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THE GREEN FUNCTION OF THE WAVE EQUATION

For a simpler derivation of the Green function see Jackson, Sec. 6.4. We will proceed by contour integration in the complex ω plane.

The Green function is a solution of the wave equation when the source is a delta function in space and time,

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\boldsymbol{r}, t; \boldsymbol{r}', t') = 4\pi\delta^d(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'). \tag{1}$$

By translation invariance, G must be a function only of the differences $\boldsymbol{r} - \boldsymbol{r}'$ and $t - t'$. We simplify the problem by setting $\boldsymbol{r}' = 0$ and $t' = 0$, so we have

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\boldsymbol{r}, t) = 4\pi\delta^d(\boldsymbol{r})\delta(t). \tag{2}$$

We also set $c = 1$ for now; we can restore it at the end by dimensional analysis.

We define the Fourier transform of G according to

$$G(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{d+1}} \int d^d k \, d\omega \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \, g(\boldsymbol{k}, \omega), \tag{3}$$

whence

$$g(\boldsymbol{k}, \omega) = \int d^d r \, dt \, e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \, G(\boldsymbol{r}, t). \tag{4}$$

Applying a Fourier transform to Eq. (2), we find that g satisfies the *algebraic* equation

$$(k^2 - \omega^2) g(k, \omega) = 4\pi. \tag{5}$$

The solution is thus

$$G(\boldsymbol{r}, t) = \frac{1}{(2\pi)^{d+1}} \int d^d k \, d\omega \, e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \, \frac{4\pi}{k^2 - \omega^2}. \tag{6}$$

We focus first on the integral over ω in Eq. (6)

$$\tilde{G}(\boldsymbol{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \frac{1}{k^2 - \omega^2}. \tag{7}$$

It begs to be evaluated as part of a contour integral. Unfortunately, the integral is undefined since it goes right through poles on the real axis at $\omega = \pm k$. We cure this disease by moving the poles slightly off the real axis by the addition of $i\epsilon$ in the right place. This can be done in several ways; any prescription must, in the limit $\epsilon \to 0$, give a solution of Eq. (2). Thus the different prescriptions must give results that differ by homogeneous solutions of Eq. (2). We will be guided in choosing an $i\epsilon$ prescription by the demands of initial conditions.

Let's try changing Eq. (7) into

$$\tilde{G}(\boldsymbol{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \frac{1}{k^2 - (\omega + i\epsilon)^2}. \tag{8}$$

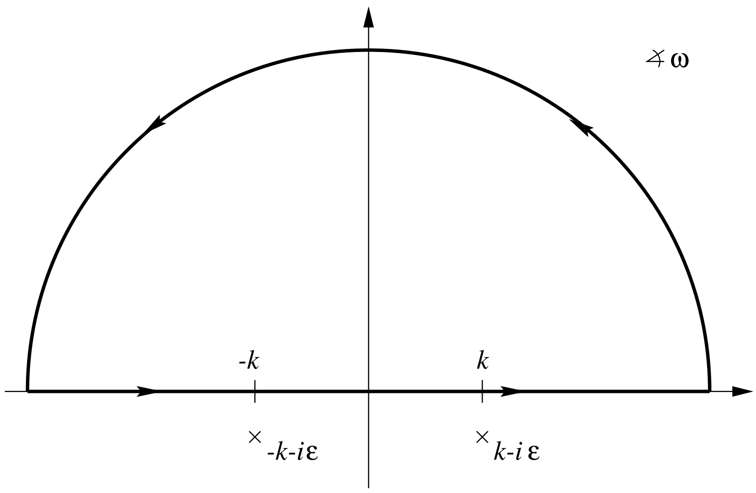

FIG. 1: Contour for the $t < 0$ case

Now the integrand has poles at $\omega = \pm k - i\epsilon$. Evaluation of the integral depends on the sign of t. We want the solution to vanish for $t < 0$ — these are the initial conditions. Let's see if this is so. For $t < 0$ we close the integral with a semicircle at $|\omega| = R$ in the upper half plane, since then the exponent is $-i\omega t = +i\omega|t| \sim -q|t|$ where $\omega = p + iq$ (see Fig. 1). The contour integral is

$$\mathcal{I} \equiv \oint d\omega\, e^{-i\omega t} \frac{1}{k^2 - (\omega + i\epsilon)^2} = \tilde{G}(\boldsymbol{k}, t) + \int_R d\omega\, e^{-i\omega t} \frac{1}{k^2 - (\omega + i\epsilon)^2}, \tag{9}$$

and we see that the integral along R indeed vanishes as $R \to \infty$. The contour integral $\mathcal{I}$ equals $2\pi i$ times the sum of the residues at the poles inside the contour—but there are *no* poles inside the contour! So indeed the contour integral is zero, and hence $\tilde{G}(\boldsymbol{k}, t) = 0$ when $t < 0$. Transforming $\boldsymbol{k} \to \boldsymbol{r}$, we see that $G(\boldsymbol{r}, t) = 0$ for $t < 0$, which is the initial condition we wanted.

Now we evaluate Eq. (8) for $t > 0$. We close the contour in the lower half plane this time (see Fig. 2). Again the integral on the semicircle vanishes; but the contour integral does *not* vanish:

$$\mathcal{I} = -2\pi i \left[Res(-k - i\epsilon) + Res(k - i\epsilon) \right] = (-2\pi i) \left(i \frac{\sin kt}{k} \right) \tag{10}$$

So for $t > 0$

$$\tilde{G}(\boldsymbol{k}, t) = 2\pi \frac{\sin kt}{k}, \tag{11}$$

and thus

$$G(\boldsymbol{r}, t) = \frac{1}{(2\pi)^d} \int d^d k\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, \frac{4\pi}{k} \sin kt. \tag{12}$$

Since $G(\boldsymbol{r}, t)$ satisfies the wave equation with the initial condition $G = 0$ for $t < 0$, it is the retarded Green function. If we change $i\epsilon \to -i\epsilon$ in Eq. (8) we will get the solution that vanishes for $t > 0$, which is the advanced Green function. Another possibility is to move one pole above the real axis and the other below; this Green function doesn't vanish in either case, and it is the Feynman propagator that appears in quantum field theory. Since all these Green functions solve the same partial differential equation, they must differ by solutions of

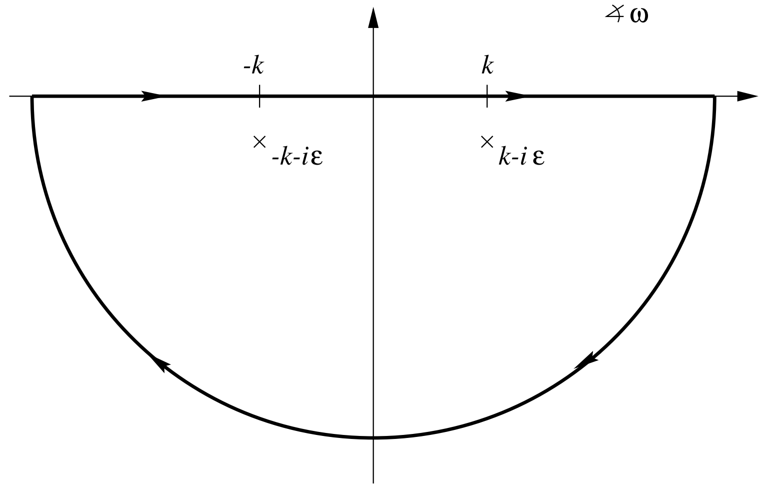

FIG. 2: Contour for the $t > 0$ case

the homogeneous equation. We choose to work with the retarded Green function because its initial conditions make physical sense.

Let us complete the evaluation of $G(\boldsymbol{r}, t)$. In the integral over $\boldsymbol{k}$ in Eq. (12), we can choose the z axis to lie along $\boldsymbol{r}$ and evaluate the integral in spherical coordinates. We will do the calculation explicitly for $d = 2$ and $d = 3$.

First for $d = 3$:

$$G(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3} \int k^2\, dk\, d(\cos\theta)\, d\phi\, e^{ikr\cos\theta} \frac{4\pi}{k} \sin kt. \tag{13}$$

We perform the integral over $\cos\theta$ first,

$$G(\boldsymbol{r}, t) = \frac{2}{\pi r} \int_0^\infty \sin kr \sin kt\, dk \tag{14}$$

$$= \frac{1}{2\pi r} \int_{-\infty}^\infty [\cos(kr - kt) - \cos(kr + kt)]\, dk. \tag{15}$$

The integral over k is easy, giving the result

$$G(\boldsymbol{r}, t) = \frac{1}{r}[\delta(r - t) - \delta(r + t)]. \tag{16}$$

Since both t and r are positive, the second delta function vanishes, and we are left with the result we know,

$$G(\boldsymbol{r}, t) = \frac{1}{r}\delta(t - r/c), \tag{17}$$

and you can check the dimensions to see that they are correct.

In $d = 2$ the result is somewhat different. In polar coordinates we have

$$G(\boldsymbol{r}, t) = \frac{1}{(2\pi)^2} \int k\, dk\, d\theta\, e^{ikr\cos\theta} \frac{4\pi}{k} \sin kt. \tag{18}$$

The angular integral gives a Bessel function, according to

$$\int_0^{2\pi} d\theta\, e^{iz\cos\theta} = 2\int_0^\pi d\theta\, \cos(z\cos\theta) = 2\pi J_0(z). \tag{19}$$

Thus

$$G(\boldsymbol{r}, t) = 2\int_0^\infty dk\, J_0(kr)\sin kt. \tag{20}$$

This is a pretty expression but it doesn't tell us much. Another way to calculate the two-dimensional Green function is to integrate the three-dimensional solution (Jackson, Problem 6.1). We return to Eq. (1), with $d = 3$, and integrate over z' in order to eliminate the third dimension:

$$\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}\right) G^{(2)}(x, y, t; x', y', t') = 4\pi\delta(x - x')\delta(y - y')\delta(t - t'). \tag{21}$$

Here we have defined

$$G^{(2)}(x, y, t; x', y', t') = \int_{-\infty}^{\infty} dz'\, G^{(3)}(\boldsymbol{r}, t; \boldsymbol{r}', t') \tag{22}$$

in terms of the three-dimensional solution $G^{(3)}$ that we derived above. [$G^{(3)}$ is a function of $\boldsymbol{r} - \boldsymbol{r}'$ and we have integrated over z'; that is why $G^{(2)}$ cannot depend on z, and why $\partial^2/\partial z^2$ has disappeared from Eq. (21).] According to Eq. (21), $G^{(2)}$ is precisely the two-dimensional Green function. Equation (22) says that we can formulate the two-dimensional problem as a superposition of $3d$ waves emitted by a line source on the z axis.

The integral is not hard to perform. We go to cylindrical coordinates and write $\rho^2 = x^2 + y^2$; we also set $x' = y' = t' = 0$ by translation. Inserting the explicit solution for $G^{(3)}$, we have $(c = 1)$

$$G^{(2)}(x, y, t) = \int_{-\infty}^{\infty} dz'\, \frac{1}{\sqrt{\rho^2 + z'^2}}\, \delta\left(t - \sqrt{\rho^2 + z'^2}\right). \tag{23}$$

The argument of the delta function is $f(z') = t - \sqrt{\rho^2 + z'^2}$, with derivative

$$|f'(z')| = \frac{z'}{\sqrt{\rho^2 + z'^2}} = \frac{z'}{t}. \tag{24}$$

The delta function imposes $t = \sqrt{\rho^2 + z'^2}$ or $z' = \sqrt{t^2 - \rho^2}$; note that the $3d$ solution imposes $t > r > \rho$ always. The result of integration is then

$$G^{(2)}(x, y, t) = \frac{1}{z'} = \frac{1}{\sqrt{c^2t^2 - \rho^2}} \tag{25}$$

for $\rho < ct$, and zero for $\rho > ct$.

The $d = 3$ solution is nonzero only on the light cone $r = ct$, where it has a delta-function singularity. The $d = 2$ solution, on the other hand, is nonzero everywhere inside the light cone and has a (square-root)$^{-1}$ singularity at we approach $\rho = ct$.