hr	5 lb/hr
water input	15 gal/hr	0
solution out	25 gal/hr	35 gal/hr
tank contents	220 gal	165 gal
weight of sugar	z lb	w lb

You may notice that the volume of solution is no longer constant for all of the tanks.

Today:

> _Systems of differential equations, Newton, nonlinear springs, predator–prey, linear systems. Preview of algebra to come._

There are several reasons for studying systems of differential equations. One is typified by Jane's Candy Factory of Lecture 9, where of course if we were accounting for 7 tanks, we would need 7 equations. Another case comes from Newton's $F = ma$ law. Each mass m needs a second order equation, and there are many masses around.

example _Two Newton's Laws give 4 first order._ Suppose we have two masses moving on the x axis pushing each other according to the system

$$\begin{cases} m_1 x_1'' = f_1(x_1, x_2) \\ m_2 x_2'' = f_2(x_1, x_2) \end{cases}$$

In fact according to another of Newton's proclamations, $f_2 = -f_1$, but we don't need this for our present purpose. Set $v_1 = x_1'$, $v_2 = x_2'$. Then we have four variables representing the positions and velocities of the two masses. The system becomes

$$\begin{cases} x_1' = v_1 \\ x_2' = v_2 \\ m_1 v_1' = f_1(x_1, x_2) \\ m_2 v_2' = f_2(x_1, x_2) \end{cases}$$

example _A nonlinear spring gives two first order._ Consider the equation $x'' + 2x' + x^3 = 0$. This equation looks somewhat like the second-order equations we studied for spring-mass systems, but it is not linear. You can think of the x^3 term as a nonlinear spring force. Specifically, in order to stretch this spring twice as far, you must pull _eight_ times as hard. In many respects, this is closer to the behavior of a real spring. We now show how to convert this equation to a system of first order equations. Set $y = x'$. Then $y' = x'' = -2x' - x^3 = -2y - x^3$. This gives us the first order system

$$\begin{cases} x' = y \\ y' = -2y - x^3 \end{cases}$$

There must be some reason for doing this, right? First order systems do have some advantages. They are much easier to think about, for one thing: to imagine what happens with $x'' + 2x' + x^3 = 0$ involves thinking about the graph of $x(t)$, and how the position x, slope x', and curvature x'' all interact. This can be pretty complicated. On the other hand, thinking about an equivalent system just involves the idea that the solution curves must follow the vector field, as described in Lecture 9. For emphasis, consider again a differential equation system

$$\begin{cases} x' = f(x, y, z) \\ y' = g(x, y, z) \\ z' = h(x, y, z) \end{cases}$$

The left sides give the velocity of a solution curve, while the right sides say that this velocity must agree with the vector field whose components are f, g, and h. In addition to this relatively simple description, most of the available software is also set up to deal with first order systems, as we saw indications of in Lecture 9.

example _Predator – Prey._ Suppose we make a model of two populations, where $x(t)$ and $y(t)$ are the population sizes. We assume that these two species depend in each other in very different ways. As a first approximation, we assume that if there were none of the y species present, that the x species would grow exponentially, and that the exact opposite

would happen to the y species if there were no x's around: we assume the y's would die out. Our first approximation is

$$\begin{cases} x' = x & \text{first approximation: exponential growth} \\ y' = -y & \text{first approximation: exponential decay} \end{cases}$$

This fits, if we are thinking of the y's as predators, and x's as prey, wolves and mice, for example. The next step is to add some terms for the interaction of these species. We assume

$$\begin{cases} x' = x - xy & \text{wolves are bad for mice} \\ y' = -y + xy & \text{mice are good for wolves} \end{cases}$$

It should be clear that we have neglected to put in various coefficients; this does not affect the ideas involved. In a homework problem we ask you to run `pplane` on this system to see what happens. For now, we will just show that it is possible to begin with a system like this one, and convert it to a single second order equation, which is however quite a complicated equation sometimes. We calculate from the first equation, $y = (x - x')/x = 1 - x^{-1}x'$. This gives $y' = x^{-2}(x')^2 - x^{-1}x''$. Then from the second equation, $y' = (-1 + x)y = (-1 + x)(1 - x^{-1}x')$. The next step would be to equate these expressions and simplify. As you can see, this will give a very nasty second order equation. The first order system definitely makes more sense here.

linear systems ——————————————————————————————

A linear system of differential equations means a system of the form

$$\begin{cases} x' = 2x + 3y \\ y' = 4x + 5y \end{cases}$$

The constants 2, 3, 4, and 5 might of course be replaced by any others. There could also be more than the two unknown functions x and y. You may notice that this seems to be a fairly simple system, since it does not include the predator – prey example, nor the nonlinear spring. We study these systems because 1) They are easy and fun to solve and make pretty pictures in the phase plane, 2) We learn some new algebra called linear algebra in the process of solving these, which is used for many things other than differential equations, and 3) They are the foundation for the harder equations.

example We'll attack the linear system above with exponential functions, since these have been useful so many times already. Try $x = ae^{rt}$, $y = be^{st}$. The system becomes

$$\begin{cases} are^{rt} = 2ae^{rt} + 3be^{st} \\ bse^{st} = 4ae^{rt} + 5be^{st} \end{cases}$$

It is not quite clear what to do next, is it? But notice what happens if we take $s = r$. Then all the exponentials cancel

$$\begin{cases} ar = 2a + 3b \\ br = 4a + 5b \end{cases}$$

You may wish to compare this to what happened with second order linear equations. As in that case, we have gotten rid of the differential equations, and now have algebraic equations to solve. It should be easier this way. These are easier to solve if you rearrange them as

$$\begin{cases} (2 - r)a + 3b = 0 \\ 4a + (5 - r)b = 0 \end{cases}$$

What happens next can be confusing, if you don't see it coming. We are not going to solve this algebraic system right now. In fact, we are going to leave differential equations for a while to study the algebra connected with systems like these. It happens to be the same

algebra which is used for many purposes not connected at all with differential equations, too. Once we know the algebra, we'll come back to the differential equation problem.

Problems ————————————————————————— **10**

1. Find a first order system corresponding to $x'' - x + x^3 = 0$. You may use $y = x'$ or something else that you choose, as the second function.

2. Using $y = x'$, find a first order system corresponding to $x'' - x + x = 0$.

3. For the system you obtained in problem 2, substitute $x = ae^{rt}$ and $y = be^{rt}$ and find the algebraic system which results. You do not have to solve the algebraic system.

4. Make a sketch of the phase plane for the system of problem 2. You should do this both by hand and by using `pplane`.

5. Run `pplane` on first order systems for the equation $x'' + 2x + x^p = 0$ for $p = 1$ and $p = 3$. What differences do you observe?

6. Find a first order system corresponding to $x'' + x^3 = 0$ and try to sketch the phase plane by hand. This is too hard to do accurately, so don't worry about it. Next, multiply the equation by x' and integrate to find a conservation law, as in Lecture 6. Now it should be easy to sketch the phase plane, because the conservation law shows you what curves the solutions run along.

7. Consider the predator–prey equations again. Our attempt to find a second order equation for this system was a mess, but it suggests something which actually works: you may notice the term x'/x in part of our derivation. This suggests $\ln(x)$, doesn't it? since $d(\ln(x))/dt = x'/x$. Show that the equation $x + y - \ln(x) - \ln(y) = c$ is a conservation law, in the sense that $dc/dt = 0$ for all solutions to the predator–prey system. Sketch a graph of the function $f(x, y) = x + y - \ln(x) - \ln(y)$. It may help to use Matlab for this sketch; type `help` or `help plotxyz` to find out how.

8. Here is another way to "discover" the conservation law of problem 7. Imagine that the parametric equations for $(x(t), y(t))$ are solved so that y is expressed as a function of x. Then by the chain rule you get $dy/dx = y'/x'$. This gives $dy/dx = (-y + xy)/(x - xy)$. Show how to solve this by separation of variables.

9. Run `pplane` on the predator–prey system and on the predator–prey system with migration

$$\begin{cases} x' = x - xy - .2 \\ y' = -y + xy \end{cases}$$

Which species is migrating here, and are they moving in or out? Write a verbal description of what the differences are for the two phase planes. Do you think there is a conservation law for the migratory case?

10. *What's rong with this?* $\begin{cases} x' = y \\ y' = -x^3 \end{cases}$, $x'' = -x^3$, $d^2x/dt^2 = -x^3$, $d^2x/x^3 = dt^2$, $dx/x^2 = dt + C$, $\ln(x^2) = t$, $x = e^{t/2}$, $y = (1/2)e^{t/2} + C$. ?!? There are at least 3 mistakes.

Today:

systems of linear algebraic equations, matrices

A system of linear algebraic equations is something like

$$\begin{cases} 2x - 3y + 5z = 0 \\ .7x - y + 3z = 0 \\ -y - 2z = 7 \end{cases}$$

You might think, based on your past experience in math courses, that there is always exactly one answer to every problem. The situation here is a bit more subtle: This system can be thought of as three planes. If you plot these planes in an (x, y, z) coordinate system, you might find that all three happen to intersect in a single point. That point is the solution, in that case.
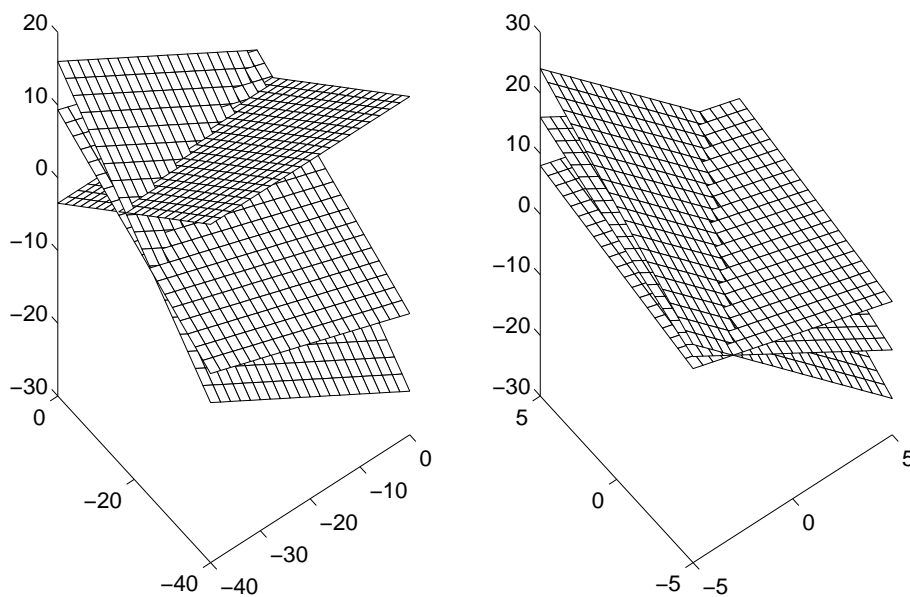

Fig. 11.1 *Planes can intersect more ways than shown. The difference is subtle; can you see it? Note that planes which intersect, do so in a line; the software did the best it could. (figure made using the* `plot` *command in Matlab)*

It may also happen that the planes meet in a line, rather than a point. Then there are infinitely many solutions. Again, it is possible that two of the planes are parallel. Then there is no solution.

For Practice: 1) There is another possibility. Can you think what it might be? 2) Which of these possibilities holds for the system $x + y = 0$, $2x + 2y = 3$, $z = 5$?

matrix ————————————————————————————————————

A matrix is an array of numbers, such as the coefficients in our first system

$$A = \begin{bmatrix} 2 & -3 & 5 \\ .7 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

44

A matrix can also be formed to hold the variables

$$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as well as the numbers on the right–hand side

$$b = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

Matrices are sometimes written using bold type, or with underlines, overlines, overarrows, or rounded brackets, or square brackets around the name of the matrix. We won't use any of those. That means that when we use the symbol A or b, you have to keep straight from the context whether we are talking about a number or matrix or what. The size of a matrix is given as *number of rows* by *number of columns*. A is 3 by 3, while u and b are 3 by 1. We introduce a new form of multiplication so that our system can be written

$$\begin{bmatrix} 2 & -3 & 5 \\ .7 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

or

$$Au = b$$

You might notice that to do this multiplication, it is helpful to move your left hand along the row of A, and your right hand down the column of u. The same concept is used to multiply matrices of any sizes, except that to multiply AB you must have the same number of columns in A as rows in B. Otherwise some poor number is left without anybody to multiply.

example *matrix multiplication*

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 4x + 5y \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 5 \\ .7 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6.7 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a + 2c & 1b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix}$$

I is the special matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or ..., one for each size. It is the *identity* matrix and you should check that always $AI = IA = A$, $Iu = u$. It is also useful to abbreviate the system $Au = b$ further by not writing the variables at all: the 3 by 4 matrix $[A\,b]$ is called the augmented matrix of the system. In general, if you have a system of m equations in n unknowns, $Au = b$, then A is m by n, b is n by 1, u is m by 1, and the augmented matrix $[A\,b]$ is m by $(n+1)$.

Geometric aspects of matrices _____

Before continuing with equation–solving, there are several remarks. Matrices like

$$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

45

which consist of one column are sometimes called column vectors, and pictured as an arrow from the origin to the point with coordinates (x, y, z). In other words, u here has exactly the same meaning as $x\vec{i} + y\vec{j} + z\vec{k}$, and you can think of it in exactly the same way. Of course we are now allowing our column vectors to contain more than three coordinates, so the three–dimensional vector notation doesn't carry over. Welcome to the fourth dimension! There are some very nice geometric ideas connected to this form of multiplication. These seem at first very different and unrelated to equation–solving. We'll give one example now, just so you realise that there are several different reasons for studying matrices. Consider the simple matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

A can be applied to any "point" (x, y) in the plane by using multiplication:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{y}{2} \end{bmatrix}$$

This means that every point moves vertically closer to the x axis. In spite of how easy this multiplication is, the effect on the plane is dramatic, as you can see in the next figure.
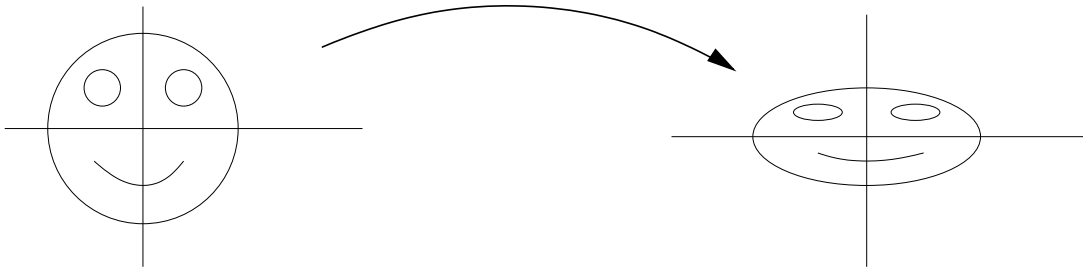

Fig. 11.2 Matrices can change little smiley faces. What would happen if the face were not centered at the origin?

Problems ─── **11**

1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, and let $x = \begin{bmatrix} a \\ b \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$. Calculate AB, $A^2 = AA$, C^3, DB, BD, Ax, $3Ax - 2Dz$, Dx, and Cy.

2. Suppose $Au = b$ and $Av = b$, and set $s = u + v$. Is it true that $As = b$? How about if $Av = 0$ instead, what then?

3. An airline company plans to make x flights per month from its main hub airport to Chicago, and y flights to New York. New York flights require 6500 lbs of fuel and 200 bags of pretzels, while Chicago flights require 5400 lbs of fuel and 180 bags of pretzels. The available monthly supply of materials at the hub consists of 260,000 lbs of fuel and 9,000 bags of pretzels, and all these are to be used if possible. Write a system of equations expressing all this. Discuss faults of this model.

4. *What's rong with this?* $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 9 & 12 \end{bmatrix}$?!?

Today:

Row operations

Now that our matrix notation is established, we turn to the question of solving the linear system $Au = b$. You may have solved such things in algebra courses in the past, by eliminating one variable at a time. This works fine for systems of 2 or maybe 3 equations. For our purposes though, it is important to learn a very systematic way of accomplishing this. We will describe a procedure which applies equally well to systems with 2000 variables! (In case you are wondering about the need for that, let me reassure you that linear algebra, of all the things you study, might be the one thing that really gets used in real life. Consider an airline company which needs to keep track of 4500 passengers, 200 planes, 300 crews, 8500 spare parts, 25 cities, 950000 pounds of jet fuel, 85600 little bags of pretzels, etc, etc. You get the picture.) Before we describe row operations, you should see what the system of equations looks like in a special case like

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

This is the augmented matrix for the system $x = 5$, $y = 7$, and $z = 9$. There is nothing to solve in this case. Consequently, the object of these row operations is to put the matrix into the form $[I\,b]$ if possible.

We now describe "row operations" for solving the system $Au = b$. There are three row operations, which are applied to the augmented matrix $[Ab]$, and have the effect of manipulating the various equations represented by the augmented matrix.

1) A row may be multiplied by a non–zero number.
2) Two rows may be interchanged.
3) A row may be replaced by its sum with a multiple of another row.

Row operations do not change the *solutions* of the system. (This is clear for the first two, and requires a bit of thought for the third one.) They only change the appearance of the equations.

example *An example with many solutions* The system

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

represents the two lines $2x + 3y = 1$ and $4x + 6y = 2$. You might notice that these are two equations for the *same* line. Therefore any point on this line should be a solution to both equations. Let's do the row operations and see what happens. We have the augmented matrix

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \end{bmatrix} \rightarrow (\text{row } 2 - 2\,\text{row } 1) \rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have replaced row 2 by (row 2)–2(row 1). Look at the bottom line. It says that $0x + 0y = 0$. That is true but not helpful. The top line says that $2x + 3y = 1$. Since this is the only requirement for a solution, there are indeed infinitely many solutions

$$\begin{cases} x = \frac{1}{2} - \frac{3}{2}y \\ y \qquad\qquad \text{anything} \end{cases}$$

example *An example with no solutions* The system

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

has augmented matrix

$$\begin{bmatrix} 2 & 3 & 0 \\ 4 & 6 & 2 \end{bmatrix} \rightarrow (\text{row } 2 - 2\,\text{row } 1) \rightarrow \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Look at the bottom line. It says that $0x + 0y = 2$. Is that possible? (No.) So this system has no solution. It describes two parallel lines.

The next example involves a bit more arithmetic, and is our original system.

example *An example with a unique solution*

$$[A\,b] = \begin{bmatrix} 2 & -3 & 5 & 0 \\ .7 & -1 & 3 & 0 \\ 0 & -1 & -2 & 7 \end{bmatrix} \rightarrow (.5\,\text{row } 1) \rightarrow \begin{bmatrix} 1 & -1.5 & 2.5 & 0 \\ .7 & -1 & 3 & 0 \\ 0 & -1 & -2 & 7 \end{bmatrix} \rightarrow (\text{row } 2 - .7\,\text{row } 1) \rightarrow$$

$$\begin{bmatrix} 1 & -1.5 & 2.5 & 0 \\ 0 & .05 & 1.25 & 0 \\ 0 & -1 & -2 & 7 \end{bmatrix} \rightarrow (\text{interchange row } 2 \text{ and row } 3) \text{ and } (-\text{row } 2) \rightarrow$$

$$\begin{bmatrix} 1 & -1.5 & 2.5 & 0 \\ 0 & 1 & 2 & -7 \\ 0 & .05 & 1.25 & 0 \end{bmatrix} \rightarrow (\text{row } 1 + 1.5\,\text{row } 2) \text{ and } (\text{row } 3 - .05\,\text{row } 2) \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 5.5 & -10.5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 1.15 & .35 \end{bmatrix} \rightarrow (\text{row } 3)/1.15 \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 5.5 & -10.5 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 1 & -.3043 \end{bmatrix} \rightarrow (\text{row } 2 - 2\,\text{row } 3) \text{ and } (\text{row } 1 - 5.5\,\text{row } 3) \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & -12.1737 \\ 0 & 1 & 0 & -7.6086 \\ 0 & 0 & 1 & .3043 \end{bmatrix}$$

The solution to our original system is therefore

$$u = \begin{bmatrix} -12.1737 \\ -7.6086 \\ .3043 \end{bmatrix}$$

Notice there is only one solution, which says that these three planes intersect in just one point. You may notice that this is the system graphed in Fig. 11.1, on the left.

A matrix is "row–reduced" if it looks like one of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 5 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3.55 & 8 \\ 0 & 1 & 2.17 & 8 \end{bmatrix}, \text{etc.}$$

i.e., the leading non–zero in each row is 1, the column of that 1 contains no other non–zeroes, and these leading 1's move to the right as you read down successive rows of the matrix. A row–reduced augmented matrix is easy to solve: For example the 3 by 4 matrix represents the system

$$\begin{cases} x + 4y = 5 \\ z = 7 \\ 0 = 0 \end{cases}$$

which has solutions

$$\begin{cases} z = 7 \\ y \qquad\qquad \text{anything} \\ x = 5 - 4y \end{cases}$$

The downright pattern of 1's allows you to solve the reduced system from the bottom up.

We point out again the term "linear". Multiplication of vectors by a matrix is linear in the sense that $A(c_1 u_1 + c_2 u_2) = c_1 A u_1 + c_2 A u_2$, where the c's are any constants. Consequently if you have two solutions u_1 and u_2 to $Au = 0$, it follows that some other solutions are the so–called "linear combinations" $c_1 u_1 + c_2 u_2$. So we say that $Au = 0$ is a linear equation, and this is linearity in the strictest sense of the word. We also say that $Au = b$ is a linear equation, but here you have to be careful to see that a linear combination of solutions is usually *not* a solution.

Problems ── 12

1. Let $A = \begin{bmatrix} 1 & 2.6 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. What single row operation has been done to A to get B?

2. What row operation can be done to A of problem 1, to produce the identity matrix I?

3. The matrix $\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ needs two more row operations done, to put it into the reduced form. What are they, and what is the reduced form? Note that the reduced form is unique, but there are two different sets of row operations which are capable of producing it.

4. Write out the system of algebraic equations corresponding to the augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Is there something odd about the third equation? How many solutions does this system have?

5. Write the augmented matrix for the following system, row reduce it, and find all the solutions.

$$\begin{cases} x - y = 0 \\ x + y = 1 \\ -x - 3y = -2 \end{cases}$$

6. The following is a reduced augmented matrix for a system of 4 equations in the five unknowns $x_1 \ldots x_5$. Find the solutions to the system.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

7. Same instructions as problem 5 for the system

$$\begin{cases} x - y + z - 2w = 0 \\ x + y + z - w = 1 \\ -x - y = 2 \end{cases}$$

8. *What's rong with this?* $\begin{cases} x + 2y = 0 \\ 3y = 0 \end{cases}$, gives $y = 0$, then $x = 0$. Therefore there are no solutions. ?!?

Today:

Matrix inverse, solutions to linear sytems by Matlab, the eigenvalue concept

matrix inverse ─────────────

We learned last time about solving a system $Ax = b$ by row operations – you row reduce the augmented matrix $[A\,b]$ and read the solutions, if there are any. There is another approach which is harder to compute but helpful to know about sometimes. It goes like this. We try to solve $Ax = b$ by analogy with the arithmetic problem $2x = 7$. In arithmetic the answer is $x = 2^{-1}7$. Here there is sometimes a matrix which deserves to be named A^{-1}, so that the solution might be expressed as $x = A^{-1}b$. A^{-1} is pronounced "A inverse". Wouldn't that be nice? Yes, it would be nice, but we'll see some reasons why it can't work in every case. It can't work in every case because, among other things, it would mean that there is a unique answer x, and we have already seen that this is not at all true! Recheck the examples in Lecture 11 if you need a reminder of this point. Some systems have no solutions, some have one, and some have infinitely many. It turns out that for some matrices A, the inverse A^{-1} does exist. All of these matrices are square, and satisfy some additional restrictions which we will list below. Before getting to that list though, return to the phrase used above, "a matrix which deserves to be named A^{-1}". What does that mean? We'll say that a matrix B is an inverse of matrix A if $AB = BA = I$. This is the key property. In fact, there can only be one matrix with this property: if B and C both have this property, then $B = BI = BAC = IC = C$. Since there is only one, we can name it A^{-1}.

sample

For $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, we have $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $BA = I$.

So
$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

sample

For $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, we have $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ and $BA = I$. So
$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 3 \end{bmatrix}$$

and
$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

It turns out that there is a formula for the inverse of a 2 by 2 matrix, but that there is no really good formula for the larger ones. It is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ IF } ad - bc \neq 0.$$

You can derive this formula yourself: try to solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = I$ for x, y, z, and w in terms of a, b, c, and d, and you will get it eventually. The number $ad - bc$ is called the "determinant" of the matrix, because it determines whether or not the matrix has an inverse.

For Practice: You should check that this agrees with the earlier examples.

We also are going to record the method used to find inverses of larger matrices. They may be found, when they exist, by row operations. You write the large augmented matrix $[A\,I]$ which is n by $2n$ if A is n by n. Then do row operations, attempting to put it into the form $[I\,B]$. If this is possible, then $B = A^{-1}$. If this is not possible, then A^{-1} does not exist. The rationale for this method is left for a later course in linear algebra, but you can consider for yourself what equations are really being solved, when you use this large augmented matrix.

   sample

   For $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ we do operations:

$[A\,I] = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{bmatrix} \rightarrow -(\text{row}\,1)$ and $(1/4)(\text{row}\,3)$ and $(\text{row}\,2 + 2\,\text{row}\,1) \rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & .25 \end{bmatrix}$$

   You can check that $\begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & .25 \end{bmatrix} = I$

Have you heard about people who "only know enough to be dangerous"? It means they have some special knowledge of something, but without enough depth to use it properly. Matlab provides a tool to put you into this category. Isn't that exciting? So it is with great trepidation that we tell you the following fact, hoping that you will not get hurt: Matlab can sometimes solve $Au = b$ if you simply type $u = A\backslash b$. That is a backward division symbol, intended to suggest that one is more–or–less dividing by A on the left. It mimics the notation $u = A^{-1}b$ although the actual method used involves row operations, not inverse. In the problems, we suggest trying this to see what it produces in several cases.

eigenvalues

Usually when you multiply a vector by a matrix, the product has a new length and direction, as we saw with the little smiley faces: If u is a vector from the origin to the left eye in Figure 10.2, then Au points in a different direction from u. This is typical. On the other hand, let v be the vector from the origin to the top of Smiley's head. In this case, we see that $Av = \frac{1}{2}v$. This is not typical, it is special, and turns out to be quite important. Try to locate other vectors with this special property, that

$$Av = \lambda v$$

A vector $v \neq 0$ which has this property, for some number λ ("lambda"), is called an eigen-vector of A, and λ is called an eigenvalue. eigen is a German word which means here that these things are characteristic property of the matrix. You may notice that we excluded the zero vector. That is because it would work for every matrix, and therefore not be of any

interest. It is ok for the eigenvalue to be 0, though. That just means that $Av = 0$ for some nonzero v. Since this is not true for all matrices, it is of interest.

Note that this eigenvector equation $Av = \lambda v$ is superficially like the system equation $Ax = b$, but there is a big difference. The difference is that b is known, whereas the right–hand side of the eigenvector equation contains the unknown v *and* the unknown λ. So the eigenvector equation is probably harder. Let's write out the eigenvector equation for the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

If we write a and b for the components of v, then $Av = \lambda v$ becomes

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

or

$$\begin{cases} 2a + 3b = \lambda a \\ 4a + 5b = \lambda b \end{cases}$$

Now go back to reread the discussion of the linear system of differential equations near the end of Lecture 10. Do you recognize the equations?

sample

Reconsider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ which squeezes smiley faces as in Fig.10.2. Any vector along the x_1 axis is unchanged, and is therefore an eigenvector with eigenvalue 1: $A \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. What happens to vertical vectors?

Problems ———————————————————————————— 13

1. Solve

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} u = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

three ways: by row operations, matrix inverse, and the Matlab \ command.

2. Same as problem 1, for

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 4 & 6 & 1 & 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3. Find the inverses of the matrices $\begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix}$ and

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \end{bmatrix}$$

Is there any restriction on a in either case?

4. Given $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and

$z = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Which of u, v, w, x, y, and z are eigenvectors of A, and what is the eigenvalue in each case?

5. Find all the eigenvalues and eigenvectors for all the identity matrices I.

6. Run `matlab` on the three examples of Lecture 12: to solve $Ax = b$, set $x = A\backslash b$. *Warning: We know ahead of time that this must fail in at least two of the three cases. Discuss the results.*

7. Write down a matrix which stretches smiley faces to 4 times their original height. Find all the eigenvectors and eigenvalues of this matrix.

8.   *What's rong with this?* $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} x = b$, $x = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{8} \end{bmatrix} b$?!?

Today:

Eigenvectors and eigenvalues

We have seen the eigenvector concept in Lecture 13: a non-zero vector x is an eigenvector for a matrix A with eigenvalue λ if $Ax = \lambda x$, and we have seen some examples. In this lecture we will outline some methods for finding eigenvectors. For our purposes in this course, we need to know the 2 by 2 case very well, the 3 by 3 somewhat, and the larger sizes just a little. We begin with a 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Consider the following statements.

1. A does not have an inverse.
2. $det\, A = 0$.
3. One row of A is a multiple of the other.
4. 0 is an eigenvalue of A, i.e. there is a non-zero x with $Ax = 0$.
5. Solutions to $Ax = b$, if they exist for a particular b, are not unique.

These statements are all equivalent, in that if one of them is true of a particular matrix A, then they all are true of A. We discussed the equivalence of 1 and 2 in Lecture 13. We won't prove all the equivalences, but the implication (4 implies 1) is the most interesting for our purpose, so let's think about that one: assume that there is a non-zero vector x such that $Ax = 0$. Now the claim is that A cannot have an inverse under this circumstance. The proof is very easy. Suppose A^{-1} exists. Multiply the equation $Ax = 0$ by A^{-1} to get $A^{-1}Ax = A^{-1}0$ or $x = 0$. This contradicts the fact that x is non-zero. Therefore A^{-1} does not exist.

Now we pull ourselves up by our own bootstraps, using these facts about eigenvalue 0 to help us find all eigenvalues. So suppose now that B is any 2 by 2 matrix and we want to solve

$$Bx = \lambda x$$

Note first that this may be rewritten as

$$(B - \lambda I)x = 0$$

where I is the identity matrix. Now we just apply the previous discussion, taking A to be $(B - \lambda I)$. The conclusion is that

$$\det(B - \lambda I) = 0$$

This is our equation for λ. You can see, or will see, that it is a quadratic equation. Once it is solved, we solve the previous one for x.

example

$$B = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

$\det(B - \lambda I) = \det(\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 1 - \lambda & 2 \\ 0 & -3 - \lambda \end{bmatrix}) = (1-\lambda)(-3-\lambda) - 0 = 0$.

This is already factored so we find $\lambda = 1$ or $\lambda = -3$. These must be the eigenvalues of B. The next step is to find the eigenvectors. Take the case $\lambda = 1$. We must solve $(B - 1I)x = 0$. Explicitly this says

$$\begin{bmatrix} 0 & 2 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These require $x_2 = 0$, and note that is the only requirement. So you can take $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ where x_1 is anything other than 0. Usually we take the simplest possible thing, which is

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We have found, tentatively,

$$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and you should check this. Then do the case $\lambda = -3$: We must solve $(B + 3I)y = 0$ or

$$\begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These require $y_2 = -2y_1$. So you can make a simple choice $y = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. We have found

$$B \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Again, you must check this. Does it work when you multiply it out? Ok. Note that we took simple cases, but that there are also other choices of eigenvectors which are multiples of these, such as $\begin{bmatrix} 5 \\ -10 \end{bmatrix}$ and $\begin{bmatrix} -3.172 \\ 6.344 \end{bmatrix}$. Check these too if you're not sure.

Note that you can make a picture of the plane showing all the eigenvectors of B, and you will find that they lie on the two lines $y = 0$ and $y = -2x$, excluding 0. These are called the eigenspaces for the eigenvalues 1 and –3, respectively, except that 0 is included in the eigenspaces even though it is not an eigenvector. Everything on the x axis remains unmoved by B, while everything on the line $y = -2x$ gets stretched by 3 and flipped to the other side of the origin. Note that having all this eigenvector information gives us a very complete understanding of what this matrix does to the plane:

For Practice: What would happen to a smiley face under the action of B?

sample

$$C = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$$

We get $\det(C - \lambda I) = \lambda^2 + 9$. This is zero only if $\lambda = \pm 3i$. For the case $\lambda = 3i$ we solve $(C - 3iI)x = 0$ which reads

$$\begin{bmatrix} -3i & 1 \\ -9 & -3i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

So $x_2 = 3ix_1$. We take $x = \begin{bmatrix} 1 \\ 3i \end{bmatrix}$. Then check

$$C \begin{bmatrix} 1 \\ 3i \end{bmatrix} = 3i \begin{bmatrix} 1 \\ 3i \end{bmatrix}$$

For the case $\lambda = -3i$, the only change is to replace i by $-i$ everywhere, and check that
$$C \begin{bmatrix} 1 \\ -3i \end{bmatrix} = -3i \begin{bmatrix} 1 \\ -3i \end{bmatrix}$$

Note that for C, one has run into complex numbers, and so the nice geometric interpretation which was available in the plane for B is no longer available. Two complex

numbers are about the same as four real numbers, so you might have to imagine something in four dimensions in order to visualize the action of C in the same sense as we were able to visualize B. It does make sense, nevertheless, that not all 2 by 2 matrices can be thought of as stretching vectors in the plane: figure out what what C does to smiley faces, if you want to understand this point better.

larger matrices

The procedure for 3 by 3 and larger matrices is similar to what we just did for 2 by 2's. The main thing which needs to be filled in is the definition of determinant for these larger matrices. We really do not want to get involved with big determinants in this class. To find eigenvalues for a 3 by 3 matrix you can look up determinants in your calculus book, or remember them from high school algebra, but our emphasis will not be on hand computation. There is also a possibility of using some routines which are built into `matlab`. These work as follows. Suppose we have a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To approximate the eigenvalues of A you can type the following commands:

```
A=[1 2 3;4 5 6;7 8 9];
eig(A)
```

To get eigenvector approximations also you may type

```
[V D]=eig(A)
```

This creates a matrix V whose columns are approximate eigenvectors of A, and a matrix D which is a "diagonal" matrix having approximate eigenvalues along the main diagonal. For example, D in this case works out to be

$$D = \begin{bmatrix} 16.1168 & 0 & 0 \\ 0 & -1.1168 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problems ——————————————————————————— **14**

1. Sketch a smiley face centered at the origin and its image under the matrix B of the text.
2. Repeat problem 1 for the matrix C of the text.
3. The last example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

seems to have 0 as an eigenvalue. Find a corresponding eigenvector.

4. Find eigenvalues and eigenvectors for the matrix $X = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$.

5. Find eigenvalues and eigenvectors for the matrix $W = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$.

6. Which of the vectors $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$ are eigenvectors for the matrix

$$Z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and what are the eigenvalues?

7. Which of the vectors in problem 6 are eigenvectors for I, the 4 by 4 identity matrix and what are the eigenvalues?

8. Which of the vectors in problem 6 are eigenvectors for $7I$ and what are the eigenvalues?

9. Which of the vectors in problem 6 are eigenvectors for the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and what are the eigenvalues?

10. If you don't know how to do the determinant of a 3 by 3 matrix, look it up somewhere, and find the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

11.   *What's rong with this?* $A = \begin{bmatrix} 3 & 0 \\ 5 & 7 \end{bmatrix}$, $\det(A - \lambda I) = \det(\begin{bmatrix} 3 - \lambda & 0 \\ 5 & 7 - \lambda \end{bmatrix}) = (3 - \lambda)(7 - \lambda) = 21 - 7\lambda - 3\lambda + \lambda^2 = \lambda^2 - 10\lambda + 21$, so use the quadratic formula … ?!?

Today:

Just when you thought it was safe! back to linear differential equations. The solution by eigenvectors.

At this point you have seen those essential parts of linear algebra which are needed for solving linear systems of differential equations of the form

$$x' = Ax$$

Here, A may be an n by n matrix and x an n by 1 column vector function of t. As in any differential equation system $x' = f(x)$, the right side represents a vector field which exists in space, and the mission of any curve $x(t)$ which intends to solve the system is to be sure that it's tangent velocity vector $x'(t)$ agrees at all times with the given vector field. In this lecture we deal only with vector fields of the form $f(x) = Ax$, and these may be solved rather completely using our new eigenvector knowledge.

For example, we know that sometimes you find an eigenvector v for which Av is a positive multiple of v. In such a case, the vector field points away from the origin and so any solution passing through point v must be headed away from the origin. Similarly negative eigenvalues give rise to solutions headed for the origin. The effects of complex eigenvalues are a little harder to guess. So let's proceed with the analysis in the following way. Suppose by analogy with many things which have gone before, that we are looking for a solution of the form $x(t) = ce^{rt}$, where r is a number as usual, but now c must be a *vector* for the assumption to make any sense. Then substitute into the differential equation to get

$$x' = rce^{rt} = Ax = Ace^{rt}$$

These will agree if

$$Ac = rc$$

Now stop and take a deep breath, and think about what you learned before about linear algebra. This equation is the same except for notation as

$$Av = \lambda v$$

isn't it? We have reached the following fact:

If $Av = \lambda v$, then one solution to $x' = Ax$ is $ve^{\lambda t}$.

sample

From the example in Lecture 14, $\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ So one solution to

$\begin{cases} x' = x + 2 \\ y' = -3y \end{cases}$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{1t}$ i.e., $x = e^t$, $y = 0$. Check these in the differential equation to be sure.

Note that we certainly don't stop with one solution. For each eigenvector you get a solution.

sample

Also for the example on page 50 we found that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector with eigenvalue –3. Consequently we have another solution $\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$, or $x = e^{-3t}$, $y = -2e^{-3t}$. Check these too.

In general we hope for a system $x' = Ax$ where A is n by n, that there will be n such solutions. Usually there are, but not always. More on this in a moment. The most important thing now is to point out how you can combine these solutions which we have found. Our equation $x' = Ax$ is linear in the strictest sense of the word: linear combinations of solutions are solutions, i.e., if $x'_1 = Ax_1$ and $x'_2 = Ax_2$, then for any constants c_1 and c_2 it is true that $(c_1 x_1 + c_2 x_2)' = A(c_1 x_1 + c_2 x_2)$. This is because differentiation and multiplication by A are both linear operations. You should write out the one-line proof of this yourself if you are not sure about it.

sample

$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$ is a solution to $x' = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} x$ for any choice

of constants c_1 and c_2. In order to meet a given initial condition, we have to solve

for the c_j. For example if we want to have $x(0) = \begin{bmatrix} 2.5 \\ -2 \end{bmatrix}$ then it is necessary to

solve $\begin{bmatrix} 2.5 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. The answer is $c_2 = 1$, $c_1 = 1.5$ and we get the

solution $x(t) = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + 1.5 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$.

Back to the question of finding n solutions for the case of n by n matrices A. Certainly if we find n eigenvectors v_j we can form solutions like

$$x(t) = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$$

The question is whether these are *all* of the solutions to the differential equation. Equivalently, is it possible to meet all the possible initial conditions using

$$x(0) = c_1 v_1 + \cdots + c_n v_n?$$

Note that this is a system of linear algebraic equations in the unknown c_j, like we saw in the previous sample. What is needed here is that the v_j should be linearly independent. This concept is slightly beyond the level of these notes, but basically it means that the vectors really point in n different directions, or that none of them may be written as a linear combination of the other ones. You will study this concept in a linear algebra course; we're only doing an introduction to it here.

sample

Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$. This was problem 10 of Lecture 14. We have $Av_1 = 3v_1$

and $Av_2 = 0$ where $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and there are no other eigenvectors

except for multiples of these. So some solutions to $x' = Ax$ are $x = c_1 v_1 e^{3t} + c_2 v_2$. This is one of the exceptional cases where you do not get $n(=3)$ eigenvectors, nor the general solution to the differential equation. See if you can find another form of solution to these differential equations; for this system it is actually easier to write the equations out and deal with them explicitly, rather than to think about eigenvectors.

1. Solve

$$\begin{cases} x' = 2x \\ y' = 3y \\ x(0) = 5 \\ y(0) = 0 \end{cases}$$

2. Solve the system of problem 1 with the initial conditions

$$\begin{cases} x(0) = 0 \\ y(0) = 5 \end{cases}$$

3. Sketch the phase plane for the system of problem 1.
4. Solve the system of problem 1 with initial conditions

$$\begin{cases} x(0) = 1 \\ y(0) = 1 \end{cases}$$

Sketch your solution onto the phase plane, or run `pplane` on it. Show by eliminating t between your formulas for $x(t)$ and $y(t)$ that $x = y^{\frac{2}{3}}$. This should help your sketch.

5. Solve

$$\begin{cases} x' = 2x \\ y' = -3y \end{cases}$$

Show that solutions lie on curves $x = cy^{-\frac{2}{3}}$, or lie along the positive or negative axes, or are zero. Sketch the phase plane.

6. Sketch the phase plane for the system

$$x' = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} x$$

which was solved in the text. Does it look more like the phase plane for problem 3 or problem 5, qualitatively?

7. Solve

$$\begin{cases} x' = x + 2y \\ y' = 4x - y \end{cases}$$

and sketch the phase plane.

8. Solve

$$\begin{cases} x' = x - y \\ y' = x + 5y \end{cases}$$

and sketch the phase plane.

9. You are given that a matrix A has eigenvectors $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$, and that $Au = 5u$, $Av = 4v$, and $Aw = 3w$. Solve $x' = Ax$, if $x(0) = \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}$. Note that $x(0) = u + 2v$.

10. *What's rong with this?* $\begin{cases} x' = Ax \\ x(0) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{cases}$, and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -14 \end{bmatrix}$ is given. So $x = 3e^{-7t} + C$.

?!?

Today:

A complex example

Today's lecture consists entirely of one example having complex eigenvalues. It is not really *that* hard, but we want to be careful and complete, the first time we work through one of these. Our system is

$$x' = Ax$$

where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

We know what has to be done to solve this; you find eigenvalues then eigenvectors of A, then the solution $x(t)$ is in the form of linear combinations of the $ve^{\lambda t}$ things. This is what we have done before, and here we will find a bit of complex arithmetic to go along with it. Here we go.

First plot the phase plane to see what to expect. You can plot a little bit of it by hand, or you can run `pplane` on our system:

$$\begin{cases} y' = y + 2z \\ z' = -2y + z \end{cases}$$
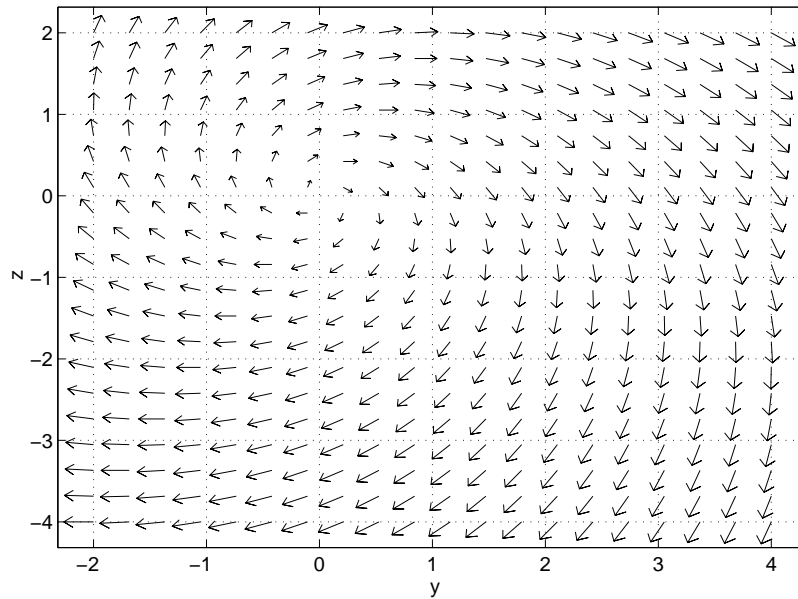
Figure 16.1 The phase plane for our example. (figure made by `pplane`*)*

Now let the analysis begin. First we find the eigenvalues of A. We have

$$\det(A - \lambda I) = \det(\begin{bmatrix} 1 - \lambda & 2 \\ -2 & 1 - \lambda \end{bmatrix}) = \lambda^2 - 2\lambda + 5$$

61

This is zero by the quadratic formula if $\lambda = (2 \pm \sqrt{4 - 4(5)})/2 = (2 \pm \sqrt{-16})/2 = 1 \pm 2i$. So we have our eigenvalues.

Now let's look for an eigenvector for $1+2i$. We must solve $(A-(1+2i)I)v = 0$. Writing a and b for the components of v, this says

$$\begin{cases} -2ia + 2b = 0 \\ -2a - 2ib = 0 \end{cases}$$

The first equation says $b = ia$. The simplest choice we can make is to take $a = 1$, $b = i$. Now something strange happens. If you check these numbers in the second equation, you will find that they work there too! In fact, you can just forget about the second equation, because it is a multiple of the first one. I *know* it might not look like a multiple of the first one, but that is because it is an imaginery multiple of the first one. Anyway we have found that

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = (1 + 2i) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Read that through again to check it, because there are lots of places you can make mistakes in complex arithmetic.

Next let us state the solution we have found so far, and expand it so we can see what it looks like.

$$\begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} = e^t (\cos(2t) + i \sin(2t)) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + i e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

That's a mess, isn't it? It is neater to leave it in the original complex exponential form, but we expanded it out so you can see exactly what the real and imaginary parts are. Now we need a second solution, so we can go through all this again with the other eigenvalue $1 - 2i$. The only thing which changes, if you read through what we just did, is that you must replace i by $-i$ wherever it occurs. This happens to work because A is a real matrix. The general solution to the system can then be expressed as

$$x(t) = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1-2i)t}$$

There is an alternate to this, sometimes preferred because it emphasises real solutions. This goes back to our discussion in Lecture 7 about real and imaginary parts of solutions to real equations. The same statement holds here, that if $x = u + vi$ is a solution to $x' = Ax$ where u, v, and A are real, then u and v are also solutions. You can prove this yourself pretty easily. Taking the real and imaginary parts we found for the first solution above allows us to express solutions in the alternate form

$$x(t) = a_1 e^t \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + a_2 e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

The relations between the constants are $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$.

It is important to think about the phase plane and see whether it matches the solutions we found. You can see that we got the increasing exponential e^t, and this has to do with the fact that solutions are moving away from the origin in Figure 16.1. Also we got sines and cosines, which oscillate positive and negative. That has to do with the rotational motion in the picture.

Problems ── **16**

1. Solve $x' = Ax$ if $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$. Find the solution if $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. Solve $x' = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} x$. What is the second order equation for the first component x_1 in this case?

3. Write out the system corresponding to $x'' = -x$ using $x' = y$ and solve it, find a conservation law for the equation and plot its level curves, and plot the phase plane for the system. How are these plots related?

4. Solve
$$\begin{cases} x' = -3x \\ y' = z \\ z' = -9y \end{cases}$$

and plot some solutions.

5. *What's rong with this?* $x = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^t(\cos(2t) + i\sin(2t))$, so re$(x) = e^t \cos(2t)$, im$(x) = e^t \sin(2t)$, and $x = e^t(c_1 \cos(2t) + c_2 \sin(2t))$. ?!?

Today:

> *Classification of linear planar systems. Spirals, centers, saddles, nodes, and other things more rare.*

Today we have a nice treat for you. Rather than doing a lot of computation, which we have been doing for several lectures, we are going to state a classification theorem. If this sounds like when the doctor says "It won't hurt at all", that's because we first have to explain the significance of what we are about to do. A classification theorem is one step higher in the world than computation, because it is something which lists all the possible things which could ever happen, in a given circumstance. Having such a theorem is considered by mathematicians to be a good thing. There are not very many classification theorems, and mathematicians wish they could find more. Here is what this one says.

> **Classification of Linear Systems in the Plane** Suppose you have a system $x' = Ax$ where A is a 2 by 2 real matrix. Draw the phase plane. Then the picture you just drew has to be essentially one of the seven pictures shown below, and there are no other possibilities. *Essentially* means that you might have to rotate or stretch your picture a little to make it fit or reverse the arrows, but there will be no more difference than that.

There now, that was not so bad, was it? Here are the pictures. The most important of these should look familiar from homework or examples in Lectures 15 and 16. The classification proceeds by considering what the various possibilities are for the eigenvalues of A.
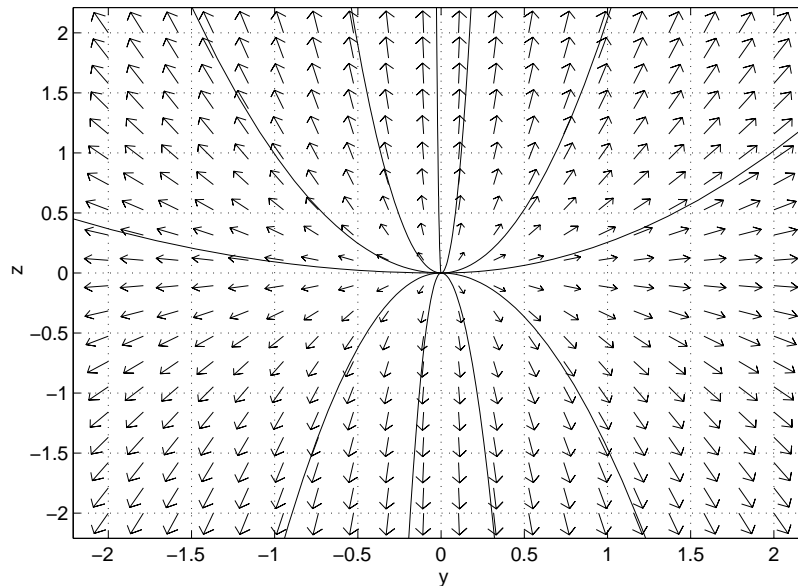
Figure 17.1 This is called a "node". It is what happens when the eigenvalues are both real numbers, and have the same sign. The case shown has both eigenvalues positive. For both negative, the arrows would be reversed. Important: solutions do not go through the origin.
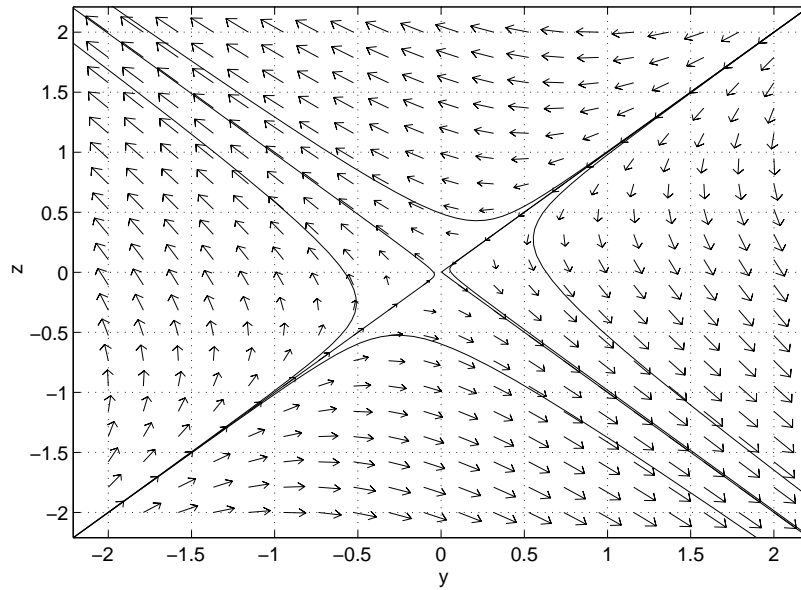
Figure 17.2 This is called a "saddle", because it is the pattern of rain running off a saddle, as seen from above. It occurs when the eigenvalues are real, but of opposite sign.
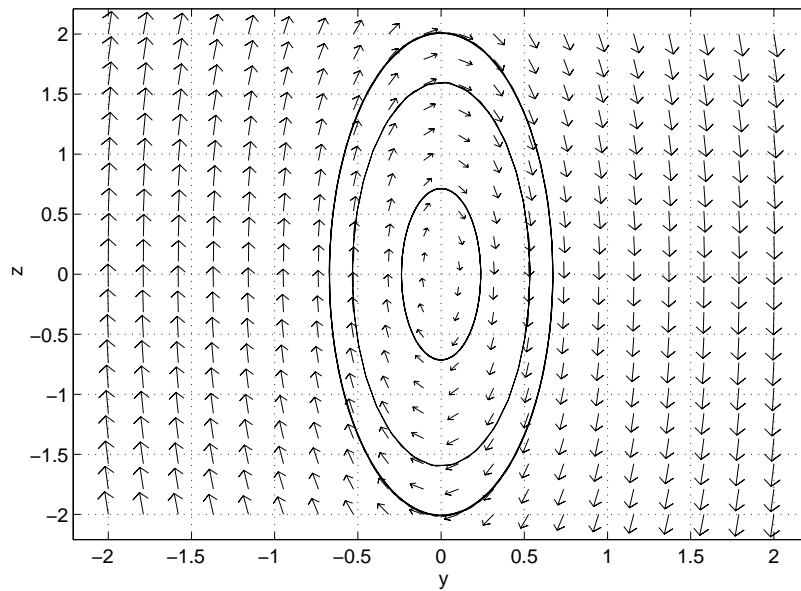

Figure 17.3 This is called a "center". It occurs when the eigenvalues are $\pm bi$, for some non-zero real number b.
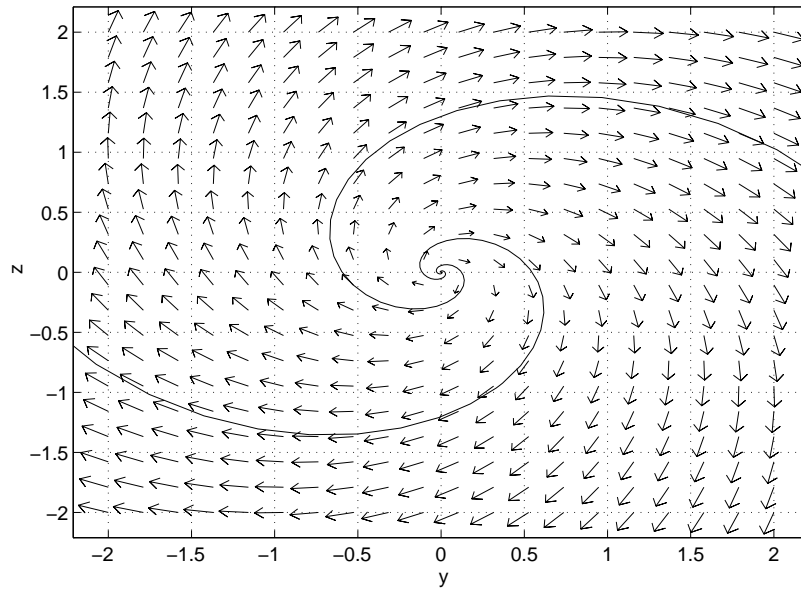
65

Figure 17.4 This is a "spiral", for obvious reasons. The eigenvalues are $a \pm bi$ with $a \neq 0$. The case $a > 0$ is shown, but if $a < 0$ the arrows are reversed. Also this can appear clockwise or counterclockwise.
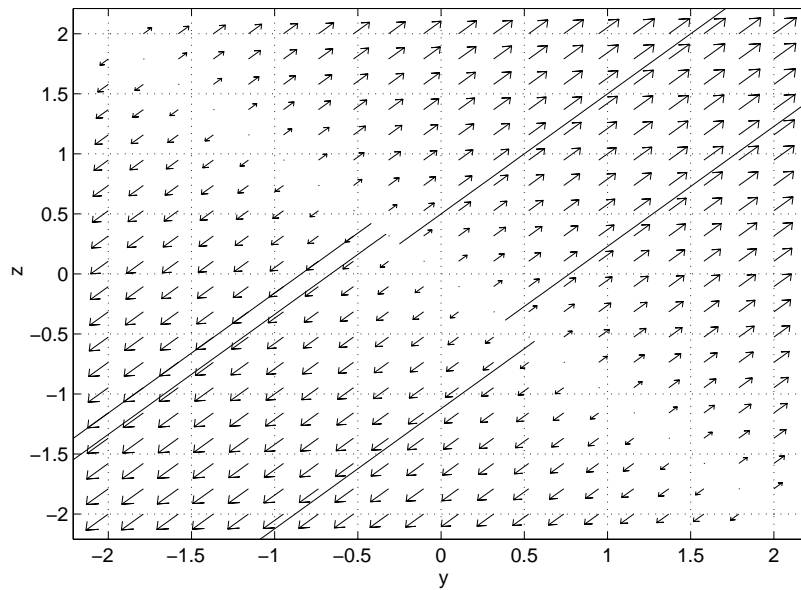
Figure 17.5 A rare case in which one eigenvalue is 0.
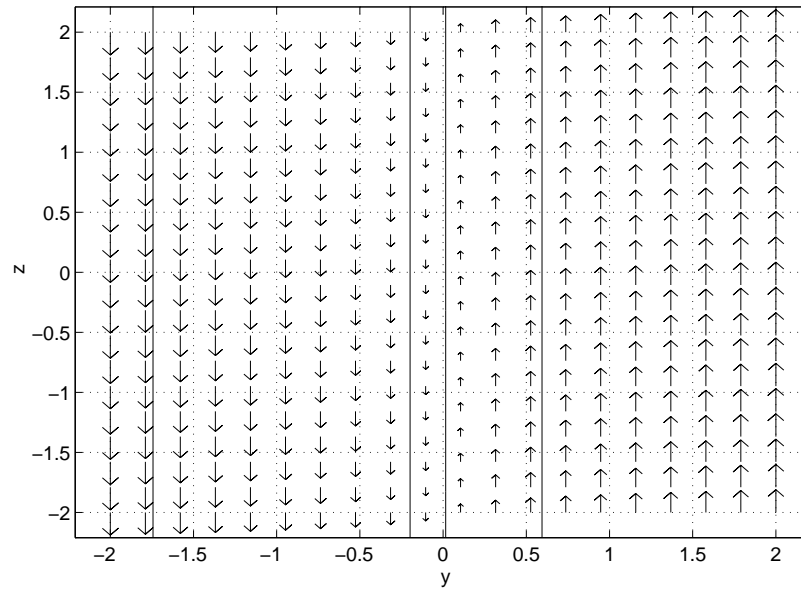
Figure 17.6 A rarer case in which both eigenvalues are 0.
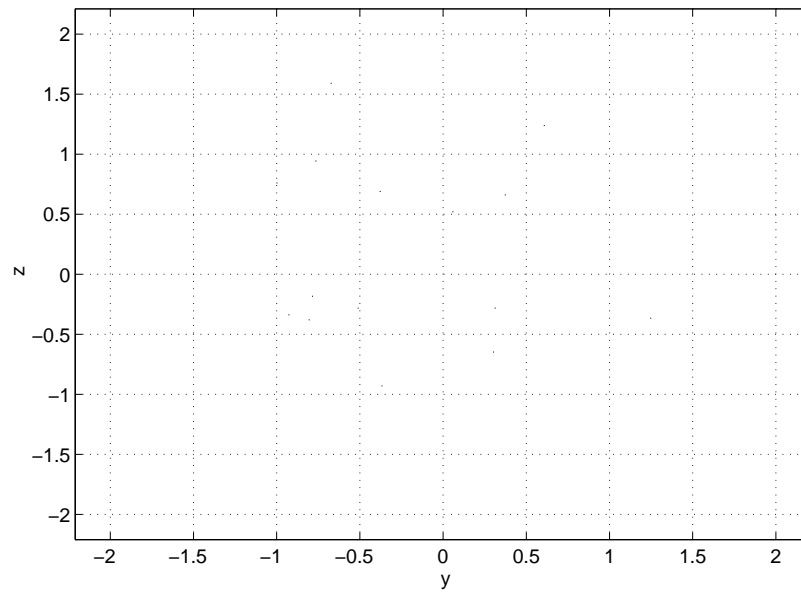
Figure 17.7 The rarest case, in which the entire matrix is 0.

Problems _____ **17**

1. If you did problems 1, 2, and 3 of Lecture 16, determine which of our phase planes in this lecture applies to each of them.

2. Make up your own system $x' = ax + by$, $y' = cx + dy$, by choosing the coefficients at random. Draw the phase plane for your system by hand or `pplane` and classify which type it is by trying to match it with one of the figures in this lecture. The theorem says that it should match.

3. Is there any other classification theorem that you can think of, anywhere in mathematics?

4. Why are there just dots in Figure 17.7, instead of curves?

5. Suppose you're solving a system $x' = Ax$ and find that the eigenvalues are $4.2 \pm 3i$. Which phase plane applies? Is the motion toward or away from the origin? How can you determine whether the motion is clockwise or counterclockwise just from the differential equation itself?

6. *What's rong with this?* Figure 9.2 and Figure 18.1 don't match any of the 7 types listed in our classification theorem. So there must be a mistake in the theorem. ?!?

Today:

  Nonlinear systems in the plane, including things which cannot happen in a linear system. Critical points. Limit cycles.

We now know about all the possible phase planes for systems of two linear equations. The next step is to see how this helps us analyze more realistic systems. There are two new ideas which enter into this study, and the first one is well illustrated by an example we looked at previously, the predator- prey system of Lecture 10. We will see that the phase portrait for the predator-prey system looks as though it has been made by selecting two of the linear phase planes from Lecture 17 and pasting them together in a certain way. In this sense, the linear systems prepare us pretty well for some of the nonlinear systems. Then we will show a second example containing entirely new behavior which can't happen at all in a linear system. So from this perspective it seems that perhaps the linear systems don't prepare us well enough at all for some of the nonlinear ones.

  We begin with the predator-prey system

$$\begin{cases} x' = x - xy \\ y' = -y + xy \end{cases}$$

which we have discussed on page 38. The phase plane for this system is shown in the figure. You may recognize some of it's elements immediately if you have done the homework problems for Lecture 17 already. It shows a cycling between the sizes of the mouse and wolf populations.
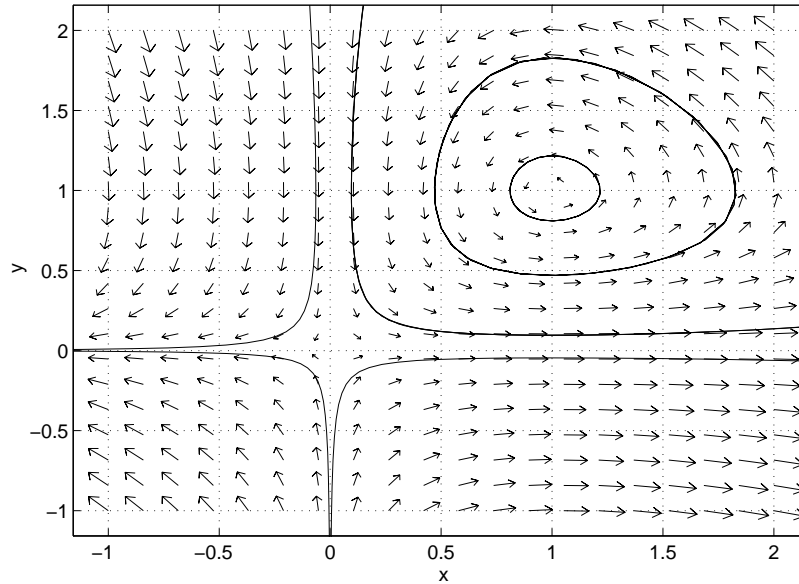

Fig. 18.1 The predator-prey system. Of course you have already noticed the part which looks like a center, and the part which looks like a saddle.

  The points about which the action seems to be centered are known as critical points, and you may notice in Figure 18.1 that these are located at (0,0) and at (1,1). There is an easy way to find the critical points for any system, and that is to notice that they are just

69

the *constant* solutions to the equations. In other words, all you have to do is set $x' = 0$ and $y' = 0$ and solve the resulting equations.

Next we modify the system to include a little bit of emigration, thinking that perhaps the mice have discovered the inadvisability of living in this neighborhood. The system is now

$$\begin{cases} x' = x - xy - .2 \\ y' = -y + xy \end{cases}$$

Observe the interesting effect this emigration has on the phase plane. What do you think it means, when that solution curve spirals outward for a while and then runs into one of the coordinate axes? It is not usually significant when a curve crosses an axis, but here we are talking about populations. When a population reaches zero, that *is* significant. The result of the emigration can therefore be extinction, or at least absence.
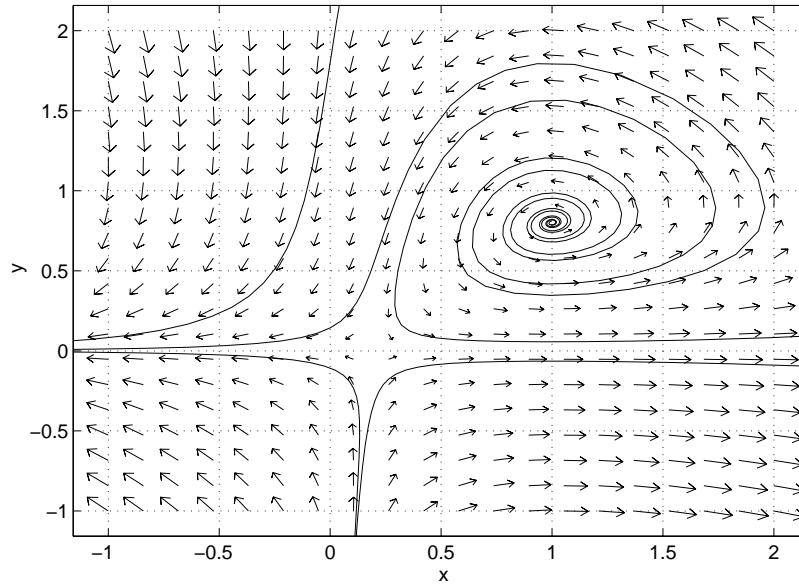
Figure 18.2 The predator-prey system with emigration of the prey. Note the part which looks like a saddle and the part which looks like a spiral.

In both of these situations we have shown negative values of x and y. This is because the system exists independent of its interpretation as a population model, and we wanted to look at the whole phase plane. If your interest is in the population aspects though, you don't care about the negative values, and the conclusion is that as soon as a solution curve touches an axis, the model no longer applies.

Our second main example is a classic known as the van der Pol equation. It is important for several reasons, partly historical and partly because it shows behavior which is impossible in a linear equation. The historical aspect is that this was originally a model of an electrical circuit, whose understanding was important in radio communications. The equation is

$$x'' + (-1 + x^2)x' + x = 0$$

You should notice that it looks just like our second order equations from Lecture 6, except that the coefficient of x' is not a constant. So our characteristic equation method will do no

good here. It converts to the system

$$\begin{cases} x' = y \\ y' = (1 - x^2)y - x \end{cases}$$

which is not of the form $z' = Az$, so our eigenvector methods will also do no good. So we let `pplane` make a picture for us to study.
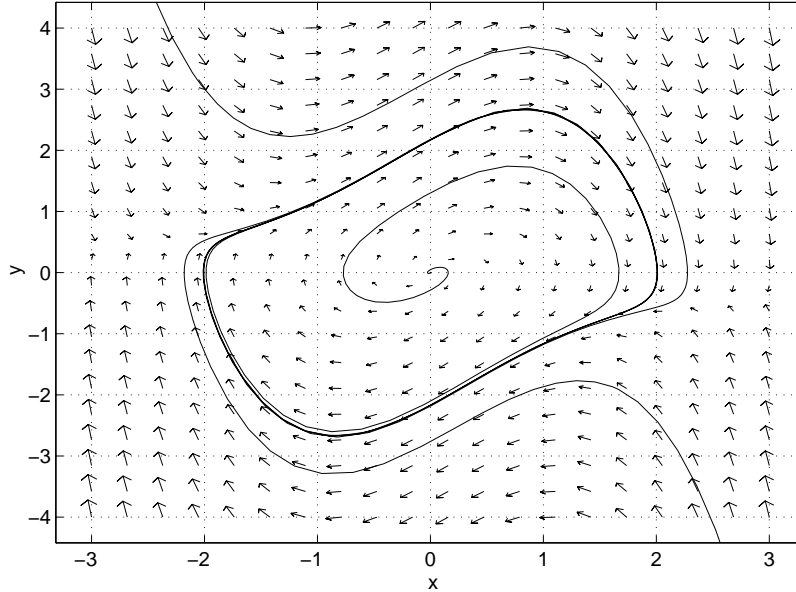

Figure 18.3 The van der Pol system contains a limit cycle. This is completely new, significant behavior not possible in a linear system.

Notice what is happening in this phase plane. There are solutions spiraling outward from the origin. That certainly can happen in a linear system as well. There are solutions coming inward also. But what is new is what happens between these. There seems to be a solution which is periodic and which the others approach. In the radio circuit what would that mean? It must mean that there is a voltage or current which cycles back and forth repeatedly. Now you recognize that this kind of thing happens when you have a center in a linear system too, but there is a difference here. Here there is only one periodic solution. This means that this radio is going to settle down near this one special solution *no matter what* the initial conditions are. That is very important. Such behavior deserves a new name; it is called a stable limit cycle. If you run this machine in your laboratory, the only thing you will observe is the limit cycle.

Problems ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯ **18**

1. Run `pplane` on the system

$$\begin{cases} x' = \sin(x + y) \\ y' = \sin(xy) \end{cases}$$

How many saddles and spirals can you find, visually?

2. Consider the system

$$\begin{cases} r' = r - r^2 \\ \theta' = 1 \end{cases}$$

The first one is the logistic equation. Look back at Lecture 2 to remind yourself that all positive solutions approach 1. What do the solutions to this system do then, if you take (r, θ) as polar coordinates in the plane? Convert the system to cartesian coordinates and run `pplane` on it. Did you find a limit cycle?

3. Study a predator-prey system in which the predators are moving out. What happens? This can be used as a model of a fishing industry, in which the predators and prey are two fish species, and "move out" means get caught. Are the results reasonable or unexpected? If the plan is to harvest the sharks so that the tuna population will increase, what would your advice be?

4. Run `pplane` on the system

$$\begin{cases} x' = \cos(x + y) \\ y' = \cos(xy) \end{cases}$$

How many limit cycles can you find, visually?

5. Find all critical points for the system

$$\begin{cases} x' = y(1 - x^2) \\ y' = x + y \end{cases}$$

Sketch the phase plane and try to figure out what type of linear behavior occurs near each of them.

6. Find all the critical points of the van der Pol system.

7. Try to sketch a phase plane which contains two saddles. You are not asked for any formulas, but just to think about what such a thing could look like. Remember the uniqueness theorem, that solutions cannot run into each other.

8. Try to sketch a phase plane containing two limit cycles.

9. *What's rong with this?* Jane's Candy Factory used the system

$$\begin{cases} x' = 10 - .25x \\ y' = 6 + .25x - .2y \end{cases}$$

The critical points are the constant solutions, so we solve $0 = 10 - .25x$ getting $x = 40$, and $0 = 6 + .25x - .2y$ getting $y = 32$. ?!?

Today:

Nonlinear systems in space. Chaos. Why we can't predict the weather.

We have seen lots of linear systems now, and we have also seen that these provide a background for some, but only some, things which can happen in nonlinear equations, in two dimensions. Today we move to three dimensions. We will see that nothing you learned about two dimensions can prepare you for three dimensions! There are aspects of systems of three equations which can be understood using the prior knowledge, but there are also things called "chaos", which pretty well suggests how different and intractable they are. This lecture is only the briefest kind of introduction to this subject, but is here because it is important for you to know about these wonders.

We begin with an example in which your knowledge of plane linear systems does help you understand a 3D system. Consider the system

$$\begin{cases} x' = -x \\ y' = z \\ z' = -y \end{cases}$$

These equations may be read as follows. The equation for x does not involve y or z, and vice versa. In fact the x equation is a simple equation for exponential decay. The y and z equations are a system corresponding to our favorite second order equation $y'' = -y$ whose solutions are sines and cosines. We are therefore dealing with a center in the (y, z) plane. Consequently we can think of this system as having a circular motion in the (y, z) plane combined with an exponential decay in the x direction. So the solutions are curves in 3-space moving around circular cylinders centered along the x axis, and which approach the (y, z) plane as time goes by. Any solution already in that plane remains there, circling.

In contrast with that system let us look at the Lorentz system

$$\begin{cases} x' = 10(y - x) \\ y' = -xz + 28x - y \\ z' = xy - \frac{8}{3}z \end{cases}$$

This very interesting system was made by Edward Lorentz who is a meteorologist at MIT, as a very simplified model of circulation in the atmosphere. Note that I said *is*; you are studying modern mathematics now! The system is nonlinear because of the xz and xy terms, and is derived from much more complicated equations used in fluid mechanics. You may be interested to know that the weather predictions we all hear on the radio and TV are to some extent derived from calculations based on observational data and the real fluid mechanics equations. Here is a phase portrait for the Lorentz system.
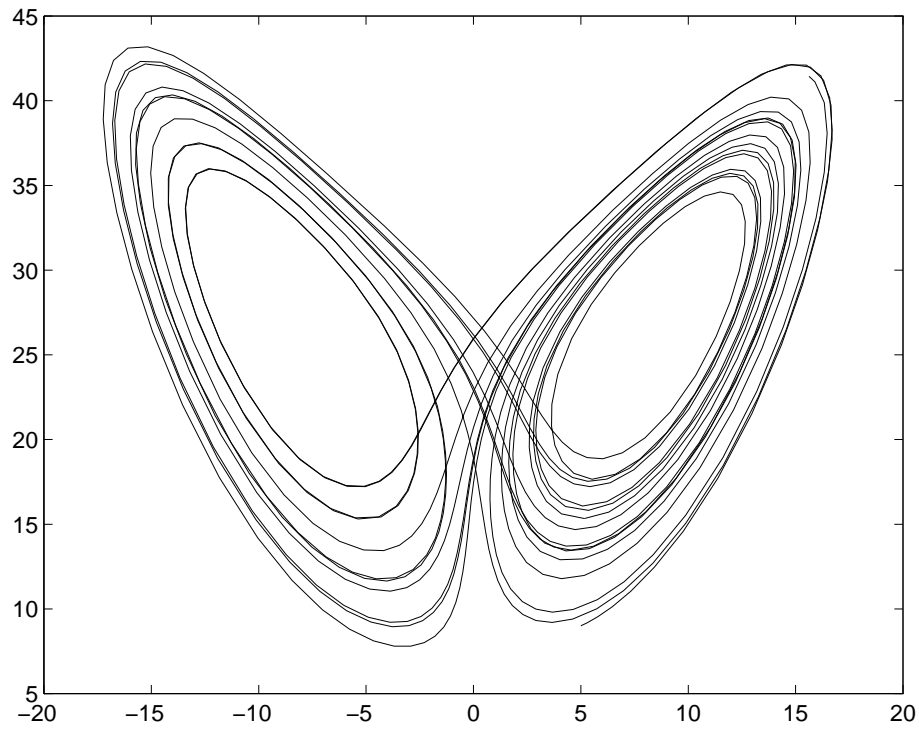
Figure 19.1 The Lorentz system. (figure made using the Matlab commands shown in problem 2)

You may notice that there seem to be two important critical points about which the solutions circulate. One finds that the solution turns a few times around the left one, then moves over to the right one for a while, then back, and so on. The only problem is that it is not easy to predict how many times it will circle each side before moving to the other! Lorentz came upon this fact as follows. He was computing this on whatever kind of computers they had in 1963, and noticed the pattern of random–looking jumps from one side to the other. These were very interesting solutions, so he reran the computation from a point part way along, to study it in more detail. In the process, he typed in as initial conditions the numbers which had been printed out by his program. These numbers were slightly rounded by comparison with the precision to which they had been computed. The result made history: the qualitative features were still there, but the solution in detail was very different, in that after a few turns the numbers of cycles around the left and right sides came out different, just because of rounding off a few decimal places in the initial conditions. That means that if you try to make a picture like the one above, it will look qualitatively the same, but will be different in detail. It also means something about weather prediction. The phrase is "If a butterfly flaps its wings in China, it can later affect the weather in the US." This is a restatement of the effect of rounding off, i.e. slightly changing, the initial conditions. Everybody has known for years that weather prediction is an art, that you really can't predict the weather very well. Part of what Lorentz did was to explain *why* you can't predict the weather: you can never know enough about the initial conditions no matter how many weather stations you build, when the equations of motion are so sensitive to initial conditions.

We also have an enormous shift in the way the universe is conceived, as a result of studying this and other chaotic systems. On the one hand there is the uniqueness theorem which says that the future is determined uniquely by the initial conditions. This is the Newtonian approach. On the other hand we now know about chaotic systems in which yes, solutions are uniquely determined by their initial conditions, but they may be so sensitive

to these conditions that as a practical matter we cannot use the predictions very far into the future. Isn't that interesting? It is a really major idea.

1. Check out the java applet `de` which is described in Lecture 3 and run some chaotic systems on it. Several are built in as examples.

2. Solve a 3D system in Matlab. You can't use `pplane` of course, but you can set up a function file to compute your vector field, like

```
% file ef.m
function vec=ef(t,w)
vec(1)=10*(w(2)-w(1));
vec(2)=-w(1)*w(3)+28*w(1)-w(2);
vec(3)=w(1)*w(2)-8*w(3)/3;
% end file ef.m
```

Then give commands like

```
[t, w] = ode45('ef',0,20,[5 7 9]);
plot(w(:,1),w(:,3))
```

This will compute the solution to the Lorentz system for $0 \leq t \leq 20$ with initial conditions $(5, 7, 9)$, and make a plot. The symbol w was used here for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, so what was plotted was x versus z. Naturally you can play with the plot command to get different views. The hardest part is understanding why $w(:,1)$ means x. You have to realize that w is a list of three lists, before it makes any sense. Change these files to solve a system of your choice. Note that you are not restricted to just three dimensions either.

3. Change the `plot` command in problem 2 to plot x versus t.

4. Change the `ef.m` file of problem 2 to solve the linear system described in the text. You will have to play with the `plot` command to get a nice perspective view, for example you can plot x versus $y + 2z$ or something like that. Does the picture fit the description given in the text?

5. _What's rong with this?_ According to this lecture, if you have a system of 3 or more variables you can get chaos. And, according to Lecture 10 on page 37, if you have a system of 2 spring-masses, you get 2 Newton's laws or a system of 4 first-order equations. Therefore 2 spring-masses are always chaotic. ?!?

Today:

> *A change from initial conditions. Boundary values. We find that we don't know everything about $y'' = -y$ after all.*

You have by now learned a lot about differential equations, or to be more specific, about initial value problems for ordinary differential equations. We have always had our initial conditions. At this time we are prepared to make a change, and consider a new kind of conditions called boundary conditions. These are interesting both on their own, and also in connection with some problems in partial differential equations.

First you have to know what a boundary is. It is nearly the same thing in mathematics as on a map: the boundary of a cube consists of its six faces, the boundary of Puerto Rico is its shore line, the boundary of the interval $[a, b]$ consists of the two points a and b. The concept of a boundary value problem is to require that some conditions hold at the boundary while a differential equation holds inside the set. Here is an example.

sample

$$\begin{cases} y'' = -y \\ y(0) = 0 \\ y(2\pi) = 0 \end{cases}$$

We are asked to solve a very familiar differential equation, but under very unfamiliar conditions. The function y is supposed to be 0 at 0 and π. The differential equation here has solutions $y = A\cos(t) + B\sin(t)$. We apply the first boundary condition, giving $y(0) = 0 = A$. So we must take $A = 0$. That was easy! Now apply the second boundary condition, giving $y(2\pi) = 0 = B\sin(2\pi)$. Well, it just so happens that the $\sin(2\pi)$ *is* zero. So the second boundary condition is fulfilled no matter what B is. Answer: $y(t) = B\sin(t)$, B arbitrary.

Notice how different this example was from our experience with initial conditions, that there were infinitely many solutions. Just the opposite thing can happen as well:

sample

$$\begin{cases} y'' = -y \\ y(0) = 0 \\ y(6) = 0 \end{cases}$$

This seems close to the previous example since only 2π has been changed to 6. This begins as before with $A = 0$. Then it just so happens that the $\sin(6)$ *is not* zero. So unlike the previous case, we have to set $B = 0$. Answer: $y(t) = 0$. There is no other possibility.

What is a physical interpretation of these problems? We know that the equation $y'' = -y$ describes an oscillator, a rock hanging from a Slinky, vibrating up and down at frequency $1/2\pi$. If you require the condition $y(0) = 0$, that just means you want the rock at the origin at time 0. That alone is not a real problem, because you can start the stopwatch when it is there. Since the frequency of oscillation is $1/2\pi$, the rock will be back at the origin after 2π seconds. If you then require the condition $y(2\pi) = 0$ then there is no problem because that happens automatically. But if you require instead that $y(6) = 0$, then you are asking for the impossible. The rock takes $2\pi = 6.283\ldots$ seconds to get back, period. The only exception is the special function 0, signifying that the rock never moved at all. In that case it will certainly be back in 6 seconds, since it is there already. That is why there is

no solution in the second case except for the 0 solution. A more down-to-earth example of a boundary value problem: "You can go out with your friends at 8:00 but you have to be home by 12:00."

There is a tendency to use x as the independent variable when one is discussing boundary value problems rather than t because usually in practise it is position rather than time, in which one is interested. We'll do that in the next example. There is also the concept of eigenvalue in connection with boundary value problems.

sample

$$\begin{cases} y'' = -cy & \text{Finding } c \text{ is part of the problem too.} \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

The "eigenvalue" here is c. We will assume we are looking for positive values of c to keep things simple, and a homework problem will be to analyze the cases in which c might be 0 or negative. To solve this equation $y'' = -cy$ we recognize that we are again looking at sines and cosines: $y(x) = A\cos(\sqrt{c}x) + B\sin(\sqrt{c}x)$. That is where our previous knowledge of differential equations comes in. If you have any doubts about these solutions, go no further! Go back to Lecture 6 and study second order linear equations a little more. You can't build a house on a foundation of sand. All set now? The boundary condition $y(0) = 0 = A\cos(0) = A$ determines $A = 0$ as before. Then we have to deal with the other boundary condition $y(\pi) = 0 = B\sin(\sqrt{c}\pi)$. Certainly if $B = 0$ this is satisfied, and we have found a solution $y(x) = 0$. But we have another parameter c to play with, so let's not give up too easily. Question: Is it possible that

$$\sin(\sqrt{c}\pi) = 0$$

for some values of c? If so, then we are golden because B can be anything and such c will be our eigenvalues. Well what do you know about where sine is zero? Certainly $\sin(0) = 0$ but if $c = 0$ then we are back at $y(x) = 0$ which is dull by now. Where else is the sine zero? The sine is zero at π, for example. That gives $\sqrt{c}\pi = \pi$, or $c = 1$. There is a good solution to our problem! $y(x) = B\sin(x)$, B arbitrary, $c = 1$. But there are other places where the sine is zero. For example $\sin(2\pi) = 0$. That gives $\sqrt{c}\pi = 2\pi$, or $c = 4$. So there is another solution $y(x) = B\sin(2x)$, B arbitrary, $c = 4$. Also $\sin(3\pi) = 0$, giving a solution $y(x) = B\sin(3x)$, B arbitrary, $c = 9$. In general we have solutions

$$\begin{cases} y(x) = B\sin(nx) & B \text{ arbitrary} \\ c = n^2 & n = 1, 2, 3, \ldots \end{cases}$$

So the eigenvalues for this problem are 1, 4, 9, 16, 25,....

significance of eigenvalues

Until now you have known eigenvalues as something connected with matrices. Suddenly the same word is being used in a different way, although in both cases there is (some operation)(something) equal to (the eigenvalue)(the same something). There is an abstraction here, that the equations $Ax = \lambda x$ and $y'' = -cy$ have something in common. We need to suggest that there is much more to eigenvalues than this. They are very important. You have gotten somehow through 18 years of living without knowing about them, but they have been all around you! For example, the color of light from your flourescent lights as you read this is determined by the frequency of the light, which is mainly a difference of atomic energy levels for the atoms in the light bulb. These energy levels are eigenvalues

of something, not of a matrix, but of Schrödinger's equation which was mentioned in Lecture 1. The music that you might be listening to as you read this comes in pitches which are eigenvalues of some equation describing the guitar strings' motion. The room you are sitting in may occasionally vibrate a little when somebody drops something heavy, and the frequency is an eigenvalue of something too. We will discuss just some of these things in this course, including the guitar strings but not the atoms nor the floor vibrations, but most of the mathematics is similar for all such things.

Problems ———————————————————————— **20**

1. Solve
$$\begin{cases} y'' = -25y \\ y(0) = 4 \\ y'(2\pi/5) = 7 \end{cases}$$

2. Find all solutions to
$$\begin{cases} y'' = -25y \\ y(0) = 4 \\ y(2\pi/5) = 7 \end{cases}$$

3. Solve
$$\begin{cases} y'' = -25y \\ y(0) = y(2\pi/5) \end{cases}$$

4. Solve
$$\begin{cases} y'' = -cy \\ y(0) = y(5) \end{cases}$$

Finding the possible values of c is part of the problem, but to keep things simple you may assume that $c \geq 0$.

5. Find all the solutions to the $c \leq 0$ cases of
$$\begin{cases} y'' = -cy \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

6. Find all the solutions to
$$\begin{cases} y'' = -cy \\ y(0) = 0 \\ y(3) = 0 \end{cases}$$

7. Find all the solutions to
$$\begin{cases} y'' = -cy \\ y(0) = 0 \\ y'(\pi/2) = 0 \end{cases}$$

8. Find all the solutions to
$$\begin{cases} y'' = -cy \\ y(0) = 0 \\ y'(3) = 0 \end{cases}$$

9. Find all the solutions to the $c \leq 0$ cases of
$$\begin{cases} y'' = -cy \\ y'(0) = 0 \\ y'(\pi) = 0 \end{cases}$$

10. Find all the solutions to
$$\begin{cases} y'' = -cy \\ y'(0) = 0 \\ y'(3) = 0 \end{cases}$$

Today:

A derivation of the heat equation from physical principles. Two, in fact.

Our survey of ordinary differential equations is now complete, and we are going to move to partial differential equations. These are equations involving a partial derivative of an unknown function of more than one variable. These are very important because they describe much of the world around us, but they are also somewhat different from ordinary differential equations. For example, you could make up an ordinary differential equation almost at random and find that many of the methods we learned will apply to it, and the software will solve it approximately and draw pictures of the approximate solutions. You *could* do that, although the motivation for doing so might not be clear. In the case of partial differential equations, almost *nobody* would just make one up, and the reason is that the equations people are already working on, which tend to be about things in the real world, are hard enough to solve as it is. There are no cut-and-dried software packages for partial differential equations either, since to some extent each problem needs its own methods. What happens is that rather than solving a lot of equations as we did with ordinary differential equations, we study just two or three partial differential equations with many different initial and boundary conditions. So partial differential equations are about real things and have names to reflect this fact. We begin with one called the heat equation.

Our first partial differential equation looks like

$$u_t = au_{xx}$$

and is the heat equation. It is an abbreviation for

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$$

The equation concerns the temperature $u(x,t)$ of a metal bar along which heat is conducted. You can imagine one end of the bar in the fire and the other in the blacksmith's hand, to emphasize that the temperature may change with time and with position along the bar. The number a is a physical constant which we will explain while we derive the heat equation.

Now let's read the heat equation carefully to see whether it makes any sense or not. Suppose you graph the temperature as a function of x, at a particular time. Maybe the graph is concave up as in Figure 21.1 on the left. What do you know from calculus about functions which are concave up? Isn't the second derivative positive? The heat equation contains the second derivative, and says that the more positive it is, the bigger the *time* derivative will be. What do you know from calculus about the first derivative being positive? Doesn't it mean that the function is increasing? So we have to imagine what that means, and conclude that the temperature must be rising at any point of the bar where the temperature graph is concave up. Similarly for the right side of Figure 21.1, if the temperature graph is concave down, the temperature must be decreasing with time. Does that seem correct? There is no hope whatsoever of understanding partial differential equations unless you think about things such as that.
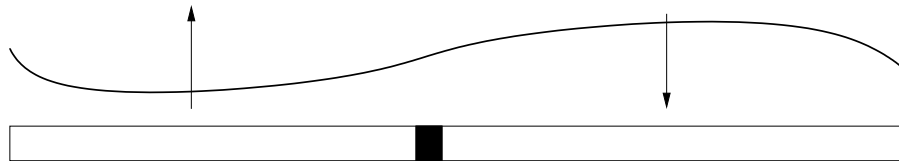

Figure 21.1 Temperature versus position along a metal bar. The arrows show the time dependence, according to the heat equation. On the left the temperature graph is concave up and the bar is warming, and on the right it is cooling. The little slab will be used in the derivation below.

As you will see from our discussion, it is not necessary to think of a metal bar. It is also possible to think of the conduction of heat in the x direction, where this axis passes through a door or wall, or any similar situation in which heat energy flows in one dimension only. If you want to talk about heat flow in a sheet of metal you will have to use the two-dimensional heat equation

$$u_t = a(u_{xx} + u_{yy})$$

If you want to talk about heat flow throughout a solid block of metal, you use the three-dimensional heat equation

$$u_t = a(u_{xx} + u_{yy} + u_{zz})$$

Our plan is to derive the one-dimensional heat equation by showing what physical principles are behind it, and by using what you might call elementary mathematics, i.e. calculus. Later we will show a different derivation for the three-dimensional heat equation using the exact same physical principles, but more sophisticated mathematics, the Divergence Theorem. In both cases we need to point out that we are assuming more familiarity with physics, for these derivations, than we have previously in these notes. On the other hand it is not so much a specific knowledge of physics as a willingness to think about heat, that you really need for the next few paragraphs. Actually solving the equation later does not make so many demands as the derivation does, but thinking about heat will help a lot there too. We begin.

Consider a little slab cut from the bar between coordinates x and $x + \Delta x$, and suppose the density of the bar is ρ mass per length. We are going to account for the heat energy contained within this little slab. We have

$$\begin{cases} \Delta x = & \text{length of the slab} \\ \rho\Delta x = & \text{mass of the slab} \end{cases}$$

physical principle # 1: There is something called specific heat, c, which reflects the experimental fact that a pound of wood and a pound of steel at the same temperature do not contain the same amount of usable heat energy. According to this principle we have

$$\{\, cu\rho\Delta x = \quad \text{the heat energy content of the slab}$$

physical principle # 2: Heat flows from hot to cold, and more specifically there is something called conductivity, k, which reflects the fact that copper conducts heat better than feathers do. According to this principle we have

$$\begin{cases} -ku_x(x, t) = & \text{the rate heat enters the left side of the slab} \\ ku_x(x + \Delta x, t) = & \text{the rate heat enters the right side} \end{cases}$$

These two principles give us two different expressions for the rate at which heat energy enters the slab. This is wonderful! Whenever you have two different expressions for the same thing, you are about to discover something important. Equate them:

$$(cu\rho\Delta x)_t = ku_x(x + \Delta x, t) - ku_x(x, t)$$

Then

$$c\rho u_t = \frac{ku_x(x + \Delta x, t) - ku_x(x, t)}{\Delta x}$$

Finally take the limit as $\Delta x \to 0$ to get

$$c\rho u_t = ku_{xx}$$

This gives the heat equation, and shows us that

$$a = \frac{k}{c\rho}$$

81

A second derivation may be accomplished in three dimensions by using the divergence theorem. We imagine heat conduction in a solid block of something. Fix any region R inside the solid and account for heat flow into R. The heat content is $\iiint_R c\rho u \, dV$ and its rate of change is $\iiint_R c\rho u_t \, dV$. The rate of heat flow through the boundary surface S is $\iint_S k\vec{\nabla}u \cdot \vec{n} \, dA$. By the divergence theorem this last integral is equal to $\iiint_R \vec{\nabla} \cdot (k\vec{\nabla}u) \, dV$. Thus we have

$$\iiint_R c\rho u_t \, dV = \iiint_R \vec{\nabla} \cdot (k\vec{\nabla}u) \, dV$$

for *all* R. Now if two functions happen to integrate to the same thing it means nothing, except that here it is true for all R. This turns out to be enough to insure that the integrands are equal, and we get

$$c\rho u_t = \vec{\nabla} \cdot (k\vec{\nabla}u) = k(u_{xx} + u_{yy} + u_{zz})$$

Problems ———————————————————————————————— **21**

1. Suppose the same temperature distribution is present in two metal bars at a particular time, where the first bar has $u_t = 3u_{xx}$ and the second has $u_t = 4u_{xx}$, due to different material. Which bar cools faster?

2. A new kind of physics is discovered on Mars. Their heat equation says $u_t = au_{xx} - u$. Do Mars bars cool faster or slower than Earth bars?

3. Amazingly enough, we have discovered that the heat equation used by the Klingons is different from either Earth or Mars. They have $u_t = au_{xx} + u$, $u_t = a(u_{xx} + u_{yy}) + u$ etc. Sometimes their pizza doesn't cool off at all. Why?

4. *What's rong with this?* The heat equation can't be right because we already learned Newton's law of cooling in Lecture 4, and it doesn't say anything about the temperature depending on x. ?!?

Today:

The meaning of boundary values. Steady-state solutions. Product solutions.

There are many, many solutions to the heat equation, by comparison with any ordinary differential equation. For example all constants are solutions. This makes sense if you think about what the heat equation is about; a metal bar can certainly have a constant temperature. There is another special class of solutions which are also easy to find. These are the steady-state solutions, i.e. the ones which do not depend on the time. Everyday experience suggests that there are such cases; consider heat conduction through the wall of a refrigerator. The kitchen is at a constant temperature, the inside of the refrigerator is at a constant temperature, and there is a continual flow of heat into the refrigerator through the walls. To find such solutions analytically we assume that u is a function of x only. What happens to the heat equation then? We have $u_t = 0$ and $u_x = u'(x)$, so the heat equation becomes $u'' = 0$. That is an ordinary differential equation, and a pretty easy one to solve. Integrating once we get $u' = a$. Integrating again gives $u = ax + b$. All straight lines are the graphs of steady-state temperature distributions? Check by substituting into the heat equation, and you will see that these indeed work. Note that these straight-line solutions are a generalization of the constant solutions.

So far we have not said much about boundary conditions or initial conditions. It is typical of partial differential equations that there are many possible choices of conditions, and it takes a lot of work to decide what the reasonable ones are. Let's say for now that we are interested in heat conduction along a bar of length l which is located in $0 \le x \le l$. Then one set of conditions which specifies a solvable problem is as follows. You may specify the temperature at the ends of the bar, and the initial temperature along the bar. For example here is one such problem.

$$\begin{cases} u_t = u_{xx} & \text{the heat equation} \\ u(0) = 40 & \text{a boundary condition at the left end} \\ u(5) = 60 & \text{a boundary condition at the right end; the length is 5} \\ u(x,0) = 40 + 4x & \text{the initial temperature} \end{cases}$$

You may consider whether these conditions seem physically reasonable or not. The bar is going to have one end kept at 40 degrees, the other at 60 degrees, and the initial temperature is known. Should this kind of information be adequate to determine the future temperatures in the bar? This is a hard type of question in general, but in this case the answer turns out to be yes. In fact we can solve this particular problem with one of our steady-state solutions.

For Practice: Find the steady-state solution to this problem.

Next consider the problem

$$\begin{cases} u_t = u_{xx} & \text{the heat equation} \\ u(0) = 0 \\ u(5) = 0 \\ u(x,0) = 40 \end{cases}$$

Here we have a bar which begins uniformly at 40 degrees, and at time zero somebody presses ice cubes against both ends of the bar and holds them there forever. What should happen? In this case the initial condition does not match the boundary conditions, so the solution, if it exists, cannot be a steady-state solution. In this case, it is not possible to just quickly reason out whether the problem has any solution, or whether it might have more than one. Physically it does seem as though the bar should gradually cool off to zero, probably faster at the ends where the ice cubes are pressed.

insulation

There is another popular boundary condition known as insulation. What does insulation do? If you put a hot pizza into one of those red insulated containers that the delivery people use, the pizza still eventually cools off, it just takes longer, the better the insulation is. So the key to understanding insulation is that we model it by the assumption that *no* energy passes through it. From our derivation of the heat equation, we know that the rate of energy conduction is a multiple of u_x. You might want to reread Lecture 21 on this particular point. Bottom line: The boundary condition at an insulated end is $u_x = 0$.

Another way to think about insulation is like this: heat flows from hot to cold. So for heat to flow, there must be a temperature difference. On the other hand if $u_x = 0$ somewhere, then the graph of temperature has slope 0, and microscopically there is no temperature difference for small position changes there.

Now what is the following problem about?

$$\begin{cases} u_t = u_{xx} \\ u_x(0) = 0 \\ u(l) = 60 \\ u(x,0) = 40 \end{cases}$$

This is a bar of length l, insulated at the left end and held at 60 degrees at the right end. The initial temperature is 40 degrees all along its length.

product solutions

There is another class of solutions to the heat equation which can be found by a method we have used many times before. We try a solution of the form $u = ce^{rt}$. This trick worked the first time in Lecture 2 on first order ordinary differential equations, and indeed the heat equation is first order with respect to time. The trick worked the second time in Lecture 9 with systems of linear equations. This time, we need to allow c to be a function of x. Substituting into the heat equation $u_t = au_{xx}$ gives $rce^{rt} = ac''e^{rt}$. These will agree provided that $ac'' = rc$. Whenever you try something like this you have to take a few steps along the path to see whether it is going to lead anywhere. In this case we have run across something that we know, so things look hopeful. What we know is how to solve $ac'' = rc$. It is a second order linear ordinary differential equation for c, and it is one of the easy kind with no c' term. To simplify it a little let's write $r = -aw^2$, then what we need is $c'' = -w^2c$. The solutions are $c = A\sin(wx) + B\cos(wx)$. Thus we have found many solutions to the heat equation of the form $u = (A\sin(wx) + B\cos(wx))e^{-aw^2t}$. We call these "product solutions" because they have the form of a function of x times a function of t.

sample

Determine the values of w such that there is a product solution to

$$u_t = au_{xx}$$

having frozen boundary conditions

$$u(0) = u(\pi) = 0$$

We compute $u(0,t) = Be^{-aw^2t}$ in the product solution above. This will be zero for all t only if $B = 0$. So far we have $u = A\sin(wx)e^{-aw^2t}$. Then we compute $u(\pi,t) = A\sin(w\pi)e^{-aw^2t}$. This will be zero for all t if we choose $w\pi$ to be any of the zeros of the sine function. These make $w = 1, 2, 3,...$We have found a list of solutions:

$$u_n(x,t) = A_n\sin(nx)e^{-an^2t}, \ n = 1, 2, 3, ...$$

1. Describe what the following problem is about.

$$\begin{cases} u_t = u_{xx} \\ u_x(0) = 0 \\ u_x(l) = 0 \\ u(x,0) = 400 \end{cases}$$

Determine whether there is a steady-state solution to this problem.

2. Describe what this problem is about.

$$\begin{cases} u_t = u_{xx} \\ u(0) = 0 \\ u(l) = 0 \\ u(x,0) = 400 \end{cases}$$

Determine whether there is a steady-state solution to this problem.

3. Find a product solution to

$$\begin{cases} u_t = u_{xx} \\ u(0) = 0 \\ u(\pi) = 0 \\ u(x,0) = 3\sin(x) \end{cases}$$

4. Find a solution to

$$\begin{cases} u_t = 5u_{xx} \\ u(0) = 0 \\ u(\pi) = 0 \\ u(x,0) = 2\sin(x) \end{cases}$$

5. Find a product solution to

$$\begin{cases} u_t = u_{xx} \\ u_x(0) = 0 \\ u_x(\pi) = 0 \\ u(x,0) = 3\cos(x) \end{cases}$$

6. Determine whether there is a product solution to

$$\begin{cases} u_t = u_{xx} \\ u_x(0) = 0 \\ u_x(\pi) = 0 \\ u(x,0) = 3 \end{cases}$$

7. Suppose that u_1 and u_2 are solutions to

$$\begin{cases} u_t = u_{xx} \\ u(0) = 0 \\ u(\pi) = 0 \end{cases}$$

Note that no initial condition has been specified. Set $s = u_1 + u_2$. Is it true that s is also a solution to this system?

8. Find a solution to

$$\begin{cases} u_t = u_{xx} \\ u(0) = 0 \\ u(\pi) = 0 \\ u(x,0) = 3\sin(x) + 5\sin(2x) \end{cases}$$

which is a sum of two product solutions. If you need help, look ahead to Lecture 23, for which this problem is intended to be a preview.

9. Plot the graphs of the product functions $u_n(x,t) = \sin(nx)e^{-an^2t}$ versus x for several values of t, for the cases $n = 1$ and $n = 2$. Describe what is going on in your graphs, in terms of temperature in a metal bar.

10. In problem 7, suppose $u(x,t) = e^{-at}y(x)$, i.e. an exponential function of t times a function of x. What boundary value problem, of the type considered in Lecture 20, must y solve?

Today:

> *More heat solution examples. What they mean. Do they make sense? Super-position.*

Today we are going to analyze a heat conduction problem which happens to be solvable using some of the product solutions found last time. Our emphasis is not on formulas, but on understanding what the solutions mean, and thinking about the extent to which this heat equation corresponds with our daily experience of temperature.

Consider the problem in which a metal bar with frozen ends has temperature initially as in Figure 23.1. We want to find out how this temperature changes as time goes by. What do you think will happen? We know from daily experience that heat flows from hot to cold, and in fact a quantitative version of that statement was a major part of our derivation of the heat equation. So probably the hot area to the left will supply energy to the cooler parts.
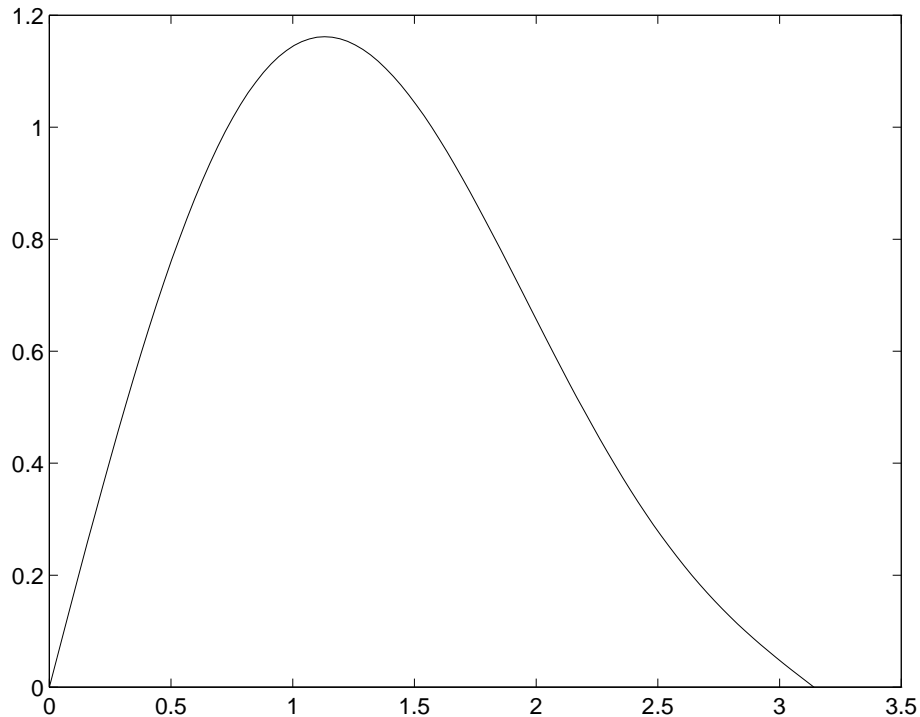
Figure 23.1 Our initial conditions. Note the spot at which the bar is hottest. What do you think will happen to the hot spot as time goes by, i.e. will it cool off, will it move to the left, to the right, etc.? (figure made by the `matlab` *commands* `x = 0:.031415:3.1415;` `plot(x, sin(x)+(1/3)*sin(2*x))`*)*

Our problem is as follows.

$$\begin{cases} u_t = u_{xx} \\ u(0, t) = 0 \\ u(\pi, t) = 0 \\ u(x, 0) = \sin(x) + \frac{1}{3}\sin(2x) \end{cases}$$

Before attacking the problem analytically let's just read it carefully to see what it is about. That may be an obvious suggestion, but it is also a way to prevent ourselves from

86

doing something idiotic. We see from the first line that we are solving the case of the heat equation in which the physical constant a has been set to 1 for convenience. We can't see what the domain is until we read the boundary conditions. The second two lines say that we are dealing with the ice cube case, where the bar ends are at 0 and π on the x axis and are kept frozen.
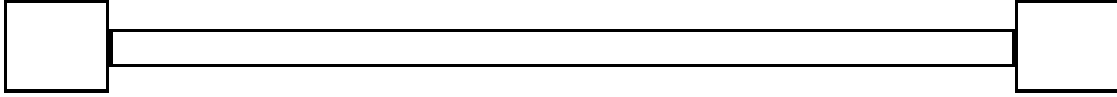
Figure 23.2 Ice cubes at the ends of the bar. You might be worried about heat escaping from all around the sides of the bar. If so, think about heat conduction through a wall rather than along a bar, or else think of the bar surrounded by perfect insulation so that the x direction is the only one of any concern.

Finally the fourth line of the problem says that the initial temperature is the sum of two terms. Graphed seperately, they look like Figure 23.3.
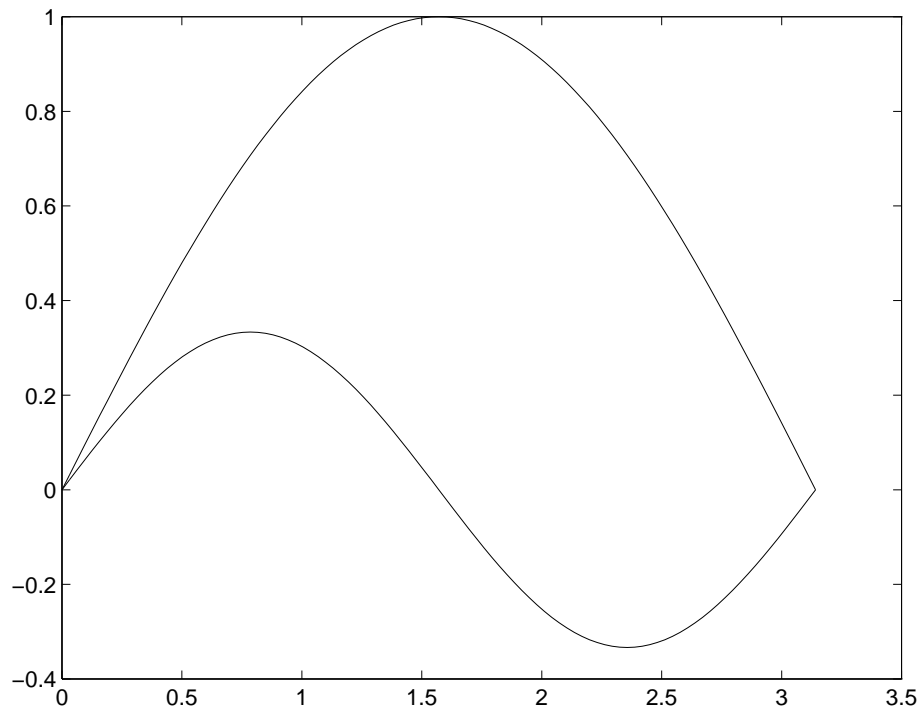
Figure 23.3 The initial conditions are a sum of these two terms. (figure made by `x = 0:.031415:3.1415; plot(x,sin(x)); hold on; plot(x,(1/3)*sin(2*x)); hold off`*)*

If you did problem 7 of Lecture 22, you may suspect why we have included Figure 23.3. Solutions to the heat equation may be added to produce new solutions. If you didn't do that problem yet, go do it now. In mathematics this is called linearity, as we have emphasized several times in these notes. In physics it is sometimes called "superposition" to crystallize the idea that two or more things are going on at the same time and place.

Now we are ready to solve this problem. There are product solutions to the heat equation of the form $u = A\sin(wx)e^{-aw^2t}$ for all values of w. Perhaps the ones we need are the cases of the sine function which appear in our initial conditions. Let's try that. Our

87

candidates are

$$\begin{cases} u_1(x,t) = A_1 \sin(x)e^{-t} & \text{and} \\ u_2(x,t) = A_2 \sin(2x)e^{-4t} \end{cases}$$

Now check to be sure that these solve the heat equation and the boundary conditions. Do that now. Does it work? Ok. Notice that it doesn't matter what the coefficients A_j are yet, *and* that the sum

$$u(x,t) = A_1 \sin(x)e^{-t} + A_2 \sin(2x)e^{-4t}$$

is also a solution to the heat equation with these frozen boundary conditions.

The next step is to attack the initial conditions. We need to compare

$$\begin{cases} u(x,t) = A_1 \sin(x)e^{-t} + A_2 \sin(2x)e^{-4t} & \text{our tentative solution} \\ u(x,0) = \sin(x) + \frac{1}{3}\sin(2x) & \text{the initial condition} \end{cases}$$

Now the question is, can this be made to work? Well, yes, it is really not hard to see at this point, since we did all the hard work already. You just take $A_1 = 1$ and $A_2 = 1/3$. This makes the initial conditions work, and we have checked everything else. So we have a solution to our problem. It is

$$u(x,t) = \sin(x)e^{-t} + \frac{1}{3}\sin(2x)e^{-4t}$$

The next step is to understand what this solution means and how it behaves as time goes by. The best way to do this is for you to sketch the graphs of the solution for various times, thinking about what happens to each of the two terms in the solution. If you do that yourself, and it is a good idea to do so, you should get something like Figure 23.4 for the individual terms, and Figure 23.5 for the solution.
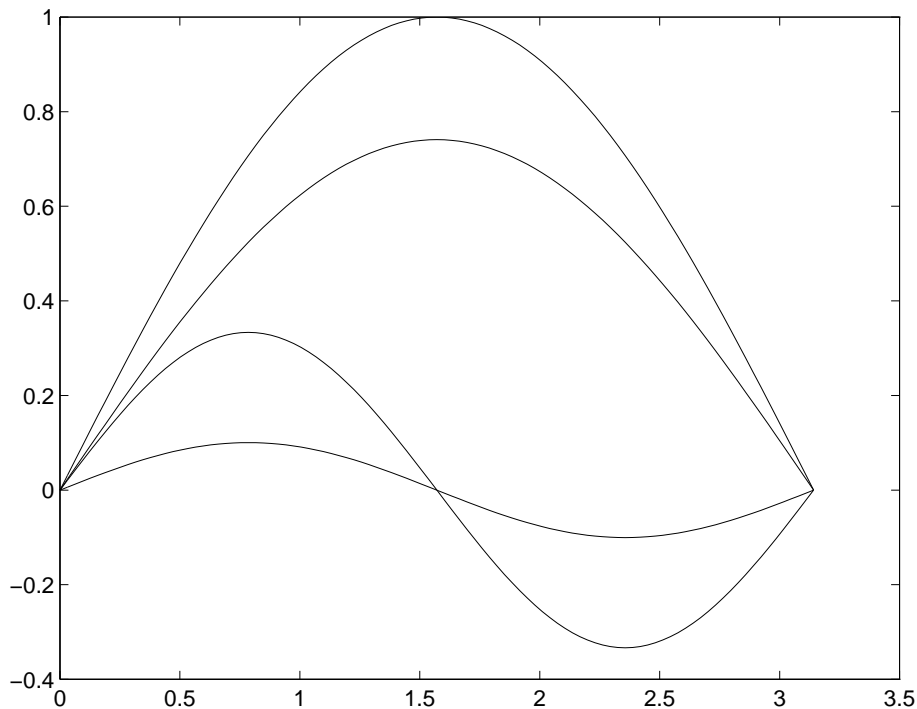
Figure 23.4 The two terms in our solution, plotted for $t = 0$ and $t = .3$ Notice how much each of these has moved, in particular that the term involving $\sin(2x)$ has moved a lot more than the $\sin(x)$ term. Why is that?
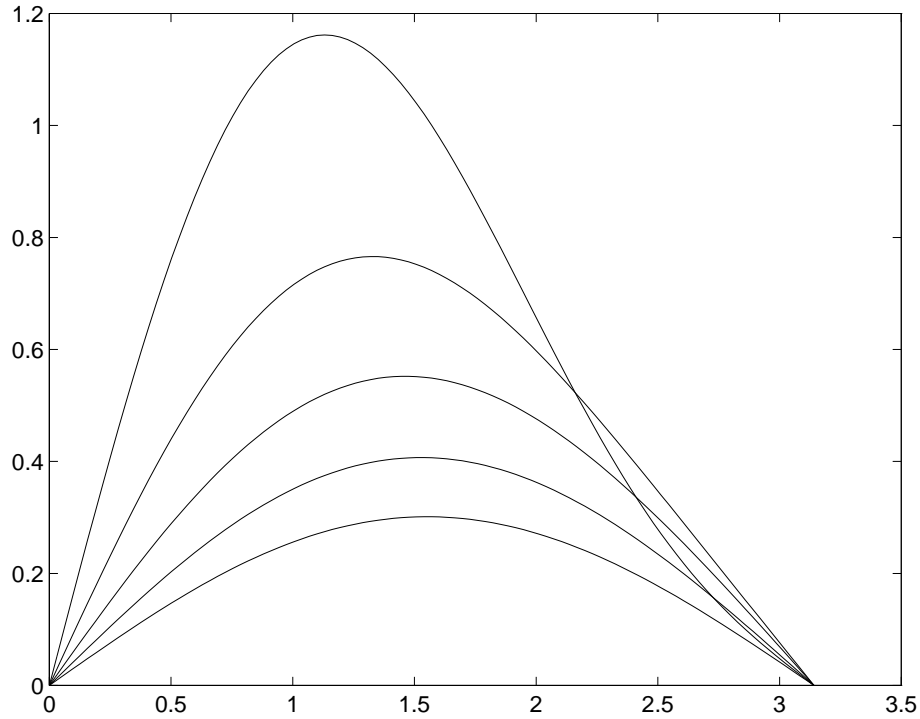
Figure 23.5 *The solution temperature at times $t = 0, .3, .6, .9, 1.2$. Now you see that the hot spot moves toward the center. Why?*

If you study the solution formula and the pictures you will see how significant the n^2 is in the time exponential part of the product solution.

Problems _____ **23**

1. Think about what should happen to a metal bar which has an initial temperature distribution consisting of alternating hot, cold, hot, etc., in a lot of narrow bands along the bar. Think of a bar which is red-hot in six places and cooler in between. Do you think it will take very long for the temperature to change? Solve

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = \sin(12x) \end{cases}$$

and sketch the solution for $t = 0, .3, .6$.

2. Solve

$$\begin{cases} u_t = u_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \\ u(x,0) = \sin(x) + .5\sin(3x) + .25\sin(5x) \end{cases}$$

Sketch the initial condition.

3. Solve

$$\begin{cases} u_t = u_{xx} \\ u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \\ u(x,0) = \cos(x) + .5\cos(3x) + .25\cos(5x) \end{cases} \qquad \text{insulated boundaries}$$

Note that the initial temperature satisfies the boundary conditions.

4. Run the `Heat Equation 1D` applet mentioned in Lecture 1. Try various initial and boundary conditions to develop some feeling for what this equation is about.

Today:

  Traveling waves

We have now solved several heat conduction problems and are about to turn to another partial differential equation known as the wave equation. You can figure out what it is about. Notice that as we were working on the heat equation we solved a fairly large number of problems distinguished from one another by boundary and initial conditions, while the equation itself never changed. This is typical of partial differential equations, that there are relatively few of them which are considered to be important, and the conditions can be changed to permit these few important equations to apply to many different settings. For the heat equation we only dealt with heat flow on a finite interval $0 \le x \le l$. Our wave equation can apply to many things also, such as the vibrations of a guitar string, but we are mainly going to talk about waves on an infinitely long string, the whole x axis. The reason for this choice is that guitar strings can be solved by methods similar to those we used for the heat equation, and we prefer to introduce new ideas here.

  The wave equation is

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u_{tt} = c^2(u_{xx} + u_{yy}) \\ u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) \end{cases}$$

in one, two, or three dimensions, respectively. Note that it is second order with respect to the time, like Newton's law. In fact it _is_ Newton's law in disguise, sometimes. We know from Lecture 6 that second order equations can oscillate, so that at least is reassuring. In one dimension u may represent the vertical position of a vibrating string.
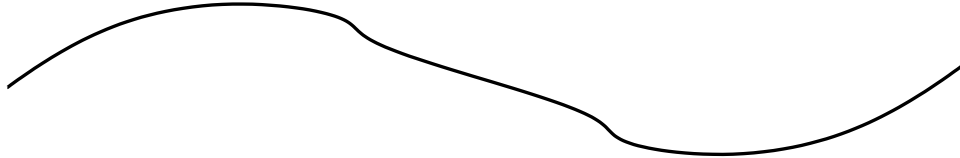

Figure 24.1 The one dimensional wave equation describes vibrations of a string, or less accurately, water waves.

  In two dimensions or three, the wave equation can represent vibrations of a drum head or of sound in the air, and other things like electromagnetic waves. It is a sound and light show all by itself.

  Let's read the one dimensional equation again. If u is position, what is u_{tt}? Think about it. Did you think about it yet? It is acceleration. Also when you draw the graph of u as a function of x as in Figure 24.1, what does u_{xx} represent? Just as for the heat equation, it is the curvature of the graph. The wave equation says that when the graph is concave up, the acceleration must be positive. Does that seem to fit with reality? Think about swinging a heavy jumprope. Better still, get a rope and try it. See whether you think the acceleration direction matches the curvature in this way.

  The wave equation for a string may be derived as follows. We apply Newton's law to a bit of the string, accounting for vertical forces.
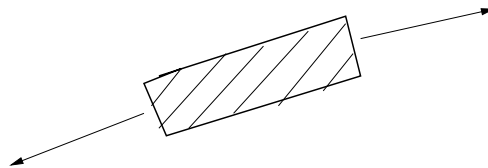

Figure 24.2 The tension in the string pulls at different angles on the two ends of any little piece of it, depending on the curvature.

The tension T in the string has a vertical component. On the left it is approximately $Tu_x(x, t)$ since the slope u_x is the tangent of the angle, which is near the sine of the angle if the angle is small. On the right it is $Tu_x(x + \Delta x, t)$. The difference of these must be the mass times the acceleration, so

$$Tu_x(x + \Delta x, t) - Tu_x(x, t) = \rho \Delta x u_{tt}$$

where ρ is the mass per length of the string. Dividing by Δx and taking the limit as Δx approaches 0 gives

$$\frac{T}{\rho} u_{xx} = u_{tt}$$

which is the wave equation with $c^2 = T/\rho$.

Now let's try to solve the wave equation, using our usual method. We try $u = f(x)e^{rt}$ in $u_{tt} = c^2 u_{xx}$. It becomes $r^2 f(x)e^{rt} = c^2 f''(x)e^{rt}$. Cancelling the time exponentials and setting $r = ac$ gives $f'' = a^2 f$. Wow! Another easy second order equation has popped out. Again we find that we must remember those second order equations. Go back and review if you don't remember. We find solutions $f(x) = e^{ax}$, $f(x) = e^{-ax}$, and linear combinations. For example if $a = 1$ we have found the solutions $u = e^x e^{ct}$ and $u = e^{-x} e^{ct}$. To understand what these are, we graphed the second one in Figure 24.3.
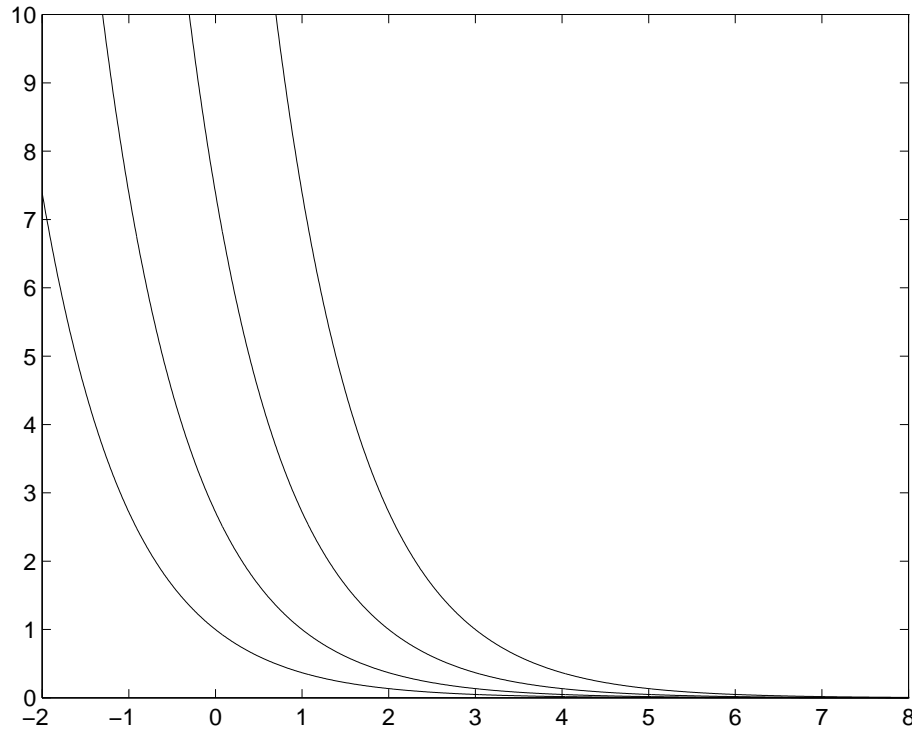
Figure 24.3 The wave $u(x, t) = e^{-(x-ct)}$, graphed for various times. Note that to keep the expression $x - ct$ constant as t increases you have to increase x also, so this wave is moving to the right. It could make a surfer cry.

So, we have found product solutions to the wave equation of the form

$$\begin{cases} u = e^{a(x+ct)} \\ u = e^{a(x-ct)} \end{cases}$$

91

for all values of a, even complex. What does this mean? It certainly means that there are an awful lot of solutions which are functions of $x \pm ct$. Now let's get crazy and generalize. Suppose you have a function

$$u(x, t) = g(x - ct) + h(x + ct)$$

not necessarily built from exponentials at all. Taking derivatives by the chain rule we find

$$\begin{cases} u_t = -cg'(x - ct) + ch'(x + ct) \\ u_{tt} = c^2 g'' + c^2 h'' \\ u_x = g'(x - ct) + h'(x + ct) \\ u_{xx} = g'' + h'' \end{cases}$$

Woah! Look what happened there, we got $u_{tt} = c^2 u_{xx}$ without assuming *anything* about g and h except that they be differentiable. Doesn't that mean you can have waves of nearly any shape moving left and right? Yes it does. They are called traveling waves.

Problems ———————————————————————————— **24**

1. Show that $\cos(x - 2t)$ and $1/(x + 2t)^2$ are solutions to $u_{tt} = c^2 u_{xx}$ for some c. Find c. Graph these waves for $t = 0$, 1, and 2.

2. The function $\sin(x + 3t) + \sin(x - 3t)$ consists of 2 traveling waves and solves

$$\begin{cases} u_{tt} = 9u_{xx} \\ u(0, t) = 0 \\ u(\pi, t) = 0 \\ u(x, 0) = 2\sin(x) \\ u_t(x, 0) = 0 \end{cases}$$

Sketch the 2 traveling waves, both separately and together, and try to see why the sum has the 0 boundary condition at $x = 0$, even though neither of the traveling waves does so separately.

3. Use the addition formula for the sine to show that the function in problem 2 is equal to $2\sin(x)\cos(3t)$. It therefore represents a vibrating string having frequency $3/2\pi$. Then solve

$$\begin{cases} u_{tt} = 9u_{xx} \\ u(0, t) = 0 \\ u(\pi, t) = 0 \\ u(x, 0) = 2\sin(2x) \\ u_t(x, 0) = 0 \end{cases}$$

Sketch the motion for this case, and convince yourself that this vibration produces a sound one octave up from that of problem 2, i.e. has twice the frequency.

4. Continuing in the tradition of problems 2 and 3, now use initial condition $u(x, 0) = 2\sin(3x)$. The note you get is the fifth above the octave, musically.

5. This problem is about using the wave equation to model an echo. Consider a function of the form $u(x, t) = f(x + ct) + f(-(x - ct))$ where that is the same f traveling in both directions. Show that u satisfies the condition $u_x(0, t) = 0$. Think of sound waves moving in $0 \le x$ and a wall at the point $x = 0$. Sketch a graph of u for several times using a function f of your choice, and try to convince yourself that what you have sketched is a representation of waves bouncing off a wall.

6. Repeat problem 5 for $u(x, t) = f(x + ct) - f(-(x - ct))$. This time you should get the boundary condition $u(0, t) = 0$, which involves a different kind of bouncing at the wall. Try to see it in your sketch. Which of problems 5 or 6 is more like a string vibration, with the string tied down at the origin? Which one is more like water waves at the edge of a swimming pool?

7. Try the `Wave Equation 1D` and `2D` applets, and explore for a while to get some feeling for how the solutions behave. Try various boundary and initial conditions.

8. *What's rong with this?* The function $u(x, t) = \cos(t)\sin(x)$ couldn't be a solution to the wave equation because it isn't a traveling wave. ?!?

1.3 A function increases where the derivative is positive, so y is increasing on (–1,0) and $(1, \infty)$.

1.4 If $y(0) = 1/2$ then y decreases as long as it is positive, so probably $y \to 0$.

1.5 $y' = (.03 + .005t)y$

2.2 $x(t) = -1/\sqrt{t + 1/9}$

2.3 $x(t) = 1/\sqrt{t + 1/9}$

3.1 Once you get `octave` downloaded or `matlab` bought, and installed, start it. Once you're in, type 'help' with no quotes, press the enter key, and follow the information given. There is also 'help demo', 'help plot', and many other things which the first 'help' will tell you about. It is important to keep trying. At first, just type simple things like '2+3' or 'cos(pi)' until you get comfortable with it. After that, learn how to make an "m-file": that is a file called something like 'myfile.m' which contains just the text of some commands. For example, `dfield` is a large m-file for `matlab`, which doesn't seem to work in `octave` (but there is the `dfield` applet, too). Here is a much shorter example.

```
2+3
t=0:.1:3.5
y=cos(t);
plot(t,y)
```

Then use the file by typing 'myfile' at the prompt. All the commands will be executed. If an error occurs just edit your m-file. This is easier usually than fixing any errors at the prompt. Once you can do this successfully, you are over the hurdle and you can use the software to help you learn. It is a nice tool to have, once you get past the first stage. The freeware `xpp`, or `xppaut`, is also excellent.

4.1 This is nasty because you have to integrate $(e^{-1.3t}x)' = e^{-1.3t}\cos t$. The answer is $x = ce^{1.3t} + (-1.3\cos t + \sin t)/(1 + (1.3)^2)$.

4.5 $x' = .03x + 3000$, $x(0) = 0$. Multiplying by integrating factor $e^{-.03t}$ you find $(e^{-.03t}x)' = 3000e^{-.03t}$. So $e^{-.03t}x = -100000e^{-.03t} + c$, $x = -100000 + ce^{.03t}$. Then $x(0) = -100000 + c = 0$ so $c = 100000$ and $x(t) = 100000(-1 + e^{.03t})$. Finally, $x(20) = 100000(-1 + e^{.6}) = 82212$. Not bad.

4.6 Here is one philosophy. We have $x' = -.028x + .028$ where $x(t)$ is the balance on an account earning 2.8% interest and deposits of \$.028 per year, and $y' = .028(y - y^2)$ where $y(t)$ is a population with a natural growth rate of 2.8% and a limiting size of 1 (million or whatever). Change the units of x to dollars per person with the idea that the analysis certainly could be applied to one individual at a time. Change the units of y to people per dollar to acknowledge the fact that the economy determines how many people can be supported. Then the substitution $x = 1/y$ is suggested by the units. This argument doesn't prove that a solution to one equation goes over into a solution to the other! Only the substitution can show whether that is true: if $y' = .028(y - y^2)$ and $x = 1/y$ then $x' = -y^{-2}y' = -y^{-2}(.028(y - y^2)) = -.028(y^{-1} - 1) = -.028(x - 1)$.

5.3 $\sec^2 y \, dy = dt$, so $y = \tan t$.

5.6 Yes. To get rid of the cosine you would have to start with a differential equation for $\tan^{-1}$ which contains only arithmetic operations +–*/. We know that $d(\tan^{-1} t)/dt = 1/(1 + t^2)$ so all you have to do is solve that numerically.

6.3 No, s solves $s'' + s = 2\cos 8t$ instead.

6.8 $x'(x'' + x^3) = 0$ integrates to $\frac{(x')^2}{2} + \frac{x^4}{4} = c$.

6.10 Somebody tried to use the characteristic equation for a nonlinear equation, which is very rong. Then even if $r^2 + 1 = 0$ had been correct, the roots should have been $\pm i$. Finally, you should never just add "$+c$" to something at random.

7.2 $r^2 + 3r + 4 = 0$, $r = (-3 \pm \sqrt{9 - 16})/2 = -3/2 \pm i\sqrt{7}$ So $y(t) = e^{-3t/2}(c_1 \cos(\sqrt{7}t) + c_2 \sin(\sqrt{7}t))$

7.3 $a + 1/a = 0$, $a^2 + 1 = 0$, $a = \pm i$. For $a + 1/a = i$ you get $a^2 + 1 = ia$ or $a^2 - ia + 1 = 0$, $a = (i \pm \sqrt{-1 - 4})/2$.

7.6 There is a factor of e^t so this grows, and a factor of $\cos(t) + i \sin(t)$ which goes around the unit circle counterclockwise, so this is a spiral.

7.7 $r^2 + br + c = (r + 2 + i)(r + 2 - i)$, so multiply it out.

8.1 $C \sin(ft + \phi) = C \sin(ft) \cos(\phi) + C \cos(ft) \sin(\phi)$. So to match this with the given $A \cos(ft) + B \sin(ft)$ you have to solve $C \cos(\phi) = B$ and $C \sin(\phi) = A$ for C and ϕ. That is like saying that (B, A) is a point in the plane and you want to find polar coordinates $(r, \theta) = (C, \phi)$ for this point. That is always possible.

8.2 This is a little harder, because the only clue you have is that in the given identity, the frequencies $7.5/2\pi$ and $.5/2\pi$ are half the sum and difference of the frequencies $8/2\pi$ and $7/2\pi$. We have

$$
\begin{cases}
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \\
\sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b) \\
A \sin(a + b) + B \sin(a - b) = (A + B) \sin(a) \cos(b) + (A - B) \cos(a) \sin(b)
\end{cases}
\qquad \text{so}
$$

For example $\sin(8t) + 2.3 \sin(7t) = 3.3 \sin(7.5t) \cos(.5t) - 1.3 \cos(7.5t) \sin(.5t)$.

8.3 From answer 8.2 we see that when you graph something like $\sin(7.5t) \cos(.5t)$ you see a slow and a fast frequency in spite of the fact that you have started with the two near frequencies $7/2\pi$ and $8/2\pi$. In the case of Figure 8.2 the near frequencies are $\sqrt{2}/2\pi$ and $3/2\pi$, so the corresponding slow and fast frequencies are $(\sqrt{2}/2\pi \pm 3/2\pi)/2$, which are about .12 and .35. The situation is more complicated than the identities in answer 8.2 though, because you really need the awful identity $A \sin(a + b) + B \sin(a - b + \phi) = (A + B \cos(\phi)) \sin(a) \cos(b) + (A - B \cos(\phi)) \cos(a) \sin(b) + (B \sin(\phi)) \sin(b)(\cos(a) - \sin(a))$. The only important thing is that the slow and fast frequencies still show up.

9.1 $x_2 = x_1 - x_1'$ from the first equation, then the second gives $x_1' - x_1'' = x_1 + 2(x_1 - x_1')$, or $x_1'' - 3x_1' + 3x_1 = 0$.

9.2 It is the same as for x_1. Either calculate it, or recognise that the formula $x_2 = x_1 - x_1'$ expresses x_2 as a linear combination of two solutions to the equation for x_1.

9.4 Assuming $y'' - 3y' + 3y = 0$ we have $z_2 = y' - y = z_1' - z_1$, $z_2' = (y' - y)' = y'' - y' = 3y' - 3y = 3z_2$. So this system is

$$
\begin{cases}
z_1' = z_1 + z_2 \\
z_2' = 3z_1
\end{cases}
$$

This is different from the system of problem 1 even though they have the same second-order equation associated with them. The point is that there is nothing unique about a system associated to a higher order equation.

11.3 All the pretzels must go to New York or Chicago, so $9000 = 200y + 180x$. All the fuel must be used on the way, so $260000 = 6500y + 5400x$. These are our system. It seems fairly clear that this is too simplified to be realistic, and in fact the numbers used are probably not realistic. If we really wanted to model costs of an airline we would certainly not consider pretzels as important as fuel. The next largest expenses after fuel would have to be identified. They might be the cost of the planes, or maintenance, or personel salaries and fringes.

11.4 This is nonsense. This kind of product has not been defined.

12.1 row 1 has been replaced by (row 1 − 1.6 row 2).

12.3 The reduced form is $\begin{bmatrix} 1 & 0 & 0 & 5.5 \\ 0 & 0 & 1 & .5 \end{bmatrix}$.

12.4 There are no solutions to this system because the last equation says $0 = 2$.

12.8 The answer is unique, and is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

13.3 $\begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a/2 \\ 0 & 1/2 \end{bmatrix}$, with no restriction on a. For the 4 by 4, if $a \neq 0$ then

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. If $a = 0$ then the 4 by 4 is not invertible.

13.5 $Ix = x$ for all x, but 0 is not an eigenvector. So, all vectors except 0 are eigenvectors of I, with eigenvalue 1.

13.6 The Matlab command works correctly only on the case which has a unique solution, and gives meaningless error messages for the other two cases.

14.3 We have to solve $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$. Row operations give $A \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, so $y = -2z$ and $x = z$. So $A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$.

14.8 For this problem and for problem 7, note that A and $7A$ have the same eigenvectors, but the eigenvalues of $7A$ are 7 times as large.

14.11 The calculation here is correct but misguided! As soon as you know that $\det(A - \lambda I) = (3 - \lambda)(7 - \lambda)$, stop, because this is already factored. There is no need to multiply it out and use the quadratic formula.

15.2 $x(t) = 0$, and $y(t) = 5e^{3t}$

15.5 $x = ae^{2t}$ and $y = be^{-3t}$. If some of a or b are 0 you get solutions running along a half axis or stationary at the origin. Otherwise $y^{-2/3} = (const)e^{(-2/3)(-3)} = (const)e^{2t} = (const)x$.

15.9 $x(0) = u + 2v$, so $x(t) = ue^{5t} + 2ve^{4t}$.

15.10 Don't add "$+C$" without a reason. This is correct otherwise.

16.3 The two plots are identical.

16.5 The real and imaginery parts were taken incorrectly because somebody forgot that the vector $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is also complex.

17.3 All rational numbers have either terminating or repeating decimal expansions, while the irrational ones have the other kind. All integers are either positive, negative, or 0. All math teachers are mean.

17.4 The system of equations is just $x' = y' = 0$, so the solutions are constant, and in the picture this means they don't go anywhere.

18.3 I tried the system $\begin{cases} x' = x - xy \\ y' = -y + xy - .2 \end{cases}$. I found that solutions can make increasing oscillations until the prey becomes extinct. This is not what one would hope for! My advice is not to fool with Mother Nature.

18.6 $(0,0)$ only.

19.5 The correct conclusion is that chaos is possible in a system of two or more masses.

20.1 general solution to $y'' = -5y$ is $y = a\cos(5x) + b\sin(5x)$. Apply boundary conditions to find $y(x) = 4\cos(5x) + (7/5)\sin(5x)$

20.2 there are no solutions

20.5 For $c = 0$ there is the solution $y = 0$. For $c < 0$ the general solution involves linear combinations of $e^{\pm\sqrt{-c}x}$ and these all get wiped out by the boundary conditions, except for $y = 0$.

20.8 If $c \leq 0$ then the only answer is $y = 0$. If $c > 0$ write it as $c = d^2$ where you may as well assume $d > 0$, and the general solution to $y'' = -d^2y$ is $y(x) = a\cos(dx) + b\sin(dx)$. The boundary condition $y(0) = 0$ forces $a = 0$. Then the condition $y'(3) = 0 = db\cos(3d)$ forces $3d = \pi/2, 3\pi/2, 5\pi/2, \ldots$. Thus $y(x) = b\cos(k\pi/6)$, for $k = 1, 3, 5, \ldots$. No restriction on b.

21.2 In $u_t = au_{xx} - u$, the $-u$ term certainly seems to be a negative influence on the time rate of change. So it seems fairly reasonable that Mars bars will cool faster than Earth bars. The only thing which might throw a monkey wrench into this argument is the case in which $u < 0$, which is certainly allowed. This is not actually a problem though, since you can always rescale temperature by adding a constant to it, keeping it positive and avoiding this problem. We should mention that this argument, though reasonable, does not actually prove anything, because the physicists aren't sure until an experiment is done, and the mathematicians aren't sure until an airtight proof is given.

22.1 This problem concerns heat conduction along a bar which is initially 400 degrees, and which has insulation all around, including the ends.

22.5 $3\cos(x)e^{-t}$

22.10 the heat equation for $u = e^{-at}y$ says that $u_t = -ae^{-at}y(x) = u_{xx} = e^{-at}y''(x)$. So we must have

$$\begin{cases} y'' = -ay \\ y(0) = 0 \\ y(\pi) = 0 \end{cases}$$

23.2 $u(x,t) = e^{-t}\sin(x) + .5e^{-3^2t}\sin(3t) + .25e^{-5^2t}\sin(5t)$

24.3 $\sin(x+3t)+\sin(x-3t) = \sin(x)\cos(3t)+\cos(x)\sin(3t)+\sin(x)\cos(3t)-\cos(x)\sin(3t) = 2\sin(x)\cos(3t)$ has the initial condition $2\sin(x)$ of problem 2. For this problem the only difference is that we want $2\sin(2x)$ instead, so the solution is $2\sin(2x)\cos((2)(3t))$. You have to multiply the $3t$ by 2 because otherwise you would not be creating traveling waves which are functions of $x \pm 3t$, as required by the wave equation $u_{tt} = 3^2u_{xx}$.

24.8 We have not claimed that all solutions to the wave equation are traveling waves, only that sums of left and right traveling waves are solutions. In this case the function is seen to be a solution to $u_{tt} = u_{xx}$ by differentiating it, or by using the idea of problem 3 to write it as a sum of left and right traveling waves: $\cos(t)\sin(x) = (1/2)(\sin(x+t)+\sin(x-t))$.