The Formal and Harmonic Structures of *Linaia-Agon*

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Abstract

Linaia-agon is a piece of music written by Iannis Xenakis for low brass trio (horn, trombone, and tuba); and the last of three works to make use of game theory—those previous being *Duel* (1959) and *Stratégie* (1959-62). These three works fall under the category of Xenakis' music known as "Strategic Music", which involves a dialectical structure where "[a certain] party's move [will] influence and condition that of the other"[1], which in turn conditions each subsequent move in the progression of the piece (or section at hand). This concept of dialectical discourse as applied to *linaia-agon* is identical; the dialogue between the different players generates certain aspects of the piece. In the context of aleatory, this musical discourse entails that certain details in the performance of the piece are unknown to the composer by design—and it is these details (and their respective aesthetic implications) that we will discuss in the following analysis.

The work is programmatic, and narrates the legend of Apollo striking down Linus ("the celebrated musician"[2]) for challenging his skill in music. However, *linaia-agon* offers a mathematical chance to Linus or "Linos", despite the outcome of this programmatic origin.

The following analysis will explore the formal organizations of *linaia-agon* in detail, analyze the pitch and motivic content, and culminate with the application of the analytical data toward a practical performance environment.

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[1]Xenakis, Iannis. *Formalized Music: Thought and Mathematics in Composition.* Bloomington Indiana: Indiana University Press, 1971, 111.
[2]Xenakis. Published score. Salabert 1972.

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1 Probabilistic Construct in the Aleatoric Formal Structure of *Linaia-Agon*

1.1 General Formal Organization

The formal construction of *linaia-agon* is variable from performance to performance in a general sense but does contain certain prerequisite interdependencies, as well as certain pre-determined organizations. There are contained within the score eight discrete sections of music (hereafter referred to as “generative constituents”) whose contents are notated exactly with the exception of local gestural figurations such as irregular and/or indeterminate modulations of pitch and dynamic. The form of the piece as heard in performance, however, has three main formal sections—those being an introduction and conclusion that are determined for performance, and a middle section that is aleatoric in form; being constructed by the discourse initiated by the dialectical structure in each of those generative constituents involved. All of the generative constituents are listed here in the order they appear in the score[3]:

1. "Linos Against Apollo"
2. "Choice of Combats"
3. "Combat α"
4. "Combat β"
5. "Combat γ"
6. "Inbetween the α, β, γ Combats"
7. "Destiny Suspens"
8. "Victory Chant and Requiem"

As far as general interdependencies and pre-determined elements in the generative constituents, there are a few:

1. The piece begins invariably with "Linos Against Apollo", which functions both toward the end of exposing the various tactics that are to be used throughout the piece (each tactic being an audibly distinct textural motive) and toward the end of introducing the inevitable conflict between Linos and Apollo—the trombone, and the tuba and/or horn, respectively.
2. The "Choice of Combats" is played next, and determines the location, duration, and possible recurrence of the next three generative constituents, combats α, β, and γ.
3. The next generative constituent is "Inbetween the α, β, γ Combats", which would seem to be eligible for interpolation at any arbitrary or contrived location between the previous system of combats α, β, and γ; with the instructions in the score saying, "some of the following [interference] beats may be interpolated inbetween the combats α, β, γ."[4]
4. The "Destiny Suspens" and "Victory Chant and Requiem" finally conclude the narrative.

1.2 Game Theory

The indeterminate elements in *linaia-agon* that are the focus of this analysis involve game theory, which is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers."[5] Specifically, the piece makes use of the two-person zero-sum game. This type of game entails a competitive situation where the gain of one member equals the loss of the other. The members of this game play by alternating "moves"—or choices of tactic—that result in various numeric amounts of loss or gain. These possible gains and loses are indicated on a matrix where each pair of

[3]Ibid.
[4]Ibid.
[5]Myerson, Roger B. *Game theory.* Harvard university press, 2013, 1.

gain and loss for each respective player is located at the intersection of the two tactics chosen by the consecutive moves of each respective player. Two-person zero-sum game matrices are integral to those generative constituents in *linaia-agon* that are aleatoric in form; and contextually, the "decision-makers" who navigate these matrices are the personae of Linos and Apollo.

Now, logically there exists an optimal strategy for each player in finite games of this sort, involving a mathematical approach that either player might utilize to minimize their possible losses. For a two-person zero-sum game (and numerous others), this optimal strategy is approached by means of the minimax theorem. Xenakis explains the process as relevant to *Duel*—which here serves as functionally identical to the process as relevant to *linaia-agon*:

> 1. The fundamental theorem of game theory (the "minimax theorem") is that the minimum score (maximin) corresponding to *X's* optimum strategy is always equal to the maximum score (minimax) corresponding to *Y's* optimum strategy.
> 2. The calculation of the maximin or minimax value, just as the probabilities of the optimum strategies of a two-person zero-sum game, comes down to the resolution of a pair of dual problems of linear programming (dual simplex method).[6]

The dual simplex method involves a series of calculations whose details are irrelevant to our analysis—but the results are integral, as they determine the probabilities that might be used by either player attempting to optimize their respective strategies. In the context of *linaia-agon*, these probabilities are represented literally in the score as the proportional chances for each tactic to be chosen—located in the margins of each matrix. While a practical approach to the performance of the piece can either involve the use of these proportional chances or alternatively pure intuitive strategic efforts on the part of the performers (both of which are valid approaches), it is the proportional chances for each tactic that reveal the statistical construct in the aleatoric form of *linaia-agon*.

1.3 An Intricate Aleatory

At the most basic level, each two-person zero-sum game matrix within the score—as previously mentioned—includes numbers in the margins that indicate the unique strategic advantage of each choice of tactic for either player. The narrative of the piece is one of conflict, and—again, as previously mentioned—this actual chance is provided mathematically by the matrices.[7] Therefore, the aleatoric element of the piece is programmatically representative of the presumably uncertain outcome of the actual conflict between the characters, quantified mathematically as the particular strengths and weaknesses of any tactic. As such, it is necessary to explore these conflicts in detail, and the probabilities of each tactic being played in each player's respective optimal strategy.

[6] Xenakis, Iannis. *Formalized Music: Thought and Mathematics in Composition.* Bloomington Indiana: Indiana University Press, 1971, 118.

[7] Xenakis. Published score. Salabert 1972.

Table 1: The “Choice of Combats” matrix

		Apollon Tuba			
		α	β	γ	
Linos Trombone	α	-3	-8	7	4
	β	2	2	-3	10
	γ	-8	12	2	1
		2	1	2	

Table 2: The “Choice of Combats” matrix with fractions and probabilities

		Apollon Tuba				
		α	β	γ		
Linos Trombone	α	-3	-8	7	4/15	$0.2\overline{6}$
	β	2	2	-3	10/15	$0.\overline{6}$
	γ	-8	12	2	1/15	$0.0\overline{6}$
		2/5	1/5	2/5		
		0.40	0.20	0.40		

Let us examine the “Choice of Combats”. Again, this generative constituent will determine the location, duration, and possible recurrence of the next three generative constituents combats α, β, and γ (Table 1). We notice at this point that each tactic is here labeled α, β, or γ. This will later correspond with the sequencing and proportional length of each generative constituent α, β, and γ during the following section of the piece, but currently each tactic is represented by a corresponding note—these are indicated in the score—to be played by the trombone or the tuba in alternation. Say for example that Apollo (the tuba) begins the game with the choice of tactic β—we will assume at this point, for the sake of illustration, that the performers are electing to navigate the matrix intuitively. Linos (who is in the pursuit of positive numbers) at this point will see the immediate ramifications of each choice α, β, and γ at the intersection of these and Apollo’s indicated tactic β. These are the numbers -8, 2, and 12, respective to the tactics α, β, and γ. Linos must then make a logical choice of tactic based on the previous choice of Apollo; perhaps selecting the tactic γ, to which Apollo must respond. In this way, the players will alternate moves thinking not only locally but globally—to ensure that the strategic efforts will maximize their respective gains, while minimizing their respective losses.

Toward the end of a mathematical approach, however, the probabilities required for the execution of each player’s optimal strategy are indicated in the margins of the “Choice of Combats” matrix. These numbers (4, 10, 1; and 2, 1, 2 for the α, β, and γ tactics of Linos and Apollo, respectively) can be looked at as both fractions and probabilities for more clarity (Table 2). While it might have been less than readily ap-

Table 3: Deriving the joint probabilities

$$[\alpha = 0.2\overline{6},\, \beta = 0.\overline{6},\, \gamma = 0.0\overline{6}] + [\alpha = 0.40,\, \beta = 0.20,\, \gamma = 0.40]$$
$$= [\alpha = 0.\overline{6},\, \beta = 0.8\overline{6},\, \gamma = 0.4\overline{6}]$$

and

$$0.5 \times [\alpha = 0.\overline{6},\, \beta = 0.8\overline{6},\, \gamma = 0.4\overline{6}]$$
$$= [\alpha = 0.\overline{3},\, \beta = 0.4\overline{3},\, \gamma = 0.2\overline{3}]$$

parent from studying the seemingly arbitrary numbers within the matrix, the strategic weightings for each choice of tactic become much more obvious with the introduction of the fraction and decimal conversions. As we can see, the probability of Linos playing the tactic β is $0.\overline{6}$, while tactic α is only $0.2\overline{6}$ and γ is only $0.0\overline{6}$.

At this point, let us remember the formal structure of *linaia-agon*. The "Choice of Combats" functions to generate a sequence of α's, β's, and γ's that will actually dictate the sequential organization and proportional length of the following generative constituents "Combat α", "Combat β", and "Combat γ". In order to determine the theoretically average proportional length of each combat, we will need to derive the joint probability for the occurrence of each tactic in the "Choice of Combats". We determine this joint probability for tactics α, β, and γ by adding the corresponding probabilities for each tactic and then dividing the result by two (Table 3). These probabilities at which we have arrived ($\alpha = 0.\overline{3}$, $\beta = 0.4\overline{3}$, $\gamma = 0.2\overline{3}$) are integral to the formal construct of *linaia-agon*, as they reveal the global probabilities for the occurrence of each of the combats α, β, and γ in the next musical section.

According to the instructions given in the score, each combat should have "a duration proportional to its frequency as calculated in the preceding succession [the Choice of Combats]"[8], which will functionally proliferate the durational relationships of the combats—as the total probabilities are now being applied to both the duration of each event, and the total number of possible repetitions of the event in the sequence. Therefore, the result of the "Choice of Combats" has quite a considerable effect upon the formal structure of the subsequent section (the succession of combats α, β, and γ).

1.4 The "Choice of Combats" in Detail

At this point let us examine a hypothetical succession of tactics from the "Choice of Combats". These tactics will be generated randomly, but with the specific probabilities that we have uncovered during the analysis of the "Choice of Combats" two-person zero-sum game matrix. I have chosen arbitrarily to begin with Apollo, and have chosen an arbitrary number of tactics. Here is the hypothetical succession:

Stages:	1	2	3	4	5	6	7	8	9	10	11	12	13
Apollo:	α		β		α		α		γ		β		γ
Linos:	α			β		β		β		γ		α	

[8]Ibid.

Let us now analyze our succession in order to compare its specific proportions to those prescribed by the game matrix. First, let us reveal the proportional frequency of each tactic:

α	β	γ	Total:
5	5	3	13

By dividing each of the α, β, and γ values by the total, we obtain statistical values that can be compared to the joint probabilities. The degree that these statistical values reflect the theoretical joint probabilities is indicative of the accuracy that number of stages might yield.

Tactic:	Joint Probability:	Hypothetical Succession:
α	$0.\overline{3}$	0.3846...
β	$0.4\overline{3}$	0.3846...
γ	$0.2\overline{3}$	0.2307...

We can see clearly that our hypothetical succession has generated a comparable distribution of tactics. But we must remember the implications of the arbitrary choices we make. Take for example the fact that I chose for the previous calculations an arbitrary number of tactics (13). This choice actually has a noticeable consequence on how accurately the practical succession reflects the average joint probability. For instance, let us choose for a moment to only allow 5 total tactics or the "Choice of Combats" (still beginning with Apollo for the sake of simplicity):

<table>
<tr><td>Stages:</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>Apollo:</td><td colspan="2">α</td><td colspan="2">α</td><td>α</td></tr>
<tr><td>Linos:</td><td colspan="3">α</td><td colspan="2">β</td></tr>
</table>

In this case, we see first that there is no tactic γ—meaning that the entire "Combat γ" would be completely omitted from the following discourse. In the pursuit of an appropriate interpretation we must avoid an outcome such as this. But as far as our discussion about the accuracy of the succession in representing the average joint probability, we now see that these probabilities are no longer recognizable with $\alpha = 0.8$, $\beta = 0.2$, and γ of course $= 0.0$. On the other hand, we can observe that a large succession will result in a more accurate representation than even the first. Take for instance the succession: [β β γ β γ β α β α β γ α γ β α α α β γ β γ γ α γ β α α]. The same processes as the previous successions generated this succession and the proportions are: 9 α's, 10 β's, and 8 γ's. The probabilities are therefore $\alpha = 0.\overline{3}$, $\beta = 0.\overline{370}$, and $\gamma = 0.\overline{296}$; and these are much closer to the average joint probability. However, a balance should be found, as a succession of almost thirty tactics would generate a very long rendition of the piece—but these and other performance considerations will be addressed later on in this analysis.

Now that the global form of the section is coming into focus, let us explore further the details of construction of the combats α, β, and γ.

1.5 "Combat α"

"Combat α" is scored for tuba (Apollo) and trombone (Linos). The musical discourse is composed of a total of four tactics for each player represented as textural motives.

Table 4: The "Combat α" matrix

		Tuba				
		≁	∵	/	silence	
Trombone	≁	-2	0	0	0	9
	∵	1	0	-3	-2	3
	/	0	-1	+1	-2	8
	silence	0	-1	-2	+1	7
		5	7	3	3	$V = -5/9$

1. The tactic "≁"[9], representing textural motives with irregular "trembling variation of pitch"—in the score initially as "quilisma de hauteur irrégulier", from French translated to "quilisma [here referencing a trembling trill or tremolo] of an irregular height".[10]
2. The tactic "∵", built of the motive containing irregular staccato structures and "quilisma d'intensité irrégulier" or (from French translated) "quilisma of an irregular intensity".[11]
3. The tactic "/" , representing the textural motive composed of various glissandi.
4. Finally the tactic "silence", representing the tactic of literal silence.

These textural motives are the same for both instruments, and are played in order to indicate a particular choice of tactic. In Figure 1 is an excerpt from "Combat α" displaying the first ten measures in each instrument's choice of tactic ("silence" is omitted from the notation). In the trombone part, we see also ascending or descending hooks representing a pizzicato glissando in the indicated direction. The entirety of each tactic is composed of textural material of similar construction.

These textural structures are used in order to indicate a particular tactic in the combat matrix. Table 4 shows the matrix as it appears in the score. Just as in the last instance, this matrix has included the proportions for the execution of the optimal strategy in the margins. Also, we are now presented with additional information, the game value ($V = -5/9$). The game value for a matrix shows the statistical bias it holds for a particular opponent, which influences the outcome of the game. For now, however, let us analyze the proportions in the margins of the matrix as probabilities:

	≁	∵	/	silence
Trombone:	$0.\overline{3}$	$0.\overline{1}$	$0.\overline{296}$	$0.\overline{259}$
Tuba:	$0.2\overline{7}$	$0.3\overline{8}$	$0.1\overline{6}$	$0.1\overline{6}$

We see that each instrument has its own unique strategy for the game, represented as the probabilities for each choice of tactic. But in order to determine the total probability for each textural structure, we must as before determine the joint probability (Table 5). Now that we have determined that ≁, ∵, /, and silence are weighted to the probabilities

[9] This non-tilde symbol (≁) is not identical to the symbol in the score, but it will suffice for our purposes.
[10] Xenakis. Published score. Salabert 1972.
[11] Ibid.

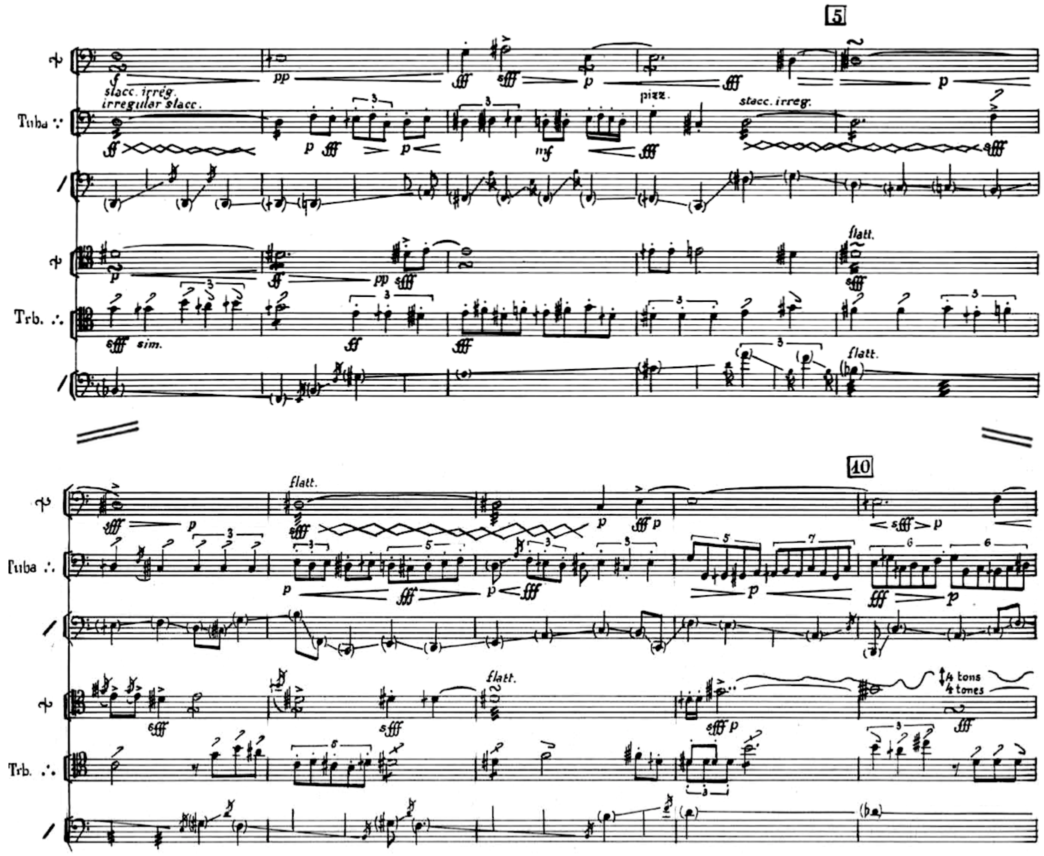

Figure 1: Detail from the published score, "Combat α", mm. 1-10

Table 5: Deriving the joint probabilities for "Combat α"

$$[\nsim = 0.\overline{3},\ \because = 0.\overline{1},\ / = 0.\overline{296},\ \text{silence} = 0.\overline{259}] + [\nsim = 0.2\overline{7},\ \because = 0.3\overline{8},\ / = 0.1\overline{6},\ \text{silence} = 0.1\overline{6}]$$
$$= [\nsim = 0.6\overline{1},\ \because = 0.5,\ / = 0.462\overline{962},\ \text{silence} = 0.425\overline{925}]$$

and

$$0.5 \times [\nsim = 0.6\overline{1},\ \because = 0.5,\ / = 0.462\overline{962},\ \text{silence} = 0.425\overline{925}]$$
$$= [\nsim = 0.30\overline{5},\ \because = 0.25,\ / = 0.231\overline{481},\ \text{silence} = 0.212\overline{962}]$$

Table 6: The "Combat β" matrix

		Horn				
		≁	∵	/	silence	
Trombone	≁	1	-2	-2	3	92
	∵	1	-1	0	-2	78
	/	0	4	-2	-2	89
	silence	-2	-1	5	1	76
		161	61	72	41	$V = 0.054$

reached in Table 5, we can begin to understand the probabilistic hierarchy governing the constituent in a general sense.

1.6 "Combat β"

In "Combat β", the tactics are represented by the same textural motives described for the previous combat, however, they are now represented by the horn and trombone. One difference is the inclusion of an alternate passage of music for the horn player when playing tactic "/". This alternate passage is more difficult to play. Other than the inclusion of this alternate passage, the differences are ones of detail—that is, the details of the construction of each motive are different, but the basic motive conveyed is the same (see Table 6). As we can see, the format is identical to the previous matrices, and now we have a game value of ($V = 0.054$). Let us now analyze the proportional relationships as probabilities and subsequently determine the joint probability of each tactic being chosen. The calculation of the joint probability contributes toward the discovery of the probabilistic hierarchy of "Combat β".

	≁	∵	/	silence
Trombone:	0.2746...	0.2328...	0.2656...	0.2268...
Horn:	0.4805...	0.1820...	0.2149...	0.1223...
Joint Probability:	0.3775...	0.2074...	0.2402...	0.1745...

1.7 "Combat γ"

Finally, there is "Combat γ", in which the tactics are represented by all three instruments. The trombone represents Linos while Apollo is represented by the horn and tuba simultaneously (see Figure 2). The tactics for "Combat γ" are now labeled in the score "I", "II", "III", and "silence". The choice of tactic for Apollo must be made in agreement by both the tuba and horn and, for the first time, the tactics are not identified solely by their textural construction. As the textural construction (i.e. irregular pitch/dynamic modulation, staccato constructions, or structures of glissandi) is considerably less unique for each respective tactic, we must rely principally on the register. The identifying textural characteristics and register are here listed:

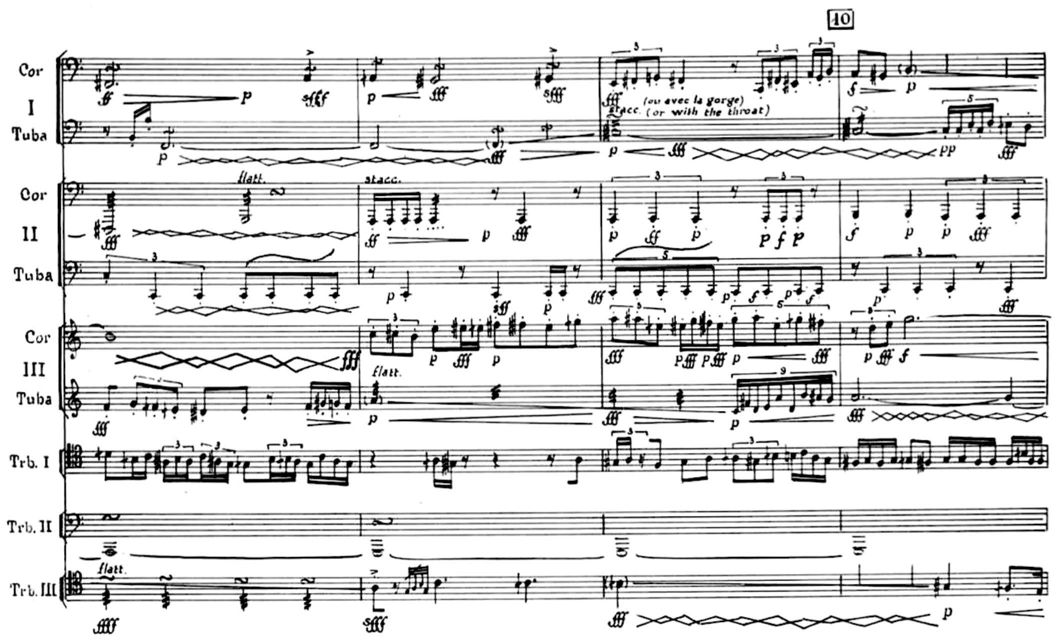

Figure 2: Detail from the published score, “Combat γ”, mm. 7-10

Table 7: The "Combat γ" matrix

		Horn and Tuba				
		I	II	III	silence	
Trombone	I	2	-6	0	1	109
	II	-3	-3	-2	2	130
	III	-5	-1	3	-4	11
	silence	-3	0	-3	-5	276
		132	235	68	91	$V = -2.03$

1. Tactic "I" for Linos (the trombone) is composed of staccato passages located in the range from F3 and G♯4.[12]
2. Tactic "II" for Linos is more texturally varied, but is within the range from E1 to C3.
3. Tactic "III" for Linos is again texturally variable, but with within the range from A♯2 to C♯4.
4. Tactic "I" for Apollo (horn and tuba) is texturally variable, within the range from A1 to C♯4.
5. Tactic "II" for Apollo is texturally variable, within the range from F1 to D3.
6. Tactic "III" for Apollo is texturally variable, within the range from C4 to F5.

The tactics for each respective character occupy certain registers, and while these registers do overlap in certain cases, the general range of each tactic per character is unique.

The two-person zero-sum game for "Combat γ" is, as before stated, navigated by the entire trio simultaneously, with the horn and tuba versus the trombone. The choice of tactic for each are indicated with the choice of music contained within the corresponding section of music ("I", "II", "III", and "silence"). See Table 7. The game value for "Combat γ" as we can see is ($V = -2.03$). At this point, we can analyze the proportions for each tactic, and determine the joint probability of each tactic:

	I	II	III	silence
Trombone:	0.2072...	0.2471...	0.0209...	0.5247...
Horn and Tuba:	0.2509...	0.4467...	0.1292...	0.1730...
Joint Probability:	0.2290...	0.3469...	0.0750...	0.3488...

Interestingly, we now see that the tactics for motive "III" and "silence" would seem to have less than ideal probabilities. In fact, it might seem that the probabilities are switched entirely. But upon further inspection, we discover a particularly logical and practical use of the probability hierarchy created by the game matrix. First, the seemingly high probability of the tactic "silence" could be explained possibly by the additional

[12]In the score, some of these ranges are defined in more detail by microtonal accidentals, but due to the lack of quarter-tone accidentals within LaTeX fonts I have generalized to the nearest half-step.

complexity of this generative constituent due to both its more texturally variable nature and the fact that all three instruments are included. This hypothesis is only strengthened by the fact that the probability for silence as voiced by the trombone constitutes about 75.2% of the total probability for the tactic, meaning that it is more probable that horn and tuba tactics are heard unobstructed by the tactics of the trombone. Second, the material for motive "III" for the tuba part in particular is exceedingly difficult to play because of the unusually high range, which might explain the relatively low probability of motive "III" in the horn and tuba part. Of course, this does not explain the even lower probability for motive "III" in the trombone part; but this might very well be a side effect from the previous considerations.

1.8 Combats α, β, and γ in General

Now that we have a sense of the game theoretic structures and other specificities for each combat, let us examine the other aleatoric elements in Combats α, β, and γ. There are several aleatoric elements not involving game theory—some implied, others explicitly stated (in the published score):

1. Back in the "Choice of Combats", the instructions in the score indicate only that the tactics are to be "played for at least 4 seconds and without interruption during a sufficient time so as to obtain a succession of α's, β's, [and] γ's"—not specifying an exact number of moves, and introducing a great deal of variety in acceptable interpretations of the total duration of the piece.
2. The durations of each combat α, β, and γ last proportionally to their frequency as calculated in the "Choice of Combats"—however, there is no specific instruction as to the general duration on which these proportional lengths are based.
3. The same "adversary" who proposed the tactic initially in the "Choice of Combats" begins each corresponding combat α, β, and γ—and even though each proposition of tactic is alternated during the "Choice of Combats", the choice of player to begin the discourse is up to the discretion of the performers.
4. The tempi are independent with the exception of "Combat γ" where Apollo is represented by the tuba and horn simultaneously.
5. The tempi may fall anywhere within half-note equals 48 beats per minute and half-note equals 96 beats per minute.
6. As far as pitch content, transpositions are permitted. In the instructions in the score Xenakis lists the formula for transposition "(k $\times$ 12 $\pm$ 3/2 r) semi-tones for $\{$r, k$\}$ = 0, $\pm$1, $\pm$2, ..." which will allow in theory any transposition by quarter tones in any possible octave.
7. The selection of a tactic will entail the selection of an arbitrary or contrived portion of the notated music for that tactic. Furthermore, it is permitted to select a single note or melodic pattern either to represent the tactic or to be omitted. The only requirement is that the textural motives for the tactic at hand remain clear, and that the repetition of structural patterns be avoided unless actually notated by the score.

8. Finally, the dynamics may be transposed relative to one another at any given moment—e.g. *mf* → *f* can be transposed to *f* → *ff*.

These aleatoric elements all deal with performance logistics involving the ensemble and the individuals. While presumably there might be some amount of aesthetic design behind any particular tuba player?s choice to take a section of music or a particular note down an octave, for example—it might possibly be toward the end of conserving his or her energy in preparation for the formidable "Destiny Suspens". Likewise, particular dynamics might be transposed to lower intensities to the same end. Also, the independence of tempi is a logical performance consideration, as the various motivic structures function not as a cooperative simultaneity, but as a representation of the various tactics that each combatant might utilize against the other. The implications of these logistical considerations are a significant factor in the interpretation and the performance of the piece.

2 Performance Considerations

2.1 An Intuitive versus Mathematical Approach

As stated earlier, a practical approach toward an interpretation of these matrices can involve both the literal "playing" of the game matrix by the two performers involved and the approximation mathematically by means of probabilistically selected choices of tactic. In *Formalized Music*, Xenakis by experimentation demonstrates that these two methods of interpretation are both acceptable in practice.[13] This is due to the fact that an exercise in choosing each move intuitively involves consideration for executing the best strategy. In a two-person zero-sum game, the best strategy is quantified mathematically as the optimal strategy. Therefore, as a person intuitively navigates the game matrix, his or her intuitive efforts approach the optimal strategy, and these two practical approaches produce satisfactory results.

For further proof of this, let us investigate the experiment of Xenakis in *Formalized Music*. First, there is the result of intuitive simulation of the game matrix. In Figure 3 we observe the result of this simulation from the game matrix of *Duel*. This graph shows the sequential progression of tactics alternating between opponents "X" and "Y", with the game value of 2.6 points in favor of opponent "X". Let us compare this to the graph of sequential progression derived mathematically from the same game matrix (Figure 4). Here we see that the game value calculated is 2.7. It is then applied that as the game values are very similar (2.6 and 2.7), the "sonic processes derived from the two experiments are, moreover, satisfactory."[14] In summary, the game matrices in *linaia-agon* may be navigated intuitively or mathematically.

[13] Xenakis, Iannis. *Formalized Music: Thought and Mathematics in Composition.* Bloomington Indiana: Indiana University Press, 1971, 118.

[14] Ibid.

Stages:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Cond. X	I	III		I		VI		I		III		VI		IV		III		III		IV	
Cond. Y	IV		III		VI		III		I		VI		III		V		II		III		IV
Scores:	2	4	1	4	2	4	1	4	2	4	1	4	1	4	2	3	2	2	1	4	2

Game value: 52/20 = 2.6 points in X's favor.

Figure 3: Simulation of game matrix from *Duel*, p. 117, *Formalized Music*

Stages:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Cond. X	I	VI		VI		II		I		II		V		IV		I		V		V	
Cond. Y	VI		VI		V		III		I		IV		VI		V		IV		VI		III
Scores:	2	4	4	4	2	3	2	4	2	3	3	3	2	2	2	3	2	3	2	2	3

Game value: 57/21 = 2.7 points in X's favor.

Figure 4: Probabilistic realization of the game matrix from *Duel*, p. 118, *Formalized Music*

2.2 The Law of Large Numbers

Let us assume that a mathematical approach is pursued—this might be one of the more manageable approaches toward an interpretation. In this case, we will need to consider the law of large numbers (LLN) in order to maximize our chances for a structurally representative rendition of the piece. There exist two forms for the LLN: the weak law and the strong law. The weak law asserts that as the number of experiments approaches infinity, the probability that the result (or sample average) deviates from the expected value approaches zero:

$$\overline{X}_n \xrightarrow{P} \mu \text{ when } n \to \infty. \tag{1}$$

That is, for any positive number ε,

$$\lim_{n\to\infty} \Pr(|\overline{X}_n - \mu| > \varepsilon) = 0. \tag{2}$$

The strong law, on the other hand, asserts that as the number of experiments approaches infinity, the probability converges almost surely to the expected value:

$$\overline{X}_n \xrightarrow{a.s.} \mu \text{ when } n \to \infty. \tag{3}$$

That is,

$$\Pr(\lim_{n\to\infty} \overline{X}_n = \mu) = 1. \tag{4}$$

Toward an interpretation of *linaia-agon*, both of these forms of the LLN are appropriate. We must apply the law to the game matrices in the piece; as each "experiment" in this case is equivalent to the indication of a tactic on any given game matrix. Of course, it would hardly be considered a reasonable interpretation if we were to continue the piece for a functional eternity (even though this would according to the LLN render the probabilities as judiciously as is possible); so we therefore need to search for a balance.

Let us now analyze the probabilities combinatorially so that we might arrive at a satisfactory general rule for the interpretation. In order to do this, we will analyze each possible sequence of tactics for n choices of tactic. As each opponent has a unique set of probabilities, we will need to analyze each separately and parallel. Let us consider the "Choice of Combats" matrix. As we have already determined, the probabilities corresponding to the optimal strategies are:

	α	β	γ
Trombone:	0.26	0.6	0.06
Tuba:	0.40	0.20	0.40

We can use these probabilities to create a table of sets (of the size $2n$) of α, β, and γ in any possible configuration. This table will include the every set, as well as the unique probability for that set's occurrence in the two-person zero-sum game, and the probability distribution that results from the set. From this table of computations (Table 8), we will determine a reasonable number of turns back and forth.

In this table, we see that the most probable outcomes tend to be reflective of the optimal strategies of each respective instrument—and furthermore the resultant joint probability—showing how even at the small scale of $n = 4$, the likelihood of significant deviation is quite small due to the propagation of the optimal strategies through combinatorial probabilities. All of this considered, one would not need to greatly exceed $n = 4$ toward the pursuit of an interpretation of the "Choice of Combats" and other game theoretic sections.

2.3 The Performance Environment

Of course, combats α, β, and γ each have four tactics, therefore complicating the combinatoric analysis above explored, but due to the fact that each of these generative constituents are to be played multiple times, and for a greater length than any of the tactics in the "Choice of Combats" this should be less than troublesome—the additional length should take care of the probabilistic structures without the need for maintenance or preparation.

A final note toward the end of a spontaneous and practical performance environment would be the exploration of methods of generating the probabilities needed for the piece. One relatively simple method involves a network of random drawings. That is to say, for each matrix in the score there should be two boxes: in which are contained objects (balls, cards, etc.) marked to indicate the various tactics at the various probabilities. This method would likely still require a "referee" and "accountant" in order to keep track of the series of tactics dictating both the selection of the combats and the details therein.

Alternatively, one could make use of pseudo-random number generators and program an application that indicates the tactics and changes of combat at any given moment to the performers. While this would likely be tedious to program, it would

Table 8: The Table of Combinations for $n = 1, 2, 3, 4$

Apollo (Tuba)					Linos (Trombone)				
				$n=1$					
Set:	Prob.:	Prob. Distribution:			Set:	Prob.:	Prob. Distribution:		
		α	β	γ			α	β	γ
α	0.4	1	0	0	α	$0.2\overline{6}$	1	0	0
β	0.2	0	1	0	β	$0.\overline{6}$	0	1	0
γ	0.4	0	0	1	γ	$0.0\overline{6}$	0	0	1
				$n=2$					
		α	β	γ			α	β	γ
$\alpha\ \alpha$	0.16	1	0	0	$\alpha\ \alpha$	0.7	1	0	0
$\alpha\ \beta$	0.08	0.5	0.5	0	$\alpha\ \beta$	0.18	0.5	0.5	0
$\alpha\ \gamma$	0.16	0.5	0	0.5	$\alpha\ \gamma$	0.018	0.5	0	0.5
$\beta\ \beta$	0.04	0	1	0	$\beta\ \beta$	0.45	0	1	0
$\beta\ \gamma$	0.08	0	0.5	0.5	$\beta\ \gamma$	0.04	0	0.5	0.5
$\gamma\ \gamma$	0.16	0	0	1	$\gamma\ \gamma$	0.004	0	0	1
				$n=3$					
		α	β	γ			α	β	γ
$\alpha\ \alpha\ \alpha$	0.064	1	0	0	$\alpha\ \alpha\ \alpha$	0.019	1	0	0
$\alpha\ \alpha\ \beta$	0.032	$0.\overline{6}$	$0.\overline{3}$	0	$\alpha\ \alpha\ \beta$	0.047	$0.\overline{6}$	$0.\overline{3}$	0
$\alpha\ \alpha\ \gamma$	0.064	$0.\overline{6}$	0	$0.\overline{3}$	$\alpha\ \alpha\ \gamma$	0.0047	$0.\overline{6}$	0	$0.\overline{3}$
$\alpha\ \beta\ \beta$	0.016	$0.\overline{3}$	$0.\overline{6}$	0	$\alpha\ \beta\ \beta$	0.118	$0.\overline{3}$	$0.\overline{6}$	0
$\alpha\ \beta\ \gamma$	0.032	$0.\overline{3}$	$0.\overline{3}$	$0.\overline{3}$	$\alpha\ \beta\ \gamma$	0.012	$0.\overline{3}$	$0.\overline{3}$	$0.\overline{3}$
$\alpha\ \gamma\ \gamma$	0.064	$0.\overline{3}$	0	$0.\overline{6}$	$\alpha\ \gamma\ \gamma$	0.0012	$0.\overline{3}$	0	$0.\overline{6}$
$\beta\ \beta\ \beta$	0.008	0	1	0	$\beta\ \beta\ \beta$	0.296	0	1	0
$\beta\ \beta\ \gamma$	0.016	0	$0.\overline{6}$	$0.\overline{3}$	$\beta\ \beta\ \gamma$	0.0296	0	$0.\overline{6}$	$0.\overline{3}$
$\beta\ \gamma\ \gamma$	0.032	0	$0.\overline{3}$	$0.\overline{6}$	$\beta\ \gamma\ \gamma$	0.00296	0	$0.\overline{3}$	$0.\overline{6}$
$\gamma\ \gamma\ \gamma$	0.064	0	0	1	$\gamma\ \gamma\ \gamma$	0.000296	0	0	1
				$n=4$					
		α	β	γ			α	β	γ
$\alpha\ \alpha\ \alpha\ \alpha$	0.026	1	0	0	$\alpha\ \alpha\ \alpha\ \alpha$	0.005	1	0	0
$\alpha\ \alpha\ \alpha\ \beta$	0.013	0.75	0.25	0	$\alpha\ \alpha\ \alpha\ \beta$	0.013	0.75	0.25	0
$\alpha\ \alpha\ \alpha\ \gamma$	0.026	0.75	0	0.25	$\alpha\ \alpha\ \alpha\ \gamma$	0.0013	0.75	0	0.25
$\alpha\ \alpha\ \beta\ \beta$	0.0064	0.5	0.5	0	$\alpha\ \alpha\ \beta\ \beta$	0.032	0.5	0.5	0
$\alpha\ \alpha\ \beta\ \gamma$	0.013	0.5	0.25	0.25	$\alpha\ \alpha\ \beta\ \gamma$	0.0032	0.5	0.25	0.25
$\alpha\ \alpha\ \gamma\ \gamma$	0.026	0.5	0	0.5	$\alpha\ \alpha\ \gamma\ \gamma$	0.00032	0.5	0	0.5
$\alpha\ \beta\ \beta\ \beta$	0.0032	0.25	0.75	0	$\alpha\ \beta\ \beta\ \beta$	0.079	0.25	0.75	0
$\alpha\ \beta\ \beta\ \gamma$	0.0064	0.25	0.5	0.25	$\alpha\ \beta\ \beta\ \gamma$	0.0079	0.25	0.5	0.25
$\alpha\ \beta\ \gamma\ \gamma$	0.013	0.25	0.25	0.5	$\alpha\ \beta\ \gamma\ \gamma$	0.00079	0.25	0.25	0.5
$\alpha\ \gamma\ \gamma\ \gamma$	0.026	0.25	0	0.75	$\alpha\ \gamma\ \gamma\ \gamma$	0.000079	0.25	0	0.75
$\beta\ \beta\ \beta\ \beta$	0.0016	0	1	0	$\beta\ \beta\ \beta\ \beta$	0.1975	0	1	0
$\beta\ \beta\ \beta\ \gamma$	0.0032	0	0.75	0.25	$\beta\ \beta\ \beta\ \gamma$	0.01975	0	0.75	0.25
$\beta\ \beta\ \gamma\ \gamma$	0.026	0	0.5	0.5	$\beta\ \beta\ \gamma\ \gamma$	0.001975	0	0.5	0.5
$\beta\ \gamma\ \gamma\ \gamma$	0.026	0	0.25	0.75	$\beta\ \gamma\ \gamma\ \gamma$	0.0001975	0	0.25	0.75
$\gamma\ \gamma\ \gamma\ \gamma$	0.026	0	0	1	$\gamma\ \gamma\ \gamma\ \gamma$	0.00001975	0	0	1

have two added benefits; the score would be maintained by the program, and the score could be monitored in progress by the audience or performers. The final score of the network of two-person zero-sum games determines the winner (Linos or Apollo). The aleatoric form is therefore a literal interpretation of the uncertain outcome of the programmatic narrative. It is the details that we have studied in this analysis, but the generalities that we will depart with—probability theory has given us an aleatory that is mathematically synonymous with external conflict; and it is this enriched area of programmatic content that has contributed greatly to the development of aleatoric music.