TREFOIL SURGERY

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Abstract. This is an expository account of a theorem of Louise Moser that describes the types of manifolds that can be constructed via Dehn surgery along a trefoil in the 3-sphere. These include lens spaces, connected sums of two lens spaces, and certain Seifert fibered spaces with three exceptional fibers. Various concepts from the topological theory of three dimensional manifolds are developed as needed.

1. Introduction

Imagine you live in some sort of three-dimensional universe. But now suppose someone (or something - maybe some sort of space worm) has carved out a tunnel in space that forms a loop. Annoyed at this you grab a loop of "space tubing" to fill in the tunnel, gluing the outside of the tube to the wall of the tunnel. Now you go back to whatever it was you were doing before, but strange things start happening. Because you weren't careful in how you did the gluing it turns out the structure of your three dimensional world has changed. Indeed, it might not even be prime[1]!

This operation, called **Dehn surgery**, is a fundamental tool in the study of three-dimensional manifolds (spaces). Our goal is to develop this theory, somewhat informally, and then use it to prove a classical theorem due to Louise Moser that describes the types of manifolds that can be derived by performing Dehn surgery on a trefoil shaped tunnel in a standard space called the three-dimensional sphere. The background material needed would normally take a couple of years of graduate level topology to master, yet the basic ideas are intuitive and visual. This material was used in shorts courses taught at Toyko Tech in January-Feburary, 2013, and Southern Illinois University July-August 2013 [12].

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[1] *Prime manifolds* will be defined later

2. Topological Manifolds

The material in this section can be found in the textbooks [5, 7]. We assume a basic familiarity with topology: open and closed sets, compactness, continuity and path connectedness. Two topological spaces X and Y are **homeomorphic** or **topologically equivalent** if there is a bicontinuous bijection $h : X \to Y$; this is denoted by $X \cong Y$. Such a function is called a **homeomorphism**. We say X can be **embedded** into Y if there is a continuous map $f : X \to Y$ such that $f : X \to f(X)$ is a homeomorphism.

An n**-dimensional manifold without boundary** M is a topological space such that for each point $x \in M$ there exists an open set containing x that is homeomorphic to an open ball in $\mathbb{R}^n$. If there are points y in M for which this fails but where there is a subset H of M containing y and a homeomorphism

$$h : H \to \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 < 1 \text{ and } x_1 \geq 0\}$$

taking y to the origin, then M is a n**-dimensional manifold with boundary**. Such points y form the **boundary** of M which is denoted ∂M. The **interior** of M is $\text{int}(M) = M - \partial M$. [2]

Examples. An n-sphere, S^n, for $n \geq 0$, is a any space homeomorphic to the unit sphere in $\mathbb{R}^{n+1}$. We will be working with S^1, S^2 and S^3. These have empty boundary. A closed n-ball, B^n, for $n \geq 1$, is any space homeomorphic to the closed unit ball in $\mathbb{R}^n$. Notice $\partial B^{n+1} \cong S^n$. For $n = 2$ we call a 2-ball a disk and denote it by D^2. Let $I = [0, 1]$. A space homeomorphic to $I \times S^1$ is called an **annulus** or a **cylinder**. A space homeomorphic to $S^1 \times S^1$ is called a **torus**, denoted T^2, while any space homeomorphic to $D^2 \times S^1$ is called a **solid torus**. We will use V to denote a solid torus although there is no standard convention. A **core** of a solid torus is a circle that maps to $(0, 0) \times S^1$ by some homeomorphism $h : V \to D^2 \times S^1$. The spaces $I \times I \times I$ and $D^2 \times I$ are 3-balls despite not being round. All of these manifolds are compact.

3. Gluing, Connected Sums and Compactification

We won't be precise in our definitions here but will proceed by examples. If we "identify" the end points of the unit interval we get a new manifold that is homeomorphic to the circle. If we take the square $I \times I$ and identify each point on the bottom edge with the point on the top edge that is above it we get a new manifold that is homeomorphic to a cylinder. We say that we have **glued** the top and bottom edges. If instead of gluing $(x, 0)$ to $(x, 1)$ we glued $(x, 0)$ to $(1 - x, 1)$ the result

[2]Manifolds are also assumed to be Hausdorff and second countable.

would be a **Möbius band**. If we glue $(x,0)$ to $(x,1)$ and $(0,y)$ to $(1,y)$, for $x,y \in I$, the result is a torus.

Exercise 1. Explain why a Möbius band is not homeomorphic to an annulus but a strip with a full (360^o) twist is.

If we take two closed disks and identify their boundaries the result is a 2-sphere. If we take two closed 3-dimensional balls, B_1 and B_2, and identify points on their boundary 2-spheres the resulting 3-manifold without boundary is a 3-sphere. See Figure 1. The identification is achieved by choosing a homeomorphism $h : \partial B_1 \to \partial B_2$ and identifying x with $h(x)$ for each $x \in \partial B_1$. It can be proven that the topological type of the result is independent of the choice of h [5].

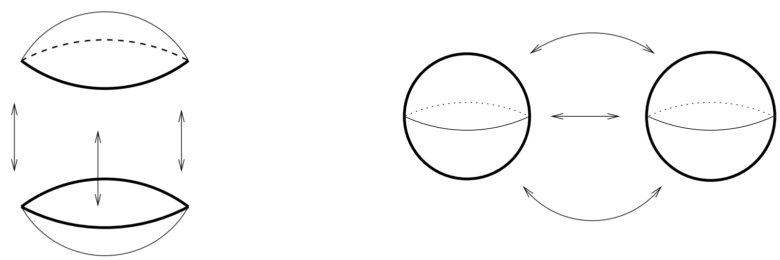

FIGURE 1. Gluing two disks gives a 2-sphere; gluing two balls gives a 3-sphere.

The 3-sphere can be constructed by gluing two solid tori. Figure 2 shows how to see this starting from gluing two 3-balls. You decompose one of the 3-balls into a solid torus and a solid cylinder (in the donut hole). We do the gluing in two steps. First glue the top and bottom disks on the cylinder to the other 3-ball. This forms a solid torus. Now glue the two solid tori together and *voila*, we have realized S^3 as the union of two solid tori.

There is another way to construct spheres that is useful. Consider the union of the real line $\mathbb{R}$ with a new point called ∞. We topologize $\mathbb{R}\cup\{\infty\}$ by choosing as the open sets all the open subsets of $\mathbb{R}$ together with sets of the form $\{\infty\} \cup O$ where $\mathbb{R} - O$ is compact. With this topology $\mathbb{R} \cup \{\infty\}$ is homeomorphic to S^1. It is called the **one point compactification** of $\mathbb{R}$. The same process can be applied to make $\mathbb{R}^2 \cup \{\infty\}$ homeomorphic to S^2 and $\mathbb{R}^3 \cup \{\infty\}$ homeomorphic to S^3.

For any two path connected 3-manifolds M_i, $i = 1, 2$, we can form the **connected sum** as follows. Select a closed 3-ball in each that does not meet the boundary (if there is one) and remove their interiors. Now choose a homeomorphism from the new boundary 2-sphere of $M_1 - \text{int } B_1$ to the new boundary 2-sphere of $M_2 - \text{int } B_2$. Glue the

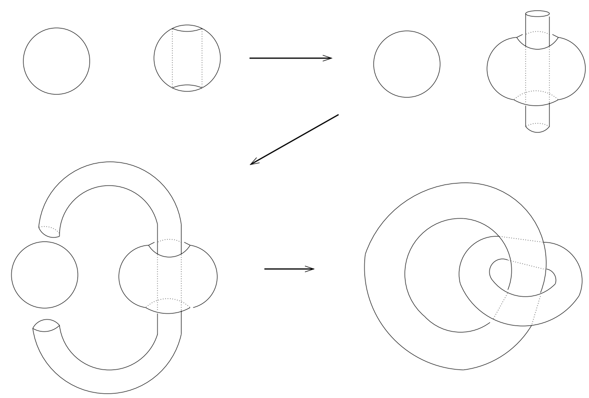

FIGURE 2. Realizing S^3 as the union of two solid tori

two 2-spheres using this homeomorphism. The new manifold is denoted $M_1 \# M_2$ and its topological type is independent of choices of the 3-balls and the homeomorphism. If the only way a manifold M can be written as a connected sum is $M \cong M \# S^3$ then we say M is **prime**. Every compact, path connected, 3-manifold without boundary can be written uniquely as a connected sum of prime 3-manifolds! [4]

4. Knots

A **knot** is a circle embedded in the interior of a 3-manifold, that is there is a homeomorphism $h : S^1 \to K \subset \text{Int } M$. A knot is said to be an **unknot** if it forms the boundary of a disk in M. Thus the unit circle U in the xy-plane in $\mathbb{R}^3$ is unknotted.

Given two knots in a 3-manifold we describe three different ways they may be regarded as equivalent. Two simple closed curves are **homotopic** if one can be continuously deformed into the other. During the deformation the curve is never cut but it can pass through itself. Two simple closed curves are **isotopic** if they are homotopic but no self crossings are permitted during the deformation. Two simple closed curves are **ambiently isotopic** if they are isotopic and the deformation can be extended to a deformation of the entire manifold. Two knots K_1 and K_2 in M are said to have the same **knot type** if they are ambiently isotopic.

Example 1. Figure 3(left) illustrates an isotopy taking a knotted curve to and unknotted circle in $\mathbb{R}^3$. However it can be show that there is no way to extend the isotopy. Thus these two curves are isotopic but not ambiently isotopic. Figure 3(right) shows two curves in a 3-manifold with boundary that is formed by gluing two solid tori along disks in their respective boundaries. They are homotopic, but not isotopic.

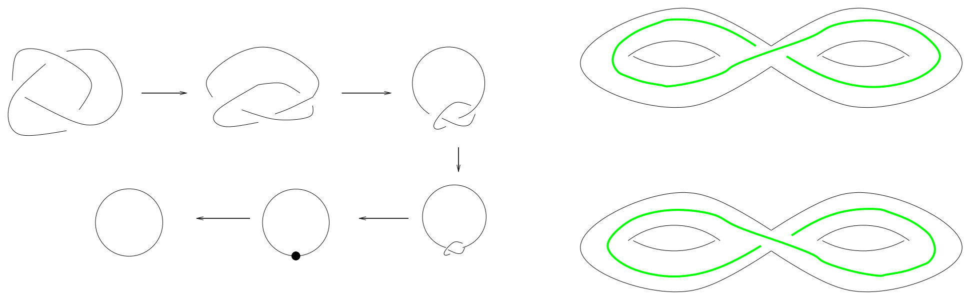

FIGURE 3. Left, isotopic but not ambiently isotopic; Right, homotopic but not isotopic

Given a knot K in a 3-manifold M a **tubular neighborhood** of K is a solid torus that misses ∂M and has K as its core [9]. It is denoted by $N(K)$. A solid torus whose core is unknotted is said to be **standardly embedded**; likewise for the boundary of such a solid torus.

Let V be a standardly embedded solid torus in S^3 and let $T = \partial V$. A knot in T that does not bound a disk in V is called a **torus knot**. The simplest torus knot, besides the unknot, is the **trefoil**; see Figure 4.

Although only unknots bound disks every knot in S^3 is the boundary of some orientable (two sided) surface. Such a surface is called a **Seifert surface**. We won't prove this fact here (see [2]) but Figure 4(upper left) shows a Seifert surface for the trefoil.

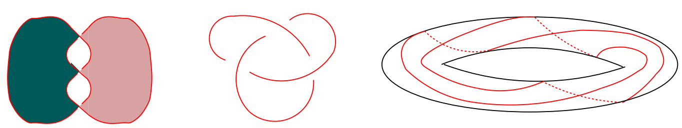

FIGURE 4. Three views of the trefoil knot

Exercise 2. Convince yourself that the three curves in Figure 4 are ambiently isotopic. Convince yourself that the Seifert surface shown is homeomorphic to a torus with the interior of a disk removed.

5. TORUS MAPS

We will be studying self homeomorphisms of the torus. These will arise as gluing maps involving a solid torus. We start setting coordinates. Let V be a solid torus. A simple closed curve on ∂V that bounds a disk inside V but does not bound any disk in ∂V is called a **meridian**. A disk inside V whose boundary is a meridian is called a **meridional disk**. (The disk must meet ∂V only along its boundary.) Any two meridians are isotopic within ∂V.

A simple closed curve in ∂V that meets a meridian in exactly one point where it passes through (we say they meet *transversely*) is called a **longitude**. While any two longitudes are isotopic within V there are infinitely many longitudes that are not isotopic in ∂V. Figure 5 shows two different longitudes on a solid torus.

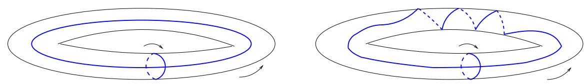

FIGURE 5. Two meridian-longitude pairs

Given a meridian-longitude pair we can orient them and set up a coordinate grid on ∂V. Usually the longitude is the first coordinate and the meridian is taken as the second.

A **preferred longitude** of a solid torus V in S^3 is a longitude that is the intersection of a Seifert surface for the core of V with ∂V. It can be shown that up to isotopy there is only one choice for the preferred longitude [2, 9].

If the torus is standardly embedded in S^3 then a preferred longitude will bound a disk in $S^3 - \text{Int } V$ and is easy to visualize. Determining a preferred longitude of knotted solid torus is not visually obvious. Figure 6 shows the preferred longitude for a trefoil. We will use this later. (A preferred longitude has *linking number* zero with the core [9].)

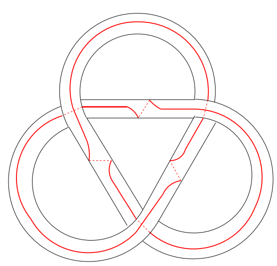

FIGURE 6. A preferred longitude of the trefoil knot based on a figure from [9].

Think of the torus T^2 as given by $I \times I$ with the opposite edges identified. A 2×2 integer matrix A induces a map from $\mathbb{R}^2$ to itself that preserves the integer lattice. If we use arithmetic modulo 1 then

A determines a map from $[0,1) \times [0,1)$ to itself. From this we can get a map from T^2 to T^2. The induced map is a one-to-one if and only if $\det A = \pm 1$. (Try to prove this!)

Of course there are many homeomorphisms from T^2 to T^2 that are not linear. However, it is always possible to "straighten" one out to get a linear map. We take $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to represent L and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to represent M.

Exercise 3. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. In Figure 7 we show the image of $I \times I$ under the action of A in $\mathbb{R}^2$ and how it wraps around the torus when using mod 1 arithmetic. Redo this for $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Try to draw each of these on a donut!

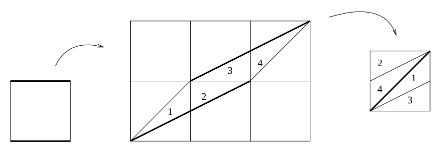

FIGURE 7. A linear torus homeomorphism

6. Homology Groups

If two groups are isomorphic we write $G_1 \approx G_2$. For every compact manifold M there is an associated Abelian group called the **first homology group**, which is denoted by $H_1(M)$. If M is homeomorphic to N then $H_1(M) \approx H_1(N)$. The formal definition is too technical for us. We can think of it as a "hole counter". A **loop** in M is the continuous image of S^1 in M. A loop can have self crossings or even be a point. If every loop is M is homotopic to a point then $H_1(M)$ is the trivial group. Thus, $H_1(S^2)$, $H_1(D^2)$ and $H_1(S^3)$ are trivial.

For an annulus A we have $H_1(A) \approx \mathbb{Z}$. Consider a loop in A that goes around, say counterclockwise, just once. This loop and all loops homotopic to it are associated to 1. The class of loops that wrap around twice are associated to 2. If we have a loop going around once and another going around twice that start and stop at the same point, we can "add" them to get a loop that goes around three times. This can be done formally to give a group structure. Loops that bound a disk in A correspond to 0 and loops that wrap around clockwise correspond to negative integers.

Take a disk D and remove the interiors of two disjoint closed disks in the interior of D. The result is a disk with two holes; call it $\ddot{D}$. Then $H(\ddot{D}) \approx \mathbb{Z}^2$. If we removed three holes the homology group is isomorphic to $\mathbb{Z}^3$ and so on.

It may not be obvious, but for a torus we have $H_1(T^2) \approx \mathbb{Z}^2$. Think about it, the meridian and longitude wrap around different holes. For a solid torus we have $H_1(S^1 \times D^2) \approx \mathbb{Z}$.

This next example will take us beyond the hole counting analogy. Let $\widetilde{M}$ be the Möbius band. It can be shown that $H_1(\widetilde{M}) \approx \mathbb{Z}$. But now we are going to introduce a new space. The boundary of a Möbius band is S^1. Thus we can glue a disk D to $\widetilde{M}$. This gives a 2-manifold without boundary. It is called the **projective plane** and is denoted P^2. You cannot visualize it in $\mathbb{R}^3$ but is exists as a mathematical object. What is $H_1(P^2)$? The boundary of the Möbius band now bounds a disk. So it must die; that is, it is in the identity equivalence class. However, the loop C that is the core of the Möbius band is still non trivial. Yet twice C is trivial; that is a loop that travels around C twice is homotopic to $\partial\widetilde{M}$. Thus, in $H_1(P^2)$ we have $2C \sim 0$. This can be used to prove that $H_1(P^2) \approx \{0, 1\}$ under mod 2 addition.

Finally, if we have two finite disjoint collections of simple closed curves in a manifold and together they form the boundary of an orientable surface then they, with suitable orientations, add up to 0. The reason for this is at the hart of the formal definition of homology groups but is too technical to present here.

7. Finitely Generated Abelian Groups

The first homology group of a compact manifold is a finitely generated Abelian group. We take a brief algebraic detour to review these. The set $\mathbb{Z}^n$ is an Abelian group under vector addition. Let A be an $n \times n$ matrix on integers. It induces a homomorphism from $\mathbb{Z}^n$ into $\mathbb{Z}^n$ via matrix multiplication. We denote the image by $A\mathbb{Z}^n$. The quotient group $\mathbb{Z}^n/A\mathbb{Z}^n$ is then a finitely generated Abelian group and all such groups have a presentation of this form. While it can happen that different matrices yield isomorphic groups there is a simple algorithm involving row and column operations that determines this [3].

For example, $\mathbb{Z}/2\mathbb{Z}$ has two elements, the even integers and the odd integers. The induced addition operation is that even plus even is even, odd plus odd is even and even plus odd is odd. We often write $\mathbb{Z}/2\mathbb{Z}$ as $\{0, 1\}$, taking addition to be addition mod 2.

Let $G = \mathbb{Z}/n\mathbb{Z}$. If $n = \pm 1$ then $n\mathbb{Z} = \mathbb{Z}$ and G is the trivial group, which has one element. In general the number of elements in G is $|n|$, unless $n = 0$. In this case $0\mathbb{Z} = \{0\}$, hence $G \approx \mathbb{Z}$, which is infinite.

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $G = \mathbb{Z}^2/A\mathbb{Z}^2$. The reader should work out that $G \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ which has six elements. If instead $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ the reader should check that $G \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, which has infinitely many elements. The following theorem is proved in [3].

Theorem 7.1. *Let A be an $n \times n$ integer matrix. Then the order of $\mathbb{Z}^n/A\mathbb{Z}^n$ is $|\det A|$ if $\det A$ is not zero and is infinite if $\det A = 0$.*

8. Dehn surgery

Let $\mathcal{M}$ be a 3-manifold and K a knot in M with tubular neighborhood $N(K)$. Now remove the interior of $N(K)$ from $\mathcal{M}$; let $\mathcal{M}' = \mathcal{M} - \text{int } N(K)$. Let V be a solid torus disjoint from $\mathcal{M}$. Let $h : \partial V \to \partial N(K) \subset M'$ be a homeomorphism. Now glue V to $\mathcal{M}'$ by using h to identify ∂V with $\partial N(K)$. Let $\mathcal{M}_{K,h} = \mathcal{M}' \cup_h V$ denote the new manifold. Its topological type depends on both K and h but not on the choice of the tubular neighborhood. It has been proven that every compact, connected, orientable 3-manifold without boundary can be constructed via Dehn surgeries on S^3; this result is known as the Lickorish-Wallace theorem. [9]

The topology of $\mathcal{M}_{K,h}$ is determined by the knot type of K and a (p, q) curve on the $\partial N(K)$ in $\mathcal{M}'$ which is the image of a median of V. The proof goes roughly like this: A meridian of V bounds a disk in V which we can thicken up to a thin ball, B. Let $\mathcal{M}'' = \mathcal{M}' \cup B$. Clearly M'' depends only on (p, q). The closure of the complement of B in V is another ball, B'. Then $\mathcal{M}_{K,h} = \mathcal{M}'' \cup B'$ where the gluing maps all of $\partial B'$ to $\partial \mathcal{M}''$. We know that the topological type of the result is independent of how this gluing is done.

If the homeomorphism is linearized then the first column of its matrix determines the image of M and hence the topological type of a Dehn surgery gluing (along with the knot used).

The simplest Dehn surgeries are those done on the unknot in S^3. Since removing an unknotted solid torus from S^3 results in a second unknotted solid torus, these Dehn surgeries are equivalent to gluing two solid tori together. We might just recover S^3, but this need not be the case. It depends on the homeomorphism. Manifolds formed in this manner are **lens spaces** [8, 9].

Let $h : \partial V_1 \to \partial V_2$ be a homeomorphism from the boundary of solid torus V_1 to the boundary of solid torus V_2. Let $\mathcal{L} = V_1 \cup_h V_2$. Select a meridian-longitude pair for each and call them (L_i, M_i), for $i = 1, 2$ Suppose $h(M_1)$ is a (p, q)-curve on V_2. Then the result of the gluing is called the (p, q)-lens space and is denoted by $\mathcal{L}(p, q)$.

We have that $\mathcal{L}(1, 0) \cong S^3$ and $\mathcal{L}(0, 1) \cong S^2 \times S^1$. Note that for $n > 1$, $\mathcal{L}(n, 0)$ and $\mathcal{L}(0, n)$ are not defined. Neither is $\mathcal{L}(0, 0)$.

Exercise 4. a. Convince yourself that $\mathcal{L}(1, q) \cong S^3$ for all q.
b. Convince yourself that $H_1(\mathcal{L}(p, q)) \approx \mathbb{Z}/p\mathbb{Z}$. It follows that if $\mathcal{L}(p_1, q_1)$ is homeomorphic to $\mathcal{L}(p_2, q_2)$ we must have $p_1 = p_2$.

Now, while (p, q) determines $\mathcal{L}(p, q)$ the relationship is not unique. It can be shown that changing the signs of p or q does not affect the topology. It is also known that $\mathcal{L}(p, q) \cong \mathcal{L}(p, q + np)$ for all integers n. Thus we may assume $0 < q < p$ except for the cases $\mathcal{L}(0, 1)$ and $\mathcal{L}(1, 0)$. There is an additional symmetry and it turns out for $0 < q_1 < p$ and $0 < q_2 < p$, with p relatively prime to each q_i, is it known that $\mathcal{L}(p, q_1) \cong \mathcal{L}(p, q_2)$ if and only if $\pm q_1 q_2^{\pm 1} = 1 \bmod p$.

9. Seifert fibered Manifolds

We define a class of manifolds as follows. Take S^2 and remove the interiors of $k + 1$ disjoint disks. Denote its boundary curves by S_i, for $= 0, 1, \ldots, k$. Then let $\mathcal{M}_0$ be this surface cross S^1. Its boundary is the disjoint union of $k + 1$ tori which we denote $T_0, T_1, \ldots T_k$. Let $V_0, V_1, \ldots V_k$ be solid tori and for $i = 0, \ldots, k$ let $h_i : \partial V_i \to T_i$ be homomorphisms. Attach the V_i's to $\mathcal{M}_0$ using the h_i's. Denote this manifold $\mathcal{M}$. It is called a **Seifert manifold**.

For each $i = 0, \ldots, k$ select a meridian M_i of V_i and on each T_i construct a coordinate system as follows. Select a point O_i on S_i. Let $F_i = \{O_i\} \times S^1$. This will be the axis of slope infinity. Select a simple closed curve $Q_i \subset T_i$ that goes through O_i and intersects, exactly once transversely, each curve $\{z\} \times S^1$, where $z \in S_i$. This will be the axis of slope zero.

Now for each i the image $h_i(M_i)$ is isotopic to a curve of slope α_i/β_i. We require this to be finite. Then it can be shown that the topological type of $\mathcal{M}$ is determined by these slopes. It is denoted by $S^2\left(\frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_k}{\beta_k}\right)$. Notice that if $k = 0$ or 1 we just get the lens spaces. Each Seifert manifold can be given a **fibration**, that is it can be decomposed as disjoint union of circles. We first look at fibrations of the solid torus.

Consider the solid cylinder $C = D^2 \times I$. Let $\mathcal{F}_c = \{\{(r,\theta)\} \times I : (r,\theta) \in D^2\}$. This gives a fibration of C by closed intervals. Let $D_i = D^2 \times \{i\}$ for $i = 0, 1$ be the bottom and top disks of C respectively. For any real number ψ let $R_\psi : D_0 \to D_1$ be given by $R_\psi(r,\theta,0) = (r, \theta + \psi, 1)$. Identify D_0 and D_1 using R_ψ as the homeomorphism to form a solid torus V. If ψ is a rational multiple of π the fibers on C become joined at their end points to form circles. The core circle will contain just one copy of I. If $\psi = 2\pi\alpha/\beta$ for coprime integers α and β then the other circles will be formed from β copies of I. Such an object is called a (α, β) **fibration of the solid torus**. The core is called the **exceptional fiber** unless $\alpha = 0$ or $\beta = 1$, in which case we say the fibration is trivial. Non-exceptional fibers are called **ordinary fibers**. If V is standardly embedded in $\mathbb{R}^3$ then it is fibered by (β, α) torus knots and its core.

Now to get a fibration for $S^2\left(\frac{\alpha_0}{\beta_0}, \ldots, \frac{\alpha_k}{\beta_k}\right)$ we start with the natural fibration of $\mathcal{M}_0 \times S^1$ where the fibers are of the form $\{*\} \times S^1$. Next we place a fibration on each V_i so that h_i will take fibers to fibers.

Two fibrations of a Seifert manifold $\mathcal{M}$ are **fiber equivalent** if there is a homeomorphism $h : \mathcal{M} \to \mathcal{M}$ that takes fibers to fibers. Seifert fibered manifolds have been completely classified up to fiber equivalence by Seifert [11]. We will not need the full classification theorem, but we do need to understand the classification of fibrations of the solid torus.

We only need to know α modulo β and we assume they are coprime. Changing the sign of either is equivalent to changing the choice of orientations for the coordinates systems. A homeomorphism is not required to preserve orientation so sign changes won't affect the fiber equivalence class. Notice that $\alpha/\beta \sim -\alpha/\beta + 1 = (\beta - \alpha)/\beta$. Thus we can assume $0 < \alpha \leq \beta/2$, where equality can only occur when $\beta = 2$. It can be shown that subject to these restrictions α and β determine a unique fiber equivalence class of the solid torus.

Fibrations of S^3. Recall our construction of S^3 as the union of two solid tori glued along their boundaries. The gluing we used identified a meridian of each torus with a longitude of the other. If we place a (α, β) fibration on one solid torus and a (β, α) fibration on the other it is easy to arrange that the gluing homeomorphism take fibers to fibers. This gives a fibration of S^3 with up to two exceptional fibers, one with index α the other with index β. Seifert showed that the only fibrations of S^3 [11]. We remark that the fibrations of S^3 have been used as a tool to study the twisting of molecular structures in "softly condensed matter" [10].

For Seifert fibered manifolds with several exceptional fibers the following equivalence is known.

$$S^2\left(\frac{\alpha_0}{\beta_0},\dots,\frac{\alpha_i}{\beta_i},\dots,\frac{\alpha_j}{\beta_j},\dots,\frac{\alpha_k}{\beta_k}\right)\cong$$

$$S^2\left(\frac{\alpha_0}{\beta_0},\dots,\left(\frac{\alpha_i}{\beta_i}\right)+1,\dots,\left(\frac{\alpha_j}{\beta_j}\right)-1,\dots,\frac{\alpha_k}{\beta_k}\right).$$

And it is obvious that

$$S^2\left(\frac{\alpha_0}{\beta_0},\dots,\frac{\alpha_k}{\beta_k}\right)\cong S^2\left(0,\frac{\alpha_0}{\beta_0},\dots,\frac{\alpha_k}{\beta_k}\right).$$

Using these we can make a normal form where the α_0/β_0 is an integer, dropped if it is zero, and $0<\alpha_i<\beta_i$ for $i=1,\dots,k$. We may also reverse all the signs by switching orientation.

10. Homology Calculations

We give some homology calculations that will be used later.

It should be no surprise to learn that $H_1(S^3)=0$. Let K be a knot in S^3 with tubular neighborhood $N(K)$. Now let $\mathcal{M}_K=S^3-\mathrm{int}(N(K))$. Then $\mathcal{M}_K$ is the **knot complement space** of K. Then $H_1(\mathcal{M}_K)\approx\mathbb{Z}$.

We will only sketch the proof. The homology group of the boundary of $N(K)$ has two generators. We take these to be a meridian M of $N(K)$ and the preferred longitude L. Now L bounds a Seifert surface and hence is null homotopic $L\sim 0$. That leaves only M and it can be shown no power of M is homologous to 0. Thus $H(M_K)\approx\mathbb{Z}$.

Take a knot complement manifold and glue in a solid torus using (p,q) surgery. Call the new manifold $\mathcal{M}_{K(p,q)}$. Now $|q|$ times the meridian of the knot is homologous to 0. The solid torus core is homologous to a preferred longitude which in turn is homologous to 0. These two facts can be used to show that $H_1(\mathcal{M}_{K(p,q)})\approx\mathbb{Z}/|q|\mathbb{Z}$.

The next result is based on a result from Seifert's paper [11].

Theorem 10.1. *Let $\mathcal{M}$ be a Seifert fibered of the type described above with k exceptional fibers with crossing slopes α_i/β_i for $i=1,\dots,k$ and α_0 an integer. Then $H_1(\mathcal{M})$ is the Abelian group with $k+1$ generators, which we denote $F,Q_1,\cdots,Q_k$, and $k+1$ relations,*

$$\alpha_0F-Q_1-\cdots-Q_k=0 \text{ and } \beta_iF+\alpha_iQ_i=0$$

for $i=1,...,k$. The F generator can be represented by any ordinary fiber and the Q_i's can be represented by the crossing curves described above.

We shall give only a rough justification. The homology of the disk with k holes is isomorphic to $\mathbb{Z}^k$. Taking the cross product with S^1 creates a new generator which we shall call F. Thus, $H_1(\mathcal{M}_0) \approx \mathbb{Z}^{k+1}$. However, we can, along with F, use as generators the Q_i's instead of the boundaries of the holes since this is just a change of basis.

Each time we glue a solid torus to a boundary component of $\mathcal{M}_0$ we get a new relation since some nontrivial curve is identified to a meridian which of course bounds a disk. Let V_i be the solid torus whose core will be the i^{th} exceptional fiber. The ordinary fiber F can be isotoped to a fiber near ∂V_i. Then the gluing prescription means that $\beta_i F + \alpha_i Q_i$ now bounds a disk inside V_i and hence we have the relations $\beta_i F + \alpha_i Q_i = 0$, for $i = 1, \ldots, k$. The sum $Q_1 + \cdots + Q_k$ is homologous to the outer boundary curve of the initial disk. Let $E = Q_1 + \cdots + Q_k$. Then the crossing curve that is glued to a meridian of V_0 is $\beta_0 F - E$ which gives the other relation.

We will make use of the following corollary.

Corollary 10.2. *For $n = 3$ the order of $H_1(M)$ is*

$$|\,\alpha_0\beta_1\beta_2\beta_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3 + \beta_1\beta_2\alpha_3\,|$$

Proof. The relations can presented in matrix from as

$$\begin{bmatrix} \alpha_0 & -1 & -1 & -1 \\ \alpha_1 & \beta_1 & 0 & 0 \\ \alpha_2 & 0 & \beta_2 & 0 \\ \alpha_3 & 0 & 0 & \beta_3 \end{bmatrix}.$$

By Theorem 7.1 we just have to compute its determinant. □

11. Surgery along a Torus Knot

We now have the tools in place to prove a classical theorem due to Louise Moser that tells us which three manifolds may result from surgery along a torus knot.

Theorem 11.1. [6] *Let K be an (r, s) torus knot in S^3 and let M be the manifold the results from preforming a (p, q) Dehn surgery along K. Set $\sigma = rsp - q$.*

1. *If $|\sigma| > 1$ then M is a Seifert manifold over S^2 with three exceptional fibers of multiplicities $\alpha_1 = s$, $\alpha_2 = r$ and $\alpha_3 = |\sigma|$. The proof will show how to compute the obstruction term and the three β_i terms.*
2. *If $\sigma = \pm 1$ then M is the lens space $\mathcal{L}(|q|, ps^2)$.*
3. *If $\sigma = 0$ then M is $\mathcal{L}(r, s)\#\mathcal{L}(s, r)$.*

Proof. THE SET UP. We will use the $\mathbb{R}^3 \cup \infty$ model for S^3. Let U be the unit circle in the xy-plane and let Z be the z-axis union $\{\infty\}$. We first partition S^3 into two solid tori, V_1' and V_2', with common boundary, where the core of V_1' is U and the core of V_2' is Z. Let M_i' and L_i' be preferred meridian-longitude pairs for V_i', $i = 1, 2$, where $M_1' = L_2'$ and $L_1' = M_2'$.

Now let K be an (r, s) torus knot on $\partial V_1' = \partial V_2'$. Let $N(K)$ be a tubular neighborhood of K that is small enough that $V_i = V_i' - \text{int}N(K)$ are still solid tori, $i = 1, 2$. Thus $V_1 \cup V_2$ is the knot complement space of K. The V_i look like the V_i' but with a trough dug out along K.

The intersection $\partial V_i' \cap \partial N(K)$ consists of two curves parallel to K. Call them K_i, $i = 1, 2$. They partition the boundary of each V_i into two annuli. Let A be the annulus between the K_i that the V_i have in common, that is $A = V_1 \cap V_2$. Let A_1 be $\partial V_1 - \text{int}A$ and A_2 be $\partial V_2 - \text{int}A$, that is A_1 and A_2 are the "bottoms" of the troughs.

Let (M_i, L_i) be meridian-longitude pairs for V_i, $i = 1, 2$ chosen by retracting M_i' and and L_i' through $N(K)$ as shown in Figure 8.

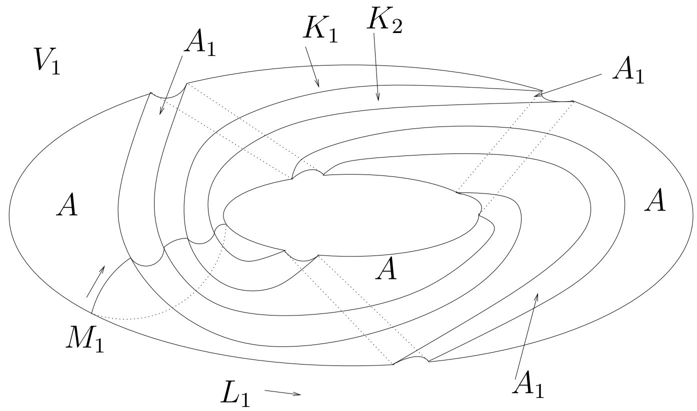

FIGURE 8. The Set Up: $\partial V_1 = A \cup A_1$; $\partial A = \partial A_1 = K_1 \cup K_2$

Let (M_3, L_3) be a preferred meridian-longitude pair for $N(K)$. Recall this means $L_3 \sim 0$ in $V_1 \cup V_2$. (See the end of Section 6.)

Next let V_4 be a new solid torus with meridian-longitude pair (M_4, L_4). This is the solid torus we shall glue to $V_1 \cup V_2$ via a homeomorphism

$$h : \partial V_4 \to \partial(V_1 \cup V_2) = \partial N(K).$$

Let $\begin{bmatrix} a & p \\ b & q \end{bmatrix}$ be the matrix representing h. Thus $h(M_4) = pL_3 + qM_3$.

The following homology calculation, done with respect to $V_1 \cup V_2$, will be used repeatedly. $K_1 \sim rZ$ and $Z \sim sM_3$ so $K_1 \sim rsM_3$. Thus $K_1 - rsM_3 \sim 0 \sim L_3$.

$$\begin{aligned} h(M_4) &= pL_3 + qM_3 \\ &\sim p(K_1 - rsM_3) + qM_3 \\ &= pK_1 - (rsp - q)M_3 \\ &= pK_1 - \sigma M_3. \qquad (*) \end{aligned}$$

Case 1. Suppose $|\sigma| \geq 2$. We augment our set up by using an (r,s) fibration of S^3 such that the knot K is a fiber and $N(K)$ is a fiber solid torus. In this fibration U and Z have multiplicities s and r respectively. We will need to figure out the fibration of V_4 such that h preserves fibers.

Now M_3 is a crossing curve on $\partial N(K)$. Therefore, the fibration of V_4 will have a fiber of multiplicity $|\sigma|$ as its core. So we have a Seifert fibered space of the form $S^2\left(\alpha_0, \frac{\alpha_1}{s}, \frac{\alpha_2}{r}, \frac{\alpha_3}{|\sigma|}\right)$.

Example 2. Suppose K is a $(3,2)$ torus knot and that the Dehn surgery is of type $(6,31)$ Thus $r = 3$, $s = 2$, $p = 6$, $q = 31$ and $|\sigma| = 5$.

By Corollary 10.2 the order of $H_1(\mathcal{M})$ is $30\alpha_0 + 15\alpha_1 + 10\alpha_2 + 6\alpha_3$. From Section 10 the order of $H_1(\mathcal{M})$ is $|q|$. Thus we want to find solutions to

$$30\alpha_0 + 15\alpha_1 + 10\alpha_2 + 6\alpha_3 = \pm 31.$$

First we use $+31$. Since $s = 2$ we know that $\beta_1 = 1$. Thus,

$$15\alpha_0 + 5\alpha_2 + 3\alpha_3 = 8.$$

Since $r = 3$ and $|\sigma| = 5$ we know $\alpha_2 \in \{1,2\}$ and $\alpha_3 \in \{1,2,3,4\}$. Clearly then $\alpha_0 \leq 0$. Suppose $\alpha_0 = 0$. Then $\alpha_2 = 1$ and $\alpha_3 = 1$ are the only solutions. If $\alpha_0 < 0$ you can check that there are no other solutions. For -31 the result is $\alpha_0 = -3$, $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 4$, which is equivalent.

Equations of this type are called *linear Diophantine equations*. There is a general procedure for solving in most number theory textbooks.Try some other choices for r, s, p and q.

Case 2. Suppose $\sigma = \pm 1$. Recall $h = \begin{bmatrix} a & p \\ b & q \end{bmatrix}$. The topological type of the Dehn surgery is determined solely by q and p. We are free to choose a and b so long as $\det h = \pm 1$. If we choose $b = rs$ and $a = 1$ we get $\det h = pb - aq = q - rsp = \sigma = \pm 1$.

From equation $(*)$ we have $h(M_4) \sim pK_1 \mp M_3$. We didn't need to study $h(L_4)$ in Case 1, but here we do.

$$h(L_4) = rsM_3 + L_3 \sim rsM_3 + K_1 - rsM_3 \sim K_1.$$

In other words the longitude on V_4 is going to a curve parallel to the knot K.

Now we glue V_4 to V_1. We claim that the result must be a solid torus. Both V_4 and V_1 can be written as $S^1 \times D^2$. For specificity we write

$$V_1 = S_1 \times D_1 \quad \text{and} \quad V_4 = S_4 \times D_4.$$

Let α be an arc in ∂D_4 and let $A_4 = S_4 \times \alpha$ be the annulus in ∂V_4 that has core L_4 and will be identified to A_1 in ∂V_1. Each copy of the disk D_1 in V_1 meets A_1 in r arcs. We can choose the homeomorphism to take each $* \times \alpha$ arc to a component of $A_1 \cap (*' \times D_1)$. Then the union $V_1 \cup V_4$ can be realized as a product $S^1 \times DD$ where DD is a disk formed by gluing r copies of D_4 to $0 \times D_1$ along copies of α. See Figure 9.

Since $V_1 \cup V_4$ is a solid torus we have that $V_1 \cup V_4 \cup V_2$ is the gluing of two solid tori and hence a lens space. We do some homology calculations to determine which lens space it is.

Remember we have four sets of meridional-longitudinal pairs. Now we need a fifth since $V_1 \cup V_4$ is a new solid torus. Call these (M_5, L_5). We will compute M_5 in terms of M_2 and L_2, that is we shall solve

$$M_5 = xM_2 + yL_2;$$

we won't need to find L_5. Then we will have the $\mathcal{L}(x, y)$ lens space.

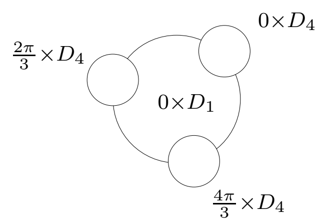

FIGURE 9. Cross section of $V_1 \cup V_4$

Looking at Figure 10 we see that

$$L_1 \sim M_2 + rM_3 \text{ and } M_1 \sim L_2 - sM_3.$$

These homology calculations are still in $V_1 \cup V_2$, the knot complement space. (Figure 10 attempts to show V_1 with a small tubular neighborhood of the core drilled out, thus creating a thick torus $(T^2 \times I)$ with a trough dug out along K, which is presented as a cube with the top

& bottom sides and the left & right sides respectively identified. Not shown is V_2 which would look like its mirror image. Recall V_1 and V_2 are glued along the annulus A. To the sides of V_1 are diagrams showing how the various meridians and longitudes are related if $r = 3$ and $s = 2$. It will likely take a good while for the reader to see this.)

Now we glue in V_4. Recall that K_1 is an (r, s) curve. Thus,

$$M_4 \sim pK_1 - \sigma M_3 \sim p(rM_1 + sL_1) - \sigma M_3 = prM_1 + psL_1 - \sigma M_3.$$

And finally,

$$\begin{aligned} M_5 &\sim M_1 - \sigma s M_4 \\ &\sim M_1 - \sigma s(prM_1 + psL_1 - \sigma M_3) \\ &\sim (1 - \sigma rsp)M_1 - \sigma ps^2 L_1 + sM_3 \\ &\sim (1 - \sigma)(L_2 - sM_3) - \sigma ps^2(M_2 + rM_3) + sM_3 \\ &= (1 - \sigma rsp)L_2 - s(1 - \sigma rsp)M_3 - \sigma ps^2 M_2 - \sigma prs^2 M_3 + sM_3 \\ &= (1 - \sigma rsp)L_2 - \sigma ps^2 M_2 + (-s + \sigma prs^2 - \sigma prs^2 + s)M_3 \\ &= (1 - \sigma rsp)L_2 - \sigma ps^2 M_2, \end{aligned}$$

where these homology calculations are in the new manifold M. If $\sigma = 1$ then $M_5 \sim -qL_2 - ps^2 M_2$; if $\sigma = -1$ then $M_5 \sim +qL_2 + ps^2 M_2$. Therefore $M \cong L(|q|, ps^2)$ as claimed.

Case 3. Suppose $\sigma = 0$. Then $q = rsp$. Since q and p can only have 1 as a common divisor, and $p > 0$ by convention, it must be that $p = 1$. Thus by equation $(*)$ $h(M_4) \sim K_1$. That is the meridian M_4 of V_4 is identified with K_1. Another meridian, M_4', of V_4 will then be identified with K_2.

We construct the union $V_1 \cup V_4 \cup V_2$ in stages. Partition V_4 into two solid cylinders C and C' by choosing two disjoint meridional disks D and D' in V_4 with $\partial D = M_4$ and $\partial D' = M_4'$. The boundary of C minus the interiors of the two disks is an annulus, call it A_3. See Figure 11.

We glue C to V_1 by attaching A_3 to A_1. This space has boundary $D \cup A \cup D'$, which must be a 2-sphere, call it S. Likewise $V_2 \cup C'$ is a manifold whose boundary is a 2-sphere, call it S'. Then M is formed from $V_1 \cup C$ and $V_2 \cup C'$ by identifying their boundary spheres. Thus M is the connected some of two manifolds. We will show that $V_1 \cup C$ and $V_2 \cup C'$ are lens spaces with an open 3-ball removed.

In fact we claim $V_1 \cup C$ is homeomorphic to the lens space $L(r, s)$ minus an open 3-ball. To see this we glue a 3-ball B to $V_1 \cup C$ and show that this space is $L(r, s)$. We do this in two steps. Partition B into a solid torus V_B and a solid cylinder C_B as shown in Figure 12. Let D_B and D_B' be the disks composing $\partial B \cap C_B$. Let $A_B = \partial B - \mathrm{int}(D_B \cup D_B')$. Attach C to C_B by identifying D_B to D_1 and D_B' to D_2. Then $C \cup C_B$

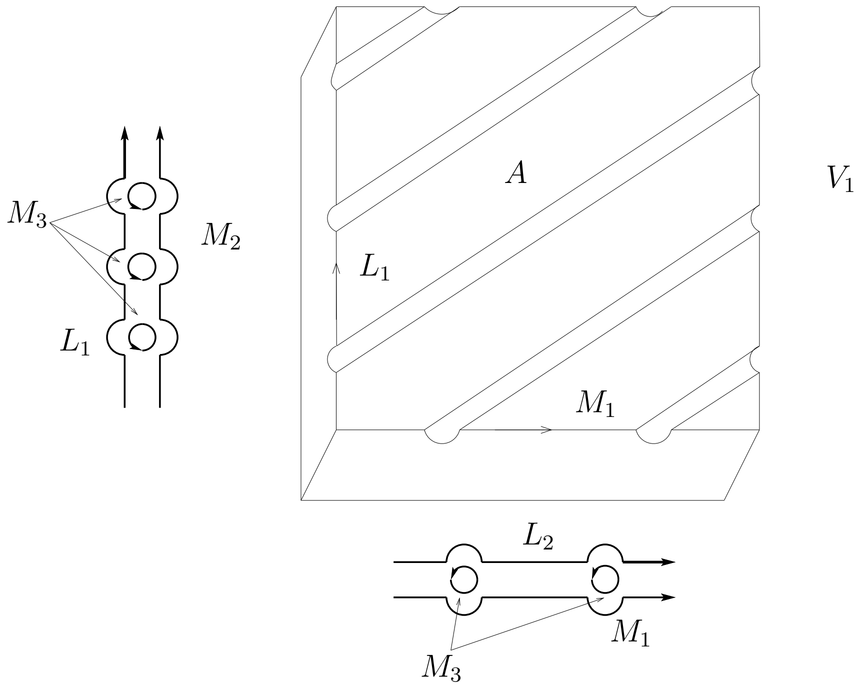

FIGURE 10. V_1 for $r = 3$, $s = 2$

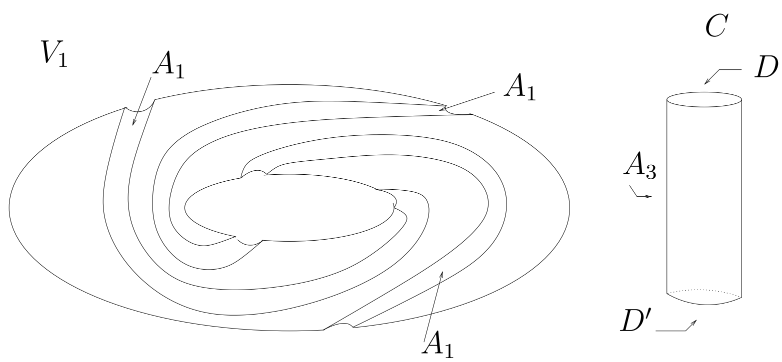

FIGURE 11. V_1 and C

is a solid torus. Now attach V_1 to V_B by identifying the annulus A_1 with A_B. As before this forms a solid torus. Thus,

$$V_1 \cup C \cup B = (V_1 \cup V_B) \cup (C \cup C_B)$$

is the union of two solid tori and is a lens space. Since a meridian ∂D_1 of $C \cup C_B$ is identified to an (r, s) curve on $\partial(V_1 \cup V_B)$ the lens space is $L(r, s)$.

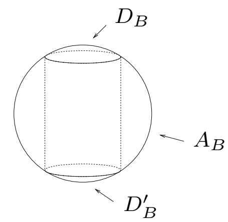

FIGURE 12. The 3-ball B partitioned

If we attach C' to V_2 we can show that this is homeomorphic to the lens space $\mathcal{L}(s, r)$ minus an open ball. Thus M is formed by taking the connected sum of $\mathcal{L}(r, s)$ and $\mathcal{L}(s, r)$. Note: It is known that $\mathcal{L}(r, s)\#\mathcal{L}(s, r)$ cannot be given a Seifert fibration. □

This concludes the proof. The figure-8 knot has just four crossings and is not a torus knot. Surgery along the figure-8 knot is the next logical topic to pursue. This turns out to be much more involved than surgery along torus knots. See [13] as a place to start. There is a large and growing literature on this topic.

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