VARIAE OBSERVATIONES CIRCA SERIES INFINITAS.

AVCTORE
Leonh. Euler.

Obſeruationes, quas hic proferre conſtitui, plerumque circa eiusmodi ſeries verſantur, quae prorſus ſunt diuerſae ab iis, quae etiamnum tractari ſunt ſolitae. Cum enim adhuc nullae aliae ſeries ſint conſideratae, niſi quarum vel terminus generalis eſſet datus, vel lex ſaltem, qua ex datis aliquot terminis ſequentes inuenire liceret; ita hic eiusmodi potiſſimum ſeries ſum contemplaturus, quae neque terminum generalem proprie ſic dictum, neque legem continuationis agnoſcant; ſed quarum natura per alias conditiones determinetur. De huiusmodi ergo ſeriebus eo magis erit mirandum, ſi ſummari poterunt, cum ad methodos ſummandi adhuc cognitas neceſſario vel terminus generalis, vel lex progreſſionis requiratur; quibus deficientibus vix alia via patere videatur, qua ad ſummas cognoſcendas pertingere queamus. Ad has autem obſeruationes me peculiaris ſeries a Cel. *Goldbach* mecum communicata deduxit, cuius ſummationem maxime admirandam *Viri Celeb.* permiſſu hic primo loco ſum demonſtraturus.

Theo-

Theorema I.

Huius seriei in infinitum continuatae

$$\tfrac{1}{3}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{15}+\tfrac{1}{24}+\tfrac{1}{26}+\tfrac{1}{31}+\tfrac{1}{35}+\text{etc.}$$

cuius denominatores vnitate aucti dant omnes numeros, qui ſunt poteſtates vel ſecundi vel altioris cuiusuis ordinis numerorum integrorum, cuiusque adeo terminus quisque exprimitur hac formula $\frac{1}{m^n-1}$, *denotantibus* m *et* n *numeros integros vnitate maiores; huius ſeriei autem ſumma eſt* $=1$.

Demonſtratio.

Hoc eſt Theorema a *Celeb. Goldbach* primum mecum communicatum, quod me etiam ad ſequentes propoſitiones manuduxit. Ex inſpectione autem huius ſeriei facile intelligitur, quam irregulariter ea progrediatur, et propterea quisque in his rebus verſatus maxime modum admirabitur, quo *Vir Celeb.* ſummam huius ſingularis ſeriei inuenit; ſequenti vero modo mihi hoc Theorema demonſtrauit. Sit

$$x=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{9}+\text{etc.}$$

deinde cum ſit

$$1=\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\text{etc.}$$

erit hanc ſeriem ab illa auferendo

$$x-1=1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{10}+\text{etc.}$$

ex denominatoribus ergo excluſi ſunt omnes poteſtates binarii cum binario ipſo; reliqui vero numeri omnes occurrunt.

Ab hac ſerie porro ſubtrahit hanc

$$\tfrac{1}{2} = \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \tfrac{1}{243} + \text{etc.}$$

et reſtabit

$$x - 1 - \tfrac{1}{2} = 1 + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{10} + \tfrac{1}{11} + \text{etc.}$$

denuoque ſubtrahit

$$\tfrac{1}{4} = \tfrac{1}{5} + \tfrac{1}{25} + \tfrac{1}{125} + \text{etc.}$$

reſtabitque

$$x - 1 - \tfrac{1}{2} - \tfrac{1}{4} = 1 + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{10} + \text{etc.}$$

Atque ſimili modo omnes reliquos terminos ſucceſſiue tollendo reperietur tandem

$$x - 1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{5} - \tfrac{1}{6} - \tfrac{1}{9} - \text{etc.} = 1$$

ſeu

$$x - 1 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{9} + \tfrac{1}{10} + \text{etc.}$$

cuius progreſſionis denominatores vnitate aucti dant omnes numeros, qui non ſunt poteſtates. Quare ſi iſta ſeries ab initio aſſumta

$$x = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \text{etc.}$$

ſubtrahatur, relinquetur

$$1 = \tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{15} + \tfrac{1}{24} + \tfrac{1}{26} + \text{etc.}$$

cuius ſeriei igitur, in qua denominatores vnitate aucti dant omnes omnino poteſtates numerorum integrorum, ſumma eſt $= 1$. Q. E. I.

Theorema 2.

Huius ſeriei in infinitum continuatae

$$\tfrac{1}{3} + \tfrac{1}{7} + \tfrac{1}{15} + \tfrac{1}{31} + \tfrac{1}{35} + \tfrac{1}{63} + \text{etc.}$$

cuius

cuius denominatores vnitate aucti dant omnes poteſtates pares, ſumma eſt $= l2$; atque huius ſeriei

$$\frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \frac{1}{80} + \text{etc.}$$

in infinitum continuatae, cuius denominatores vnitate aucti dant omnes poteſtates impares, ſumma aequalis eſt $1 - l2$. Quarum ſerierum prioris terminus quisque eſt $\frac{1}{(2m-2)^n - 1}$, poſterioris vero terminus quilibet hac continetur formula $\frac{1}{(2m-1)^n - 1}$; retinentibus m et n praecedentes valores.

Demonſtratio.

Conſideretur ſequens ſeries, cuius ſumma ponatur x;

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \text{etc.}$$

Iam cum ſit

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{etc.}$$

ſublata hac ſerie ab illa prodibit ſequens

$$x - 1 = \frac{1}{6} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{18} + \text{etc.}$$

a qua ſubtrahatur

$$\frac{1}{5} = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \text{etc.}$$

erit

$$x - 1 - \frac{1}{5} = \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{18} + \text{etc.}$$

ſimili modo ob

$$\frac{1}{9} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \text{etc.}$$

erit

$$x - 1 - \frac{1}{5} - \frac{1}{9} = \frac{1}{12} + \frac{1}{14} + \frac{1}{18} + \text{etc.}$$

Omni-

Omnibus ergo terminis hoc modo ſublatis prodibit

$$x = 1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \text{etc.}$$

cuius denominatores conſtituunt ſeriem naturalem numerorum imparium exceptis iis, qui vnitate aucti ſunt poteſtates, vti ex formatione huius ſeriei intelligitur. Cum vero ſit

$$l2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \text{etc.}$$

atque

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \text{etc.}$$

erit $x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \text{etc.} - l2.$

Illo ergo pro x inuento valore ſublato ab iſto, in quo omnes omnino numeri impares occurrunt, reſtabit

$$0 = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35}\ \text{etc.} - l2,$$

ſeu iſta

$$l2 = \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \frac{1}{35} + \text{etc.}$$

cuius ſeriei denominatores ſunt ii numeri impares, qui vnitate aucti dant omnes poteſtates pares. Huius ergo ſeriei ſumma eſt $l2$, prout in propoſitione eſt aſſertum. Q. E. Vnum.

Cum vero praecedens Theorema ſit

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \text{etc.}$$

vbi denominatores vnitate aucti dant omnes numeros, qui ſunt poteſtates tam pares quam impares, habebitur illa ſerie ab hac demta

$$1 - l2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \text{etc.}$$

cuius denominatores adeo ſunt ii numeri pares, qui vnitate aucti dant omnes poteſtates impares. Q. E. Alterum.

Theo-

Theorema 3.

Posito π *pro peripheria circuli, cuius diameter est* 1, *erit*

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} - \frac{1}{120} - \frac{1}{124} - \frac{1}{168} - \frac{1}{224} + \frac{1}{244} - \frac{1}{288} - \text{etc.}$$

cuius seriei denominatores sunt numeri pariter pares, vnitate vel maiores vel minores quam potestates numerorum imparium. Illae autem fractiones, quarum denominatores vnitate excedunt potestates, signum habent $+$ *reliquae signum* -1.

Demonstratio.

Cum sit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

cuius seriei eae fractiones, quarum denominatores vnitate deficiunt a numeris pariter paribus, signum habent -1, reliquae signum $+$. Ad illam seriem vero addatur haec geometrica

$$\frac{1}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \text{etc.}$$

erit

$$\frac{\pi}{4} + \frac{1}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

a qua subtrahatur

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \text{etc.}$$

erit

$$\frac{\pi}{4} + \frac{1}{4} - \frac{1}{4} = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \text{etc.}$$

in qua serie nec 3, et 5 nec eorundem potestates amplius insunt, simili modo 7 eiusque potestates tollentur addendo hanc seriem

$$\frac{1}{8} = \frac{1}{7} - \frac{1}{49} + \text{etc.}$$

eritque

$$\frac{\pi}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{8}=1-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\text{etc.}$$

Pari modo tollendo reliquos terminos, qui non ſunt poteſtates; ſimul enim poteſtates tolluntur, prodibit tandem

$$\frac{\pi}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{8}+\frac{1}{12}-\frac{1}{10}+\frac{1}{16}-\frac{1}{16}+\frac{1}{20}-\frac{1}{20}+\frac{1}{24}-\frac{1}{28}-\text{etc.}=1$$

ſeu

$$\frac{\pi}{4}=1-\frac{1}{8}-\frac{1}{24}+\frac{1}{28}-\frac{1}{48}-\frac{1}{80}-\frac{1}{120}\ \text{etc.}$$

ob binos terminos ſeſe plerumque deſtruentes, ita vt ſoli ſuperſint, qui ſolitarii erant: ſolitariae autem erant eae fractiones, quarum denominatores, qui ſemper prodierant pariter pares, vnitate vel aucti vel minuti poteſtates numerorum imparium efficiebant. Signa vero horum terminorum legem ſeruant praeſcriptam. Q. E. D.

Theorema 4.

Denotante π vt ante peripheriam circuli cuius diameter eſt $=1$, erit

$$\frac{\pi}{4}-\frac{3}{4}=\frac{1}{28}-\frac{1}{124}+\frac{1}{244}+\frac{1}{344}\ \text{etc.}$$

cuius ſeriei denominatores omnes ſunt numeri pariter pares vnitate vel maiores vel minores, quam poteſtates numerorum imparium non quadratae; atque illae fractiones quarum denominatores vnitate ſuperant tales poteſtates, ſignum habent $+$; *reliquae quarum denominatores deficiunt ab huiusmodi poteſtatibus non quadratis ſignum habent* $-$.

Demonſtratio.

Per Theorema praecedens tertium inuenimus

$$\frac{\pi}{4}=1-\frac{1}{8}-\frac{1}{24}+\frac{1}{28}+\frac{1}{48}-\frac{1}{80}-\frac{1}{120}-\frac{1}{124}\ \text{etc.}$$

in qua ſerie primo occurrunt denominatores, qui ab omnibus

bus quadratis imparibus vnitate deficiunt, eaeque fractiones omnes habent signum idem −. Cum vero sit

$$\frac{1}{8} + \frac{1}{24} + \frac{1}{48} + \frac{1}{80} + \frac{1}{120} + \frac{1}{168} + \text{etc.} = \frac{1}{4}$$

habebitur loco harum fractionum omnium substituendo $\frac{1}{4}$ sequens forma

$$\frac{\pi}{4} = 1 - \frac{1}{4} + \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} \text{ etc.}$$

seu

$$\frac{\pi}{4} - \frac{3}{4} = \frac{1}{28} - \frac{1}{124} + \frac{1}{244} + \frac{1}{344} \text{ etc.}$$

cuius seriei denominatores sunt numeri pariter pares vel vnitate maiores vel minores quam potestates numerorum imparium non quadratae, ob quadratas iam exclusas, atque prout vnitate sunt vel maiores vel minores, fractiones etiam habent signum + vel −. Q. E. D.

Corollarium 1.

Ad seriem ergo continuandam omnium numerorum imparium, qui non sunt potestates, sumendae sunt potestates exponentium imparium, eaeque vnitate vel augendae vel minuendae, quo prodeant numeri pariter pares, qui erunt denominatores seriei inuentae: seruata signorum regula.

Corollarium 2.

Cum autem omnis numerus impar vel sit $4m-1$ vel $4m+1$, potestates autem exponentium imparium a $4m-1$ ortorum, si vnitate augeantur, illae autem, quae a $4m+1$ oriuntur si vnitate minuantur, dent numeros pariter pares; aequabitur $\frac{\pi}{4} - \frac{3}{4}$ seriei terminorum qui omnes in hac forma $\frac{1}{(4m-1)^{2n+1}+1}$ continentur, demta ſerie terminorum in hac forma $\frac{1}{(4m+1)^{2n+1}-1}$ contentorum; vbi loco m et n omnes numeri integri affirmatiui accipi debent praeter eos, qui vel $4m-1$ vel $4m+1$ faciunt poteſtates.

Corollarium 3.

Aequabitur ergo $\frac{\pi}{4}-\frac{3}{4}$ aggregato ſequentium ſerierum infinitarum

$$\frac{\pi}{4}-\frac{3}{4}=\left\{\begin{array}{l}
\frac{1}{3^3+1}+\frac{1}{3^5+1}+\frac{1}{3^7+1}+\frac{1}{3^9+1}+\text{etc.}\\
-\frac{1}{5^3-1}-\frac{1}{5^5-1}-\frac{1}{5^7-1}-\frac{1}{5^9-1}+\text{etc.}\\
+\frac{1}{7^3+1}+\frac{1}{7^5+1}+\frac{1}{7^7+1}+\frac{1}{7^9+1}+\text{etc.}\\
+\frac{1}{11^3+1}+\frac{1}{11^5+1}+\frac{1}{11^7+1}+\frac{1}{11^9+1}+\text{etc.}\\
-\frac{1}{13^3-1}-\frac{1}{13^5-1}-\frac{1}{13^7-1}-\frac{1}{13^9-1}+\text{etc.}\\
+\frac{1}{15^3+1}+\frac{1}{15^5+1}+\frac{1}{15^7+1}+\frac{1}{15^9+1}+\text{etc.}\\
\qquad\qquad\text{etc.}\quad\text{etc.}
\end{array}\right.$$

Corollarium 4.

Hac ergo ſerie eousque continuata, donec denominatores fiant maiores quam 100000 habebitur $\frac{\pi}{4}=\frac{3}{4}+\frac{1}{28}-\frac{1}{124}+\frac{1}{244}+\frac{1}{344}+\frac{1}{1332}+\frac{1}{2188}-\frac{1}{2196}-\frac{1}{3124}+\frac{1}{3376}-\frac{1}{4912}+\frac{1}{6860}-\frac{1}{9260}+\frac{1}{12168}+\frac{1}{16808}+\frac{1}{19684}-\frac{1}{24388}+\frac{1}{29792}-\frac{1}{35936}+\frac{1}{42876}-\frac{1}{50652}+\frac{1}{59320}-\frac{1}{68920}-\frac{1}{78124}+\frac{1}{79508}-\frac{1}{91124}$.

Corollarium 5.

Cum omnes denominatores per 4 diuidi poſſint, erit $\pi=3+\frac{1}{7}-\frac{1}{31}+\frac{1}{61}+\frac{1}{86}+\frac{1}{333}+\frac{1}{547}-\frac{1}{549}-\frac{1}{781}+\frac{1}{844}$ etc.

Quae

Quae ſeries ideo notari meretur, quod eius duo primi termini iam dent Archimedis proportionem peripheriae circuli ad diametrum.

Theorema 5.

Retinente π priorem ſignificationem, erit

$$\frac{\pi}{4} - l2 = \underbrace{\frac{1}{26} + \frac{1}{28}} + \underbrace{\frac{1}{242} + \frac{1}{244}} + \underbrace{\frac{1}{342} + \frac{1}{344}} + \text{etc.}$$

cuius ſeriei haec eſt lex, vt numeri medii inter binos denominatores binario differentes, ſcilicet 27, 243, 343, *etc. ſint poteſtates exponentium imparium ortae a numeris imparibus, quae vnitate auctae ſint per* 4 *diuiſibiles ſeu numeri pariter pares.*

Demonſtratio.

Cum per Theorema tertium ſit

$$\frac{\pi}{4} = 1 - \frac{1}{8} - \frac{1}{24} + \frac{1}{28} - \frac{1}{48} - \frac{1}{80} \text{ etc.}$$

fractionum ſigno $-$ affectarum denominatores ſunt numeri pariter pares vnitate deficientes a poteſtatibus numerorum imparium; fractionum vero ſigno $+$ affectarum denominatores ſunt quoque pariter pares vnitate ſuperantes poteſtates numerorum imparium; atque praeterea ſit per Theorema ſecundum

$$1 - l2 = \frac{1}{8} + \frac{1}{24} + \frac{1}{26} + \frac{1}{48} + \frac{1}{80} + \text{etc.}$$

cuius ſeriei denominatores deficiunt vnitate ab omnibus potentiis numerorum imparium; haec ſeries complectetur omnes terminos illius ſigno $-$ affectos, et praeterea fractiones denominatores habentes impariter pares vnitate deficientes a poteſtatibus numerorum imparium. Quare

 ſi haec

si haec series ad illam addatur prodibit

$$\frac{\pi}{4} - l2 = \frac{1}{26} + \frac{1}{28} + \frac{1}{242} + \frac{1}{244} + \frac{1}{342} + \frac{1}{344} + \text{etc.}$$

cuius binae fractiones erunt ita comparatae, vt prioris denominator sit numerus impariter par, posterioris binario maior pariter par, mediusque numerus inter binos huiusmodi denominatores sit potestas numeri imparis; quae ergo potestas vnitate aucta dare debet numerum pariter parem. Q. E. D.

Corollarium 1.

Quia hae potestates numerorum imparium ita sunt comparatae, vt vnitate auctae fiant per 4 diuisibiles, erunt eae potestates imparium dimensionum, quae oriuntur a numeris huius formae $4m-1$, qui ipsi non sunt potestates.

Corollarium 2.

Si ergo omnes sumantur numeri huius formae $4m-1$, quae non sunt potestates, eorumque capiantur omnes potestates exponentium imparium; hae potestates vnitate tam auctae quam minutae dabunt omnes fractionum seriei inuentae denominatores.

Corollarium 3.

Si binae fractiones in vnam coalescant erit

$$\frac{\pi}{4} = l2 + \frac{2.27}{26.28} + \frac{2.243}{242.244} + \frac{2.343}{342.344} + \text{etc.}$$

Quae series formabitur sumendis omnibus fractionibus, quae oriuntur ex hac forma $\frac{2(4m-1)^{2n+1}}{(4m-1)^{4n+2}-1}$ substituendo

loco

loco m et n omnes numeros integros ſucceſſiue praeter eos ipſius m valores, qui reddant $4m-1$ poteſtatem.

Theorema 6.

Seriei huius

$$\frac{1}{15}+\frac{1}{63}+\frac{1}{80}+\frac{1}{255}+\frac{1}{624}+ etc.$$

cuius denominatores vnitate aucti dant omnia quadrata, quae ſimul ſunt altiores poteſtates; huius inquam ſeriei in infinitum continuatae ſumma eſt $\frac{7}{4}-\frac{\pi^2}{6}$: *denotante* π *peripheriam circuli, cuius diameter* $=1$.

Demonſtratio.

Hoc quoque Theorema a *Cel. Goldbachio*, verum ſine demonſtratione accepi, atque iisdem quibus ante veſtigiis inſiſtens hanc inueni demonſtrationem. Cum ante aliquot annos incidiſſem in huius ſeriei

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{24}+ \text{etc.}$$

ſummam $=\frac{\pi^2}{6}$, hanc ipſam ſeriem ita ſum contemplatus

$$\frac{\pi^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+ \text{etc.}$$

Iam cum ſit

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+ \text{etc.}$$

atque

$$\frac{1}{8}=\frac{1}{9}+\frac{1}{81}+\frac{1}{729}+ \text{etc.}$$

ſimilique modo

$$\frac{1}{24}=\frac{1}{25}+\frac{1}{625}+ \text{etc. et } \frac{1}{35}=\frac{1}{36}+ \text{etc.}$$

ſi loco harum ſerierum geometricarum ſubſtituantur ſummae prodibit

$$\frac{\pi^2}{6}=1+\frac{1}{3}+\frac{1}{8}+\frac{1}{24}+\frac{1}{35}+\frac{1}{48}+\frac{1}{99} \text{ etc.}$$

cuius feriei denominatores vnitate aucti dant omnes numeros quadratos, praeter eos, qui fimul fint alius fpeciei poteftates. Cum autem fumendis omnino quadratis vnitate minutis fit

$$\frac{3}{4} = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \frac{1}{48} + \frac{1}{63} + \frac{1}{80} + \text{etc.}$$

proueniet ab hac fuperiorem feriem fubtrahendo

$$\frac{7}{4} - \frac{\pi^2}{6} = \frac{1}{16} + \frac{1}{63} + \frac{1}{80} + \frac{1}{255} + \text{etc.}$$

qui denominatores vnitate aucti dant omnes numeros quadratos, qui fimul funt alius generis poteftates. Q.E.D.

Conftituunt haec fex theoremata alterum obferuationum iftarum partem, quibus fcilicet feries additione fubtractioneue terminorum ortae funt confideratae. Sequentia vero theoremata circa feries, quorum termini in fe inuicem multiplicantur, verfabuntur; neque minus erunt admirabilia, quam praecedentia, cum in iis pariter lex progreffionis tantopere fit irregularis. Difcrimen autem in hoc potiffimum erit pofitum, quod in theorematis praecedentibus progreffio terminorum fecuta fit feriem poteftatum, quae per fe eft maxime irregularis; in his autem termini progrediantur fecundum numeros primos, quorum progreffio non minus eft abftrufa.

Theorema 7.

Factum continuum in infinitum ex his fractionibus $\frac{2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19}{1\cdot2\cdot4\cdot6\cdot10\cdot12\cdot16\cdot18}$ *etc. vbi numeratores funt omnes numeri primi, denominatores vero vnitate deficiunt a numeratoribus. Hoc factum inquam aequale eft fummae huius feriei*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \textit{etc.}$$

eftque adeo infinitum.

Demon-

Demonſtratio.

Nam ſit

$$x = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \text{etc.}$$

erit

$$\tfrac{1}{2}x = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \text{etc.}$$

qua ſerie ab illa demta reſtat

$$\tfrac{1}{2}x = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \text{etc.}$$

in qua nulli amplius denominatores pares inſunt. Ab hac denuo auferatur iſta ſeries

$$\tfrac{1}{2}\cdot\tfrac{1}{3}x = \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{15} + \tfrac{1}{21} + \text{etc.}$$

reſtabit

$$\tfrac{1}{2}\cdot\tfrac{2}{3}x = 1 + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \tfrac{1}{13} + \text{etc.}$$

in cuius denominatoribus nec per 2 nec per 3 diuiſibiles reperiuntur. Quo autem etiam numeri per 5 diuiſibiles egrediantur, ſubtrahatur iſta ſeries

$$\tfrac{1.2}{2.3}\cdot\tfrac{1}{5}x = \tfrac{1}{5} + \tfrac{1}{25} + \tfrac{1}{35} + \text{etc.}$$

reſtabitque

$$\tfrac{1.2.4}{2.3.5}x = 1 + \tfrac{1}{7} + \tfrac{1}{11} + \tfrac{1}{13} + \text{etc.}$$

Atque ſimili modo tollendis omnibus terminis tum per 7 tum per 11 etc. omnesque numeros primos diuiſibilibus tandem reperietur

$$\frac{1.2.4.6.10.12.16.18.22.\text{ etc.}}{2.3.5.7.11.13.17.19.23.\text{ etc.}}\,x = 1.$$

Quare cum ſit

$$x = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \text{etc.}$$

erit

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \text{etc.} = \frac{2.3.5.7.11.13.17.19.23\text{ etc.}}{1.2.4.6.10.12.16.18.22\text{ etc.}}$$

cu-

cuius expressionis numeratores constituunt progressionem numerorum primorum, denominatores vero unitate ab iis deficiunt. Q. E. D.

Corollarium 1.

Expressiones ergo $\frac{2.3.5.7.11.13.\,etc.}{1.2.4.6.10.12.\,etc.}$ valor est infinitus, et posito absolute infinito $=\infty$, erit istius expressionis valor $= l\infty$, quod infinitum inter omnes infiniti potestates est minimum.

Corollarium 2.

Cum vero haec expressio $\frac{4.9.16.25.36.49\,etc.}{3.8.15.24.35.48\,etc.}$ finitum habeat valorem scilicet 2; sequitur infinities plures esse numeros primos, quam quadratos, in serie omnium omnino numerorum.

Corollarium 3.

Verum etiam hinc intelligitur infinities pauciores extare numeros primos, quam numeros integros; haec enim expressio $\frac{2.3.4.5.6.7\,etc.}{1.2.3.4.5.6\,etc.}$ absolute infinitum habet valorem, cum similis a numeris tantum primis ortae valor sit logarithmus istius infiniti.

Theorema 8.

Si ex serie numerorum primorum sequens formetur expressio

$$\frac{2^n.\ 3^n.\ 5^n.\ 7^n.\ 11^n.\ etc.}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\ etc.}$$

erit

erit eius valor aequalis summae huius seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + etc.$$

Demonſtratio.

Sit

$$x = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

erit

$$\frac{1}{2^n}x = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \text{etc.}$$

vnde oritur

$$\frac{(2^n - 1)}{2^n}x = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \text{etc.}$$

Porro eſt

$$\frac{(2^n - 1)}{2^n} \cdot \frac{1}{3^n}x = \frac{1}{3^n} + \frac{1}{15^n} + \frac{1}{21^n} + \text{etc.}$$

vnde fiet

$$\frac{(2^n-1)(3^n-1)}{2^n \cdot 3^n}x = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.}$$

Similibus ergo operationibus pro ſingulis numeris primis inſtitutis omnes ſeriei termini praeter primum tollentur, reperieturque

$$1 = \frac{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\text{ etc.}}{2^n \quad 3^n \quad 5^n \quad 7^n \quad 11^n \text{ etc.}}x$$

et loco x ſerie reſtituta fit

$$\frac{2^n \quad 3^n \quad 5^n \quad 7^n \quad 11^n \text{ etc.}}{(2^n-1)(3^n-1)(5^n-1)(7^n-1)(11^n-1)\text{ etc.}} = 1$$

$$= 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

Q. E. D.

Corollarium 1.

Cum posito $n = 2$ fit $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = \frac{\pi^2}{6}$ denotante π peripheriam circuli, cuius diameter est 1, erit

$$\frac{4 . 9 . 25 . 49 . 121 . 169 . \text{etc.}}{3 . 8 . 24 . 48 . 120 . 168 . \text{etc.}} = \frac{\pi^2}{6},$$

seu

$$\frac{\pi^2}{6} = \frac{2 . 2 . 3 . 3 . 5 . 5 . 7 . 7 . 11 . 11 . \text{etc.}}{1 . 3 . 2 . 4 . 4 . 6 . 6 . 8 . 10 . 12 . \text{etc.}}$$

Corollarium 2.

Cum praeterea posito $n = 4$ fit

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{\pi^4}{90}.$$

erit

$$\frac{\pi^4}{90} = \frac{4 . 4 . 9 . 9 . 25 . 25 . 49 . 49 . 121 . 121 . \text{etc.}}{3 . 5 . 8 . 10 . 24 . 26 . 48 . 50 . 120 . 122 . \text{etc.}}$$

Hac igitur expressione per illam diuisa prodit

$$\frac{\pi^2}{15} = \frac{4 . 9 . 25 . 49 . 121 . 169 . \text{etc.}}{5 . 10 . 26 . 50 . 122 . 170 . \text{etc.}}$$

Theorema 9.

Si quadrata numerorum primorum imparium omnium resoluantur in duas partes vnitate a se inuicem differentes, harumque partium impares sumantur pro numeratoribus, pares vero pro denominatoribus seriei ex factoribus compositae valor huius expressionis erit $\frac{5 . 13 . 25 . 61 . 85 . 145 . \text{etc.}}{4 . 12 . 24 . 60 . 84 . 144 . \text{etc.}} = \frac{3}{2}$

Demon-

Demonſtratio.

Per theorematis praecedentis Coroll. 1. habemus

$$\frac{\pi^2}{6} = \frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ 289.\ \text{etc.}}{3.\ 8.\ 24.\ 48.\ 120.\ 168.\ 288.\ \text{etc.}}$$

At in Coroll. 2. ſequentem elicuimus aequationem

$$\frac{\pi^2}{15} = \frac{4.\ 9.\ 25.\ 49.\ 121.\ 169.\ 289.\ \text{etc.}}{5.\ 10.\ 26.\ 50.\ 122.\ 170.\ 290.\ \text{etc.}}$$

Quarum expreſſionum ſi illa per hanc diuidatur, π ex calculo egredietur, habebiturque

$$\frac{5}{2} = \frac{5.\ 10.\ 26.\ 50.\ 122.\ 170.\ 290.\ \text{etc.}}{3.\ 8.\ 24.\ 48.\ 120.\ 168.\ 288.\ \text{etc.}}$$

cuius expreſſionis numeratores ſunt vnitate maiores quam quadrata numerorum primorum, denominatores vero vnitate minores. Si ergo vtrinque per $\frac{5}{3}$ diuidatur ſingulaeque fractiones per 2 deprimantur, habebitur

$$\frac{3}{2} = \frac{5.\ 13.\ 25.\ 61.\ 85.\ 145.\ \text{etc}}{4.\ 12.\ 24.\ 60.\ 84.\ 144.\ \text{etc.}}$$

vbi numeratores ſunt vnitate maiores quam denominatores respondentes, atque quisque numerator cum ſuo denominatore facit quadratum numeri primi imparis, ob ſublatum diuiſione quadratum numeri primi paris 2. Q. E. D.

Theorema 10.

Si π vt hactenus ſignificet peripheriam circuli, cuius diameter eſt $= 1$, erit

$$\frac{\pi^3}{32} = \frac{80.\ 224.\ 440.\ 624.\ 728.\ \text{etc.}}{81.\ 225.\ 441.\ 625.\ 729.\ \text{etc.}}$$

cuius expreſſionis denominatores ſunt quadrata numerorum imparium non primorum, numeratores vero vnitate minores.

Demonſtratio.

A *Walliſio* habetur ſequens expreſſio pro π, nempe

$$\frac{\pi}{4} = \frac{8.\ 24.\ 48.\ 80.\ 120.\ 168.\ \text{etc.}}{9.\ 25.\ 49.\ 81.\ 121.\ 169.\ \text{etc.}}$$

 quae

quae fractiones ex omnibus omnino quadratis imparibus formantur. Per Coroll. 1. vero Theor. 8 erat

$$\frac{\pi^2}{6} = \frac{4 \cdot 9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot \text{etc.}}{3 \cdot 8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdot \text{etc.}}$$

siue

$$\frac{\pi^2}{8} = \frac{9 \cdot 25 \cdot 49 \cdot 121 \cdot 169 \cdot 289 \cdot \text{etc.}}{8 \cdot 24 \cdot 48 \cdot 120 \cdot 168 \cdot 288 \cdot \text{etc.}}$$

quae fractiones ex solis numerorum primorum imparium quadratis sunt formatae. Si iam hae duae expressiones in se mutuo ducantur prodibit

$$\frac{\pi^3}{32} = \frac{80 \cdot 224 \cdot 440 \cdot 624 \cdot 728 \cdot \text{etc.}}{81 \cdot 225 \cdot 441 \cdot 625 \cdot 729 \cdot \text{etc.}}$$

quae ideo fractiones quadrata numerorum imparium non primorum sequuntur. Q. E. D.

Theorema II.

Sumto π pro peripheria circuli, cuius diameter est 1, erit

$$\frac{\pi}{4} = \frac{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \text{etc.}}{4 \cdot 4 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdot 20 \cdot 24 \cdot \text{etc.}}$$

cuius expressionis numeratores constituunt progressionem numerorum primorum, denominatores vero sunt numeri pariter pares vnitate vel maiores vel minores quam numeratores respondentes.

Demonstratio.

Cum sit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} \text{ etc.}$$

erit

$$\frac{1}{3} \cdot \frac{\pi}{4} = \frac{1}{3} - \frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \text{etc.}$$

quibus seriebus additis fit

$$\frac{4}{3} \cdot \frac{\pi}{4} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} \text{ etc.}$$

Deinde

Deinde eſt

$$\frac{1}{5}\cdot\frac{4}{3}\cdot\frac{\pi}{4}=\frac{1}{5}+\frac{1}{25}-\frac{1}{35}-\frac{1}{55}+\text{etc.}$$

qua ſublata prodit

$$\frac{4}{5}\cdot\frac{4}{3}\cdot\frac{\pi}{4}=1-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\text{etc.}$$

in qua ſerie nulli amplius occurrunt denominatores vel per 3 vel per 5 diuiſibiles. Simili modo tollentur omnes per 7 diuiſibiles addendo

$$\frac{1}{7}\cdot\frac{4\cdot 4}{5\cdot 3}\cdot\frac{\pi}{4}=\frac{1}{7}-\frac{1}{49}-\frac{1}{77}\ \text{etc.}$$

prodibit autem

$$\frac{8\cdot 4\cdot 4}{7\cdot 5\cdot 3}\cdot\frac{\pi}{4}=1-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}\ \text{etc.}$$

Perſpicitur autem denominatores per numerum primum huius formae $4n-1$ diuiſibiles tolli additione, vnde iſte nouus factor $\frac{4n}{4n-1}$ accedit: denominatores vero per numerum primum formae $4n+1$ diuiſibiles tolli ſubtractione, vnde nouus factor iſte $\frac{4n}{4n+1}$ adiicietur. Horum ergo factorum ſucceſſiue addendorum donominatores erunt numeri primi; numeratores vero numeri pariter pares vnitate vel maiores vel minores quam denominatores. Hoc ergo modo ſi auferantur omnes termini ſeriei initio aſſumtae, prodibit tandem

$$\frac{\text{etc.}\ 24\cdot 20\cdot 16\cdot 12\cdot 12\cdot 8\cdot 4\cdot 4\cdot}{\text{etc.}\ 23\cdot 19\cdot 17\cdot 13\cdot 11\cdot 7\cdot 5\cdot 3\cdot}\ \frac{\pi}{4}=1.$$

Ex qua oritur

$$\frac{\pi}{4}=\frac{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot\ \text{etc.}}{4\cdot 4\cdot 8\cdot 12\cdot 12\cdot 16\cdot 20\cdot 24\cdot\ \text{etc.}}\qquad \text{Q. E. D.}$$

Theorema 12.

Si omnes numeri primi impares in duas partes vnitate a ſe inuicem differentes diuidantur atque partes pares *ſumantur pro numeratoribus, impares vero pro denominatoribus, erit factum continuum*

$$\frac{2.\ 2.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 7.\ 9.\ 9.\ 11.\ \text{etc.}} = 2$$

Demonſtratio.

Cum ſit per Theorema praecedens $\frac{\pi}{4} = \frac{3.\ 5.\ 7.\ \text{etc.}}{4.\ 4.\ 8.\ \text{etc.}}$ erit

$$\frac{16}{\pi^2} = \frac{4.\ 4.\ 4.\ 4.\ 8.\ 8.\ 12.\ 12.\ 12.\ 12.\ 16.\ 16.\ \text{etc.}}{3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 11.\ 11.\ 13.\ 13.\ 17.\ 17\ \text{etc.}}.$$

At ex Coroll. 1. Theor. 8. ſi per $\frac{3}{4}$ multiplicetur habetur

$$\frac{\pi^2}{2} = \frac{3.\ 3.\ 5.\ 5.\ 7.\ 7\ 11.\ 11.\ 13.\ 13.\ \text{etc.}}{2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ 12.\ 14.\ \text{etc.}}$$

quarum expreſſionum vtraque per numeros primos impares formatur. Si ergo hae in ſe inuicem multiplicentur, prioris denominator deſtruet numeratorem poſterioris, atque praeterea tam ex illius numeratore quam huius denominatore medietas terminorum auferetur. Prodibit ſcilicet

$$2 = \frac{4.\ 4\ 8.\ 12.\ 12.\ 16.\ 20.\ 24.\ \text{etc}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ 22.\ \text{etc.}}$$

vbi numeratores ſunt numeri pariter pares, denominatores vero impariter pares vtrique vnitate vel maiores vel minores quam numeri primi impares. Si ergo ſingulae fractiones per binarium deprimantur, numerator continebit numeros pares, denominator vero impares, atque bini reſpondentes et vnitate a ſe inuicem different, et coniuncti numerum primum conſtituent. Habebitur igitur

$$2 = \frac{2.\ 2.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 7.\ 9.\ 9.\ 11.\ \text{etc.}}$$

Q. E. D.

Theo-

Theorema 13.

Si omnes numeri impares non primi in duas partes diuidantur vnitate a ſe inuicem diſtantes harumque pares pro numeratoribus impares vero pro denominatoribus accipiantur, erit

$$\frac{\pi}{4} = \frac{4.\ 8.\ 10.\ 12.\ 14.\ 16.\ 18.\ 20.\ 22.\ 24.\ \text{etc.}}{5.\ 7.\ 11.\ 13.\ 13.\ 17.\ 17.\ 19.\ 23.\ 25.\ \text{etc.}}$$

Demonſtratio.

Cum per *Walliſii* quadraturam circuli ſit

$$\frac{\pi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11.\ 13.\ \text{etc.}}$$

cuius expreſſionis ſi ſinguli numeratores ad ſuos reſpondentes denominatores addantur, prodeunt omnes omnino numeri impares. At quoniam ſimilis expreſſio, ſi ex numeris imparibus primis tantum formatur, aequatur binario, vti in praecedente Theoremate eſt demonſtratum, quo erat

$$2 = \frac{2.\ 2.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ \text{etc.}}{1.\ 3.\ 3.\ 5.\ 7.\ 9.\ 9.\ 11.\ \text{etc.}}$$

ſi illa expreſſio per hanc diuidatur, proueniet

$$\frac{\pi}{4} = \frac{4.\ 8.\ 10.\ 12.\ 14.\ 16.\ 18.\ 20.\ 22.\ 24.\ \text{etc.}}{5.\ 7.\ 11.\ 13.\ 13.\ 17.\ 17\ 19.\ 23.\ 25.\ \text{etc.}}$$

quae ſimiliter ex numeris imparibus non primis formatur. Numeratores ſcilicet erunt numeri pares, denominatores vero impares vnitate a numeratoribus diſtantes, atque ſinguli numeratores ad ſuos reſpondentes denominatores additi dabunt omnes numeros impares non primos. Q. E. D.

Theorema 14.

Denotante vt hactenus π circuli peripheriam cuius diameter eſt 1, *dico fore*

$$\frac{\pi}{2} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ 23.\ 29.\ 31.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ 22.\ 30.\ 30.\ \text{etc.}}$$

cuius expreſſionis numeratores conſtituunt ſeriem numerorum primorum imparium, denominatores vero ſunt numeri impariter pares vnitate) vel minores vel maiores quam numeratores reſpondentes.

Demonſtratio.

Per Coroll. Theor. 8. ſi per $\frac{3}{4}$ multiplicetur eſt

$$\frac{\pi^2}{8} = \frac{3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 11.\ 11.\ 13.\ 13.\ \text{etc.}}{2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 10.\ 12.\ 12.\ 14.\ \text{etc.}}$$

in qua numeratores ſunt numeri primi impares bis poſiti, denominatores vero numeri tam pariter pares quam impariter pares, vnitate vel maiores vel minores quam ipſi numeri primi. Deinde Theoremate 11. demonſtrauimus eſſe

$$\frac{\pi}{4} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ 23.\ \text{etc.}}{4.\ 4.\ 8.\ 12.\ 12.\ 16.\ 20.\ 24.\ \text{etc.}}$$

in qua expreſſione numeratores ſunt numeri primi impares ſemel poſiti, denominatores vero numeri pariter pares vnitate diſtantes a numeris primis, ita vt haec expreſſio inpraecedente ſit contenta. Quare ſi illa expreſſio per hanc diuidatur prodibit

$$\frac{\pi}{2} = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ \text{etc.}}$$

in qua numeri impares primi numeratores conſtituunt; denominatores vero ſunt numeri impariter pares vnitate vel maiores vel minores quam numeratores. Q. E. D.

Theorema 15.

Denotante π peripheriam circuli, cuius diameter eſt 1, *erit*

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \frac{1}{15} - \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23}$$
$$+ \frac{1}{25} + \frac{1}{27} - \frac{1}{29} + \frac{1}{31} + \frac{1}{33} - \frac{1}{35} - \frac{1}{37}\ \text{etc.}$$

cuius ſeriei denominatores ſunt numeri impares omnes, ra-

tio

tio signorum autem hoc nititur fundamento. Numeris primus huius formae $4n-1$ *datur signum* $+$; *numeris primis autem huius formae* $4n+1$ *signum* -1. *Deinde numeris compositis id tribuitur signum, quod ipsis ratione compositionis ex primis cum suis signis secundum multiplicationis regulam competit.*

Demonstratio.

Quemadmodum vsitatis hic operationibus haec series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{15}+\text{etc.}$$

conuersa est in hanc expressionem $\frac{3.\ 5.\ 7.\ 11.\ \text{etc.}}{4.\ 4.\ 8.\ 12.\ \text{etc.}}$, ita viciſſim methodus poteſt excogitari, qua hanc expreſſionem

$$\frac{3.\ 5.\ 7.\ 11.\ 13.\ \text{etc.}}{4.\ 4.\ 8.\ 12.\ 12.\ \text{etc.}}$$

in ſeriem

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}\ \text{etc.}$$

transmutare liceat. Atque ſi haec methodus ad expreſſionem theoremate praecedente inuentam

$$\frac{\pi}{2}=\frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ \text{etc.}}$$

adhibeatur, iſta expreſſio abibit in iſtam ſeriem propoſitam

$$1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}\ \text{etc.}$$

cuius propterea ſumma eſt $\frac{\pi}{2}$. Ad idem a poſteriore colligere licet, ponendo

$$x=1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}+\tfrac{1}{15}-\tfrac{1}{17}\ \text{etc.}$$

eritque

$$\tfrac{1}{3}x=\tfrac{1}{3}+\tfrac{1}{9}-\tfrac{1}{15}+\tfrac{1}{21}+\tfrac{1}{27}+\tfrac{1}{33}-\text{etc.}$$

atque ſubtrahendo prodibit

$$\tfrac{2}{3}x=1-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}\ \text{etc.}$$

Deinde

Deinde cum ſimili modo ſit

$$\frac{1}{3}\cdot\frac{2}{3}x = \frac{1}{5} - \frac{1}{25} + \frac{1}{35} + \frac{1}{55} \text{ etc.}$$

prodibit addendo

$$\frac{6.\ 2}{5.\ 3}x = 1 + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} \text{ etc.}$$

Atque ſimiliter omnibus tollendis terminis praeter primum 1 inuenietur

$$x = \frac{3.\ 5.\ 7.\ 11.\ 13.\ 17.\ 19.\ \text{etc.}}{2.\ 6.\ 6.\ 10.\ 14.\ 18.\ 18.\ \text{etc.}} = \frac{\pi}{2}.$$

Atque hinc ſimul ratio ſignorum ſeriei propoſitae colligitur eadem ipſa, quam deſcripſimus. Q. E. D.

Corollarium.

Summa ergo ſeriei propoſitae

$$1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} \text{ etc.}$$

duplo maior eſt quam ſumma huius ſeriei

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ etc.}$$

Quare cum ipſae fractiones ſint vtrinque eaedem, ſolis ſignis effectum eſt, vt altera alterius ſit dupla.

Theorema 16.

Poſito π vt hactenus pro peripheria circuli cuius diameter eſt 1, erit

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{6} + \frac{1}{10} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} + \frac{1}{18} + \frac{1}{20} \text{ etc.}$$

Fractionum autem affirmatiuarum denominatores ſunt vnitate minores quam numeri impares non poteſtates; fractionum autem negatiuarum denominatores ſunt vnitate maiores. Cuiusque autem fractionis ſignum congruit cum ſigno numeri imparis vel vnitate maioris vel minoris non poteſtatis in praecedente Theoremate.

Demon-

Demonſtratio.

Haec ipſa ſeries oritur ex conuerſione praecedentis ſecundum modum in Theorematis 1. 2. 3. vſitatis, quo continuo progreſſiones geometricae vel adduntur vel demuntur, quoad ſolus primus terminus ſuperſit. Q.E.D.

Theorema 17.

Si numeris imparibus primis huius formae $4n - 1$ tribuatur ſignum $+$, reliquis huius formae $4n + 1$ ſignum $-$, numeris vero compoſitis ea ſigna, quae ipſis per regulas multiplicationis ex primis competunt; erit

$$\frac{3\pi}{8} = 1 + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{25} + \frac{1}{33} - \frac{1}{35} - \frac{1}{39} + \frac{1}{49} - \frac{1}{51} \text{ etc.}$$

qui denominatores ſunt facta vel ex duobus vel ex 4 vel ex 6 vel etc. numeris primis.

Demonſtratio.

Cum enim per Theorem. 15 ſit

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} - \frac{1}{17} + \frac{1}{19} \text{ etc.}$$

vbi denominatores ſunt omnes numeri impares et ſigna eam tenent legem quam praeſcripſimus; atque praeterea ſit

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{ etc.}$$

quarum ſerierum ii termini ſigna habent eadem, quorum denominatores ſunt facta vel ex binis vel quaternis, vel ſenis etc. numeris primis. His ergo ſeriebus additis ſoli, iſti termini remanebunt, diuiſioneque facta per 2 erit

$$\frac{3\pi}{8} = 1 + \frac{1}{9} - \frac{1}{15} + \frac{1}{21} + \frac{1}{25} + \frac{1}{33} \text{ etc.}$$

quae eſt ipſa ſeries propoſita, atque ex lege ſignorum ſimul ſequitur, eas fractiones habere ſignum $+$, quarum denominatores in hac forma $4n+1$ continentur; reliquas ſignum $-$. Q. E. D.

Corollarium.

Si ab huius theorematis ſerie ſubtrahatur iſta:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

prodibit

$$\frac{\pi}{8} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} \text{ etc.}$$

cuius denominatoris ſunt vel ipſi numeri primi vel facta ex ternis vel quinis vel etc. ii vero qui ſunt formae $4n-1$ ſignum habent $+$ reliqui formae $4n+1$ ſignum $-$.

Theorema 18.

Si omnibus numeris primis tribuatur ſignum $-$, cuique vero numero compoſito id ſignum quod ipſi ſecundum multiplicationis regulas competit, atque ex omnibus numeris ſequens formetur ſeries

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \text{etc.}$$

erit eius ſumma in infinitum continuatae $= 0$.

Demonſtratio.

Sit enim $x =$ ſummae iſtius ſeriei ſeu

$$x = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} \text{ etc.}$$

erit per operationes in poſterioribus theorematis adhibitas

$$\frac{3}{2}x = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \text{ etc.}$$

atque ſimili modo

$$\frac{3}{2} \cdot \frac{4}{3} x = 1 - \frac{1}{5} - \frac{1}{7} - \frac{1}{11} - \frac{1}{13} \text{ etc.}$$

Denique hac operatione infinities repetita erit

$$\frac{3 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdot 14 \cdot \text{etc.}}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot \text{etc.}} x = 1.$$

At cum ſit per Theor. ſeptimum

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10} \text{ etc.} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} \; l\infty:$$

facile intelligitur et noſtrum ipſius x coefficientem eſſe infinite:

infinite magnum. Quare quo factum aequale esse possit $\frac{1}{}$ erit $x=0$, et hanc ob rem habebitur

$$0=1-\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{6}-\tfrac{1}{7}-\tfrac{1}{8}\text{ etc.}$$

cuius denominatores, qui sunt vel ipsi primi vel facta ex ternis, quinis, etc. habent signum $-$ reliqui signum $+$. Q.E.D.

Corollarium I.

Modus ergo apparet, quo in progressione harmonica signa in singulis terminis sint distribuenda, vt summa totius seriei fiat $=0$.

Corollarium 2.

Cum inuenerimus $x=0$, erit quoque $\frac{1}{2}x=0$, et hanc ob rem habebimus quoque

$$0=1-\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}-\tfrac{1}{13}+\tfrac{1}{15}\text{ etc.}$$

in qua numeri tantum impares occurrunt; descriptamque ratione signorum tenent legem.

Theorema 19.

Summa seriei reciprocae numerorum primorum

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{13}+\textit{ etc.}$$

est infinite magna; infinities tamen minor, quam summa seriei harmonicae $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$ *etc. Atque illius summa est huius summae quasi logarithmus.*

Demonstratio.

Ponatur $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{ etc.}=A$ atque $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.}=B$ et $\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{5^3}+\text{etc.}=C$. atque ita porro omnes potestates peculiaribus litteris designando; erit posito e pro numero cuius logarithmus hyperbolicus est 1

 e^{A+}

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\text{etc.}} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\ \text{etc.}$$

Nam eſt

$$A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D\ \text{etc.} = l\,\frac{2}{1}+l\,\frac{3}{2}+l\,\frac{5}{4}+l\,\frac{7}{6}+\ \text{etc.}$$

ideoque

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D} = \frac{2.\,3.\,5.\,7.\ \text{etc.}}{1.\,2.\,4.\,6.\ \text{etc.}} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ \text{etc.}$$

per Theor. 7. At non ſolum B, C, D, etc. habebunt valores finitos, ſed etiam $\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ etc. valorem habet finitum. Quare quo

$$e^{A+\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+\text{etc.}}\ \text{fit} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \text{etc.} = \infty.$$

oportet vt A ſit infinite magnum, cumque ideo eius reſpectu ſequentes termini $\frac{1}{2}B+\frac{1}{3}C+\frac{1}{4}D+$ etc. euaneſcant, erit

$$e^{A} = e^{\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\text{etc.}} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ \text{etc.}$$

Atque conſequenter erit

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\ \text{etc.}$$
$$= l\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{7}+\ \text{etc.}\right)$$

illius ergo ſeriei ſumma erit infinities minor quam huius, atque cum huius ſumma ſit $= l\infty$ erit

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ \text{etc.} = l.\, l\infty.$$

Q. E. D.

DE