THE JOURNAL OF FINANCE • VOL. XLVII, NO. 4 • SEPTEMBER 1992

Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model

FRANCIS A. LONGSTAFF and EDUARDO S. SCHWARTZ*

ABSTRACT

We develop a two-factor general equilibrium model of the term structure. The factors are the short-term interest rate and the volatility of the short-term interest rate. We derive closed-form expressions for discount bonds and study the properties of the term structure implied by the model. The dependence of yields on volatility allows the model to capture many observed properties of the term structure. We also derive closed-form expressions for discount bond options. We use Hansen's generalized method of moments framework to test the cross-sectional restrictions imposed by the model. The tests support the two-factor model.

THERE ARE ESSENTIALLY TWO approaches to the modeling of the term structure of interest rates in continuous time. The equilibrium approach pioneered by Cox, Ingersoll, and Ross (CIR 1981, 1985a, and 1985b) starts from a description of the underlying economy and from assumptions about the stochastic evolution of one or more exogenous factors or state variables in the economy and about the preferences of a representative investor. General equilibrium considerations are used to endogenously derive the interest rate and the price of all contingent claims. The arbitrage approach starts from assumptions about the stochastic evolution of one or more interest rates and derives the prices of all contingent claims by imposing the condition that there are no arbitrage opportunities in the economy (for example, see Vasicek (1977) and Brennan and Schwartz (1979)). A variation of this approach looks at the initial yield curve and considers movements in the yield curve (Ho and Lee

* College of Business, The Ohio State University, and The Anderson Graduate School of Management, UCLA, respectively. We are grateful for the suggestions and comments of Avi Bick, Fischer Black, Riccardo Cesari, K. C. Chan, Michael Hemler, Andrew Karolyi, Chester Spatt; workshop participants at the University of British Columbia, the University of Colorado at Boulder, Columbia University, the University of Geneva, Harvard University, the University of Iowa, the Kansallis Foundation for Financial Research, New York University, the University of St. Gallen, UCLA, Washington University; and participants at the Research Institute of the Finnish Economy, the 1991 European Finance Association meetings, the Second Summer Symposium of the European Science Foundation Network in Financial Markets, the Second International Conference of the Centre for Research in Finance—IMI Group, the 1991 Western Finance Association meetings, and the 1991 Western Economic Association meetings. The first author acknowledges financial support from the Charles A. Dice Center for Research in Banking and Financial Economics. We are particularly grateful for the comments of two anonymous referees and the editor, René Stulz. All errors are our responsibility.

(1986)), or changes in forward rates (Heath, Jarrow, and Morton (1988)) consistent with no arbitrage possibilities.

The equilibrium approach has several clear advantages over the arbitrage approach. For example, both the term structure and its dynamics are endogenously determined in the equilibrium approach. Furthermore, the functional forms of the factor risk premiums (market prices of risk) are also obtained as part of the equilibrium. In contrast, the arbitrage approach provides no guidance as to the form of the factor risk premiums. Moreover, the arbitrary choice of the functional form can lead to internal inconsistencies or arbitrage opportunities.[1]

In this paper, we develop a two-factor general equilibrium model of the term structure of interest rates using the CIR (1985a) framework and apply it to the valuation of discount bonds and other interest-rate-sensitive contingent claims. In particular, we derive closed-form expressions for discount bond prices and for discount bond option prices. The two factors of the model are the short-term interest rate and the instantaneous variance of changes in the short-term interest rate. This feature has the important advantage of allowing contingent claim values to reflect both the current level of interest rates as well as the current level of interest rate volatility. Furthermore, recent evidence by Dybvig (1989) suggests that these variables are the two most important factors in explaining movements in the term structure.

One of the primary motivations for developing a multifactor model of the term structure is that single-factor models imply that the instantaneous returns on bonds of all maturities are perfectly correlated—a property that is clearly inconsistent with reality. Two-factor arbitrage models have been suggested by Brennan and Schwartz (1979) who use the short-term and long-term interest rates, by Schaefer and Schwartz (1984) who use the long-term interest rate and the spread between the long-term and short-term interest rates, and by Heath, Jarrow, and Morton (1988) who use two unspecified factors that affect all forward rates. Closer to our approach is the two-factor equilibrium model suggested by Cox, Ingersoll, and Ross (1985b), who use an exogenously specified process for uncertain inflation in addition to the short-term interest rate. The CIR model has the advantage of allowing nominal securities to be priced. Our model, however, has the advantage that the second factor (interest rate volatility), its dynamics, and its factor risk premium are all endogenously determined. Furthermore, using interest rate volatility as the second state variable is intuitively appealing since volatility is a key variable in pricing contingent claims such as interest rate options and bonds. The explicit dependence of yields on interest rate volatility allows the model to capture many of the observed properties of the term structure such as humps, troughs, and the relation between term premia and interest rate volatility.

We test the cross-sectional restrictions imposed on the term structure by the two-factor model using the generalized method of moments (GMM)

[1] For a more detailed discussion of this issue, see Cox, Ingersoll, and Ross (1985b), Section 5.

approach of Hansen (1982). Using Treasury-bill yields ranging from three months to five years in maturity, we find that these cross-sectional restrictions cannot be rejected. We also test the restrictions imposed by the CIR (1985b) single-factor model, which is nested within the two-factor model as a special case.

The plan of our paper is as follows. In Section I, we develop the general equilibrium framework using a two-state variable version of the model developed in Cox, Ingersoll, and Ross (1985a). In Section II, we obtain closed-form expressions for discount bond prices and examine the properties of the term structure obtained. Closed-form expressions for discount bond options are derived in Section III. In Section IV, we provide the empirical results of the paper, and we conclude in Section V.

I. The General Equilibrium Framework

In this section, we develop a general equilibrium framework for valuing interest-rate-sensitive contingent claims using a two-state-variable version of the continuous-time economy modeled by Cox, Ingersoll, and Ross (1985a). In doing this, we make the following assumptions about production and preferences. First, all physical investment is performed by a single stochastic constant-returns-to-scale technology which produces a good that is either consumed or reinvested in production. The realized returns on physical investment are governed by the stochastic differential equation

$$\frac{dQ}{Q} = (\mu X + \theta Y)\, dt + \sigma\sqrt{Y}\, dZ_1, \tag{1}$$

where μ, θ, and σ are positive constants, X and Y are state variables, and Z_1 is a scalar Wiener process. In this specification, expected returns are driven by two economic factors, X and Y. The first factor X represents the component of expected returns that is unrelated to production uncertainty, while Y represents the component common to both. An advantage of this specification is that expected returns and production volatility are not required to be perfectly correlated. This feature is consistent with the empirical evidence of Merton (1980), French, Schwert, and Stambaugh (1987), and others.[2] The dynamics of the state variables X and Y are governed by

$$dX = (a - bX)\, dt + c\sqrt{X}\, dZ_2, \tag{2}$$

$$dY = (d - eY)\, dt + f\sqrt{Y}\, dZ_3, \tag{3}$$

where $a, b, c, d, e, f > 0$, and Z_2 and Z_3 are scalar Wiener processes. Since changes in X are assumed to represent technological changes that are unrelated to production uncertainty, we require that Z_2 be uncorrelated with

[2] Using stock market data, Merton (1980) and French, Schwert, and Stambaugh (1987) find that variation in expected returns is not due entirely to return volatility.

Z_1 and Z_3. Following Cox, Ingersoll, and Ross (1985b), we also require that $\theta > \sigma^2$, which guarantees that the riskless rate is non-negative.[3]

We assume that there is a fixed number of identical individuals with time-additive preferences of the form

$$E_t\left[\int_t^\infty \exp(-\rho s)\ln(C_s)\,ds\right], \tag{4}$$

where $E[\cdot]$ is the conditional expectation operator, ρ is the utility discount factor, and C_s represents time-s consumption. Finally, we assume the existence of perfectly competitive, continuous markets for riskless borrowing and lending as well as other types of contingent claims.

In this setting, the representative investor's decision problem is equivalent to maximizing (4) subject to the budget constraint

$$dW = W\frac{dQ}{Q} - C\,dt, \tag{5}$$

where W denotes wealth, by selecting an optimal level of consumption and reinvesting unconsumed wealth in physical production. As in Merton (1971), the investor's derived utility of wealth function (value function) is partially separable and has the simple form

$$J(W,X,Y,t) = \frac{\exp(-\rho t)}{\rho}\ln(W) + G(X,Y,t). \tag{6}$$

Standard results can be used to show that optimal consumption is ρW.[4] Substituting optimal consumption and (1) into (5) gives the following equilibrium dynamics for wealth:

$$dW = (\mu X + \theta Y - \rho)W\,dt + \sigma W\sqrt{Y}\,dZ_1. \tag{7}$$

Together, W, X, and Y form a joint Markov process—the current values of W, X, and Y completely describe the state of the economy and the joint distribution of future investment returns. Note that the CIR (1985b) single-factor setting can be nested within this framework by imposing the restriction $\mu = 0$ in (1).

A simple rescaling of the state variables gives $x = X/c^2$ and $y = Y/f^2$. Let $H(x,y,\tau)$ denote the value of a contingent claim with maturity τ and boundary conditions that do not depend on wealth. Given this framework, Theorem 3 of Cox, Ingersoll, and Ross (1985a) can be applied to obtain the fundamental partial differential equation satisfied by the contingent claim

$$\begin{aligned}(x/2)H_{xx} &+ (y/2)H_{yy} + (\gamma - \delta x)H_x \\ &+ (\eta - \xi y - (-J_{WW}/J_W)Cov(W,Y))H_y - rH = H_\tau,\end{aligned} \tag{8}$$

[3] See Cox, Ingersoll, and Ross (1985b) footnote 6.

[4] For example, let $t' \to \infty$ in Cox, Ingersoll, and Ross (1985b) footnote 5.

where $\gamma = a/c^2$, $\delta = b$, $\eta = d/f^2$, $\xi = e$, r is the instantaneous riskless rate, and $\mathrm{Cov}(W, Y)$ is the instantaneous covariance of changes in W with changes in Y. The utility-dependent term in the coefficient of H_y represents the market price of the risk of changes in the level of production uncertainty which is governed by Y.[5] Because of the separable form of the derived utility of wealth function, however, (3), (6), and (7) can be used to show that

$$(-J_{WW}/J_W)Cov(W, Y) = \lambda y, \tag{9}$$

where λ is a constant. Thus, the market price of risk is proportional to y. It is important to observe that this form of the risk premium is endogenously determined by the model, rather than exogenously imposed. This feature ensures that the risk premium is consistent with the absence of arbitrage—a property which cannot be guaranteed for partial equilibrium models that assume a specific functional form for the risk premium. Once r is specified in terms of the state variables, the value of the contingent claim can be determined by solving the partial differential equation in (8) with respect to the appropriate boundary and initial conditions.

This approach results in contingent claim values that are expressed in terms of the unobservable state variables. Rather than leaving contingent claim values in this form, however, we make a simple change of variables that allows us to express contingent claim prices in terms of two intuitive and readily estimated economic variables. These factors are the short-term interest rate r and the variance of changes in the short-term interest rate, which we designate by V.[6] An important advantage of expressing contingent claim values in terms of these two variables is that it allows contingent claim prices to reflect both the current level of interest rates as well as the current level of interest rate volatility.[7] This enables us to use more information about the term structure than the current level of r in pricing contingent claims. Furthermore, since the volatility of interest rates is a key determinant of many contingent claim values, this approach has the potential to lead to valuation expressions that are more consistent with actual prices than models that exclude interest rate volatility.

The equilibrium riskless interest rate can be obtained by applying Theorem 1 of Cox, Ingersoll, and Ross (1985a) or the results in Breeden (1986) which relate the riskless rate to the expected rate of change in marginal utility. Because of the logarithmic form of the derived utility of wealth function, the short-term interest rate is simply the expected return on production minus

[5] Because Z_2 is uncorrelated with Z_1 and Z_3, changes in X cannot be hedged and their risk is unpriced.

[6] The instantaneous variance is actually $V\,dt$. Consistent with standard usage, however, we omit the dt term in discussing the instantaneous variance since the implicit time horizon is clear from the context.

[7] This change of variables is similar to that used in the stock index futures model of Hemler and Longstaff (1991). However, their model uses the volatility of stock market returns rather than the volatility of changes in the riskless rate.

the variance of production returns

$$r = \alpha x + \beta y, \tag{10}$$

where $\alpha = \mu c^2$ and $\beta = (\theta - \sigma^2)f^2$, which is non-negative for all feasible values of the state variables.[8] The instantaneous variance of changes in the riskless rate V can be obtained by applying Ito's Lemma to the expression for r in (10) and taking the appropriate expectation. The resulting expression is

$$V = \alpha^2 x + \beta^2 y, \tag{11}$$

which is also non-negative for all feasible values of the state variables.[9] Together, (10) and (11) form a simple system of two linear equations in x and y. Provided that $\alpha \neq \beta$, this system is globally invertible and we can solve for x and y in terms of r and V, obtaining

$$x = \frac{\beta r - V}{\alpha(\beta - \alpha)}, \tag{12}$$

$$y = \frac{V - \alpha r}{\beta(\beta - \alpha)}. \tag{13}$$

This mapping allows us to make the change of variables from x and y to r and V.

The dynamics of r and V can be obtained by applying Ito's Lemma to (10) and (11) and making the corresponding change of variables

$$dr = \left(\alpha\gamma + \beta\eta - \frac{\beta\delta - \alpha\xi}{\beta - \alpha} r - \frac{\xi - \delta}{\beta - \alpha} V\right) dt + \alpha\sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}}\, dZ_2 + \beta\sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}}\, dZ_3, \tag{14}$$

$$dV = \left(\alpha^2\gamma + \beta^2\eta - \frac{\alpha\beta(\delta - \xi)}{\beta - \alpha} r - \frac{\beta\xi - \alpha\delta}{\beta - \alpha} V\right) dt + \alpha^2\sqrt{\frac{\beta r - V}{\alpha(\beta - \alpha)}}\, dZ_2 + \beta^2\sqrt{\frac{V - \alpha r}{\beta(\beta - \alpha)}}\, dZ_3. \tag{15}$$

The requirement that the original parameters be positive implies that α, β, γ, δ, η, and ξ are also positive.[10]

[8] Since the single good in the economy can be used for consumption or investment, it is appropriate to view the commodity as storable. Hence, a lower bound of zero for the riskless rate is consistent with the basic properties of this economy.

[9] If $2a > c^2$ and/or $2d > f^2$, zero is an inaccessible value for X and/or Y. In this situation, the riskless rate and the instantaneous variance are strictly positive rather than just non-negative. See Feller (1951) for a derivation of these conditions.

[10] Only six parameters are necessary to describe the evolution of the interest rate and volatility processes. This is because the parameters μ, θ, σ^2, a, c, d, and f affect the dynamics only through the parameters α, β, γ, and η.

These dynamics, in conjunction with (10) and (11), have a number of important implications for r and V. For example, the two processes are interdependent. Specifically, the stochastic evolution of r depends on V, and vice versa. Together r and V form a joint Markov process. Focusing first on r, observe that (10) implies that r can take any value in the range from zero to infinity since both of the original state variables follow square root processes. The riskless rate, however, has a long-run stationary (unconditional) distribution with mean

$$E[r] = \frac{\alpha\gamma}{\delta} + \frac{\beta\eta}{\xi}, \tag{16}$$

and variance

$$Var[r] = \frac{\alpha^2\gamma}{2\delta^2} + \frac{\beta^2\eta}{2\xi^2}. \tag{17}$$

The stationary density of r corresponds to the density of a linear combination of independent gamma variates. The distribution and properties of linear combinations of gamma variates are described in Johnson and Kotz (1970), Chapter 29, pp. 154–168. From (14), the instantaneous variance of changes in r is $(\alpha\beta r - \alpha V + \beta V - \alpha\beta r)\,dt/(\beta - \alpha) = V dt$ as expected.

A similar analysis shows that V can also take any value from zero to infinity. From (14) and (15), it can be shown that changes in r and in V are positively correlated. Furthermore, the actual value of the correlation coefficient can range from zero to one. This feature is consistent with the well-known positive relation between interest rates and interest rate volatility.[11] Because of this positive correlation, however, the values of r and V are linked. For example, if r equals zero, then V must also equal zero. This is because r can only equal zero if both X and Y equal zero, which implies that V must also equal zero. Because of this link, the value of V is constrained to be between the values αr and βr. This condition also ensures that the terms under the square root signs in (14) and (15) are non-negative. The fact that V can take any value within these upper and lower bounds illustrates one of the important advantages of this two-state-variable framework over single-state-variable settings. In many single-state-variable term structure models, the variance of changes in r is proportional to r—the variance V is completely determined by the value of r.[12] In contrast, our framework allows V to depend on more than the current value of r. As before, V also has a stationary distribution with mean

$$E[V] = \frac{\alpha^2\gamma}{\delta} + \frac{\beta^2\eta}{\xi}, \tag{18}$$

[11] For example, see Chan, Karolyi, Longstaff, and Sanders (1991).

[12] For example, see Vasicek (1977), Brennan and Schwartz (1977), Dothan (1978), Cox, Ingersoll, and Ross (1985b), and Longstaff (1989).

and variance

$$Var[V] = \frac{\alpha^4\gamma}{2\delta^2} + \frac{\beta^4\eta}{2\xi^2}. \tag{19}$$

The unconditional distribution of V is also that of a linear combination of gamma variates.

II. The Equilibrium Term Structure

In this section, we obtain closed-form expressions for discount bond prices and examine their implications for the properties of the term structure. Let $F(r, V, \tau)$ denote the value of a riskless unit discount bond with τ periods until maturity. The equilibrium value of $F(r, V, \tau)$ can be determined by solving (8), subject to the maturity condition that the bond value equals one when $\tau = 0$, and then making a change of variables to r and V. A separation of variables approach results in the following solution for $F(r, V, \tau)$:

$$F(r, V, \tau) = A^{2\gamma}(\tau)B^{2\eta}(\tau)\exp(\kappa\tau + C(\tau)r + D(\tau)V), \tag{20}$$

where

$$A(\tau) = \frac{2\phi}{(\delta + \phi)(\exp(\phi\tau) - 1) + 2\phi},$$

$$B(\tau) = \frac{2\psi}{(\nu + \psi)(\exp(\psi\tau) - 1) + 2\psi},$$

$$C(\tau) = \frac{\alpha\phi(\exp(\psi\tau) - 1)B(\tau) - \beta\psi(\exp(\phi\tau) - 1)A(\tau)}{\phi\psi(\beta - \alpha)},$$

$$D(\tau) = \frac{\psi(\exp(\phi\tau) - 1)A(\tau) - \phi(\exp(\psi\tau) - 1)B(\tau)}{\phi\psi(\beta - \alpha)},$$

and

$$\nu = \xi + \lambda,$$

$$\phi = \sqrt{2\alpha + \delta^2},$$

$$\psi = \sqrt{2\beta + \nu^2},$$

$$\kappa = \gamma(\delta + \phi) + \eta(\nu + \psi).$$

The discount bond price is a function of the variables r, V, and τ, and depends on the six parameters α, β, γ, δ, η, and ν.[13] The exponential form of (20) makes the closed-form solution tractable and easy to compute. Substitut-

[13] The market price of risk parameter λ and the parameter ξ enter the equation only through their sum ν and need not be separately specified. Not only does this reduce the number of parameters needed to specify contingent claim prices, but it eliminates the need to estimate the market price of risk as a separate parameter.

ing $\tau = 0$ into (20) shows that the maturity condition $F(r, V, 0) = 1$ is satisfied. The discount bond price does not equal one when $r = V = 0$.[14] This is because the dynamics for r and V imply that these variables immediately return to positive values if they ever reach zero. Thus, the forward rate must be strictly positive if $r = V = 0$, which implies $F(0, 0, \tau) < 1$ for $\tau > 0$. Finally, although no boundary condition is imposed as the state variables approach infinity, the discount bond has the economically realistic feature of converging to zero as $r \to \infty$ and $V \to \infty$.

This equilibrium model has many important implications for the behavior of discount bond prices and the term structure. For example, the partial derivative of $F(r, V, \tau)$ with respect to r can be either negative or positive. In particular, this partial derivative is always negative for small values of τ, but can become positive for bonds with longer maturities. This property contrasts with single-state-variable models of the term structure such as Vasicek (1977), Dothan (1978), the CIR (1985b) model in which bond prices are always decreasing functions of the short-term riskless interest rate. Intuitively, the reason for this property is that an increase in r—holding V fixed — can imply a decrease in the level of production uncertainty. In turn, this can imply a reduction in the required term premium, a corresponding reduction in the expected rate of return for the bond, and, therefore, an increase in the price of the bond.

Differentiating $F(r, V, \tau)$ with respect to V shows that the sign of this partial derivative is also indeterminate. Specifically, this partial derivative can be positive for all τ, negative for all τ, or can take on opposite signs for different τ. The reason for this property is again related to the effect of a change in V on the term premium. It is straightforward to show that this derivative approaches zero as $\tau \to 0$. Thus, the prices of instantaneously maturing bonds are unaffected by changes in V. This makes sense, of course, since we would expect the yields of these bonds to be determined only by r. The partial derivative of $F(r, V, \tau)$ with respect to τ is negative. This implies that forward rates are uniformly positive in this model.

From (20), we obtain the following expression for the yield on a τ-maturity bond Y_τ:

$$Y_\tau = -(\kappa\tau + 2\gamma \ln A(\tau) + 2\eta \ln B(\tau) + C(\tau)r + D(\tau)V)/\tau. \quad (21)$$

For a given maturity, the yield is a linear function of r and V. Using l'Hôpital's rule, it can be shown that the limit of Y_τ as $\tau \to 0$ is r. As $\tau \to \infty$, the yield to maturity converges to a constant independent of the current value of r and V

$$\gamma(\phi - \delta) + \eta(\psi - \nu). \quad (22)$$

Intuitively, this is because both r and V have long-run stationary distributions. Thus, as $\tau \to \infty$, the influence of current values of r and V becomes less

[14] The properties of r and V imply that r cannot reach zero unless V also reaches zero. Similarly, r cannot approach infinity unless V also approaches infinity.

relevant for their future value—the present value of infinitely distant future cash flows is unaffected by the current term structure.

Because discount bond prices depend on both the short-term interest rate and the current volatility of interest rates, the yield curve can take a greater variety of shapes than is possible for single-factor models of the term structure. For example, (21) allows the yield curve to be monotone increasing, to be monotone decreasing, to have a hump, to have a trough, and to have both a hump and a trough. Since the sign of the partial derivative of $F(r, V, \tau)$ with respect to r is indeterminate, changes in the short-term rate can have very complex effects on the overall yield curve. An increase in r can dramatically increase the slope of the yield curve for some maturities, while decreasing the slope for other maturities. In some cases, changes in r may induce "twists" into the yield curve such as when a monotone increasing yield curve becomes inverted for intermediate maturities.

Since V can vary even when r is held fixed, changes in the level of interest rate volatility can also have significant effects on the slope and volatility of the yield curve. This is important because an often-cited failing of single-factor term-structure models is that they lead to term structures that are too flat or do not capture the variation in longer-maturity yields. Changes in V can also have dramatic effects on the shape of the term structure. For example, varying V while holding r fixed can result in term structure that change from humped to monotonic. Generally, changes in V have the largest effect on intermediate-maturity yields.

From (8) and (9), the instantaneous expected return for a discount bond equals $r + \lambda y F_y / F$. Using the chain rule and changing variables, this can be re-expressed as

$$r + \lambda \frac{(\exp(\psi\tau) - 1)B(\tau)}{\psi(\beta - \alpha)}(\alpha r - V). \tag{23}$$

The instantaneous term premia for discount bonds can be obtained directly from this expression by simply subtracting r. For fixed τ, the resulting term premia are linear functions of both r and V. It is easily shown that term premia are always non-negative when $\lambda < 0$. For small τ, term premia are increasing functions of r. For large values of τ, however, the opposite can be true. Similarly, the relation between term premia and V is indeterminate. If the market price of risk parameter λ is zero, term premia are zero and the Local Expectations Hypothesis holds. This is consistent with Cox, Ingersoll, and Ross (1981), who show that the Local Expectations Hypothesis holds in a continuous-time economy if production returns are uncorrelated with changes in the state variables. Since the original state variable X is by assumption uncorrelated with production returns, the restriction $\lambda = 0$ implies that the state variable Y is also uncorrelated with production returns. If both state variables are uncorrelated with production returns, r and V must also be uncorrelated with production returns, and the Local Expectations Hypothesis holds. In general, if $\lambda \neq 0$, then there are two sources of variation in term premia. This is consistent with recent empirical evidence by Shiller (1979), Startz (1982), Shiller, Campbell, Schoenholtz (1983), Fama (1984), Mankiw (1986), Shiller (1986), and Froot (1989) who document the existence of time varying term premia. The dependence of term premia on V is particularly consistent with the findings of Campbell (1986), Engle, Lilien, and Robins (1987), Lauterbach (1989), and Simon (1989), who show that excess returns on bonds are related to term-structure volatility.

A number of recent papers have addressed the topic of the term structure of interest rate volatility. For example, Schaefer and Schwartz (1987) explicitly model the term structure of bond-return volatility in developing a model of bond option prices. The term structure of bond-return volatility implied by our model can be determined by first applying Ito's Lemma to (20) and then taking the appropriate expectations of the stochastic component of bond returns. The resulting expression for the instantaneous volatility of bond returns is

$$\left(\frac{\alpha\beta\psi^2(\exp(\phi\tau)-1)^2 A^2(\tau) - \alpha\beta\phi^2(\exp(\psi\tau)-1)^2 B^2(\tau)}{\phi^2\psi^2(\beta-\alpha)}\right) r + \left(\frac{\beta\phi^2(\exp(\psi\tau)-1)^2 B^2(\tau) - \alpha\psi^2(\exp(\phi\tau)-1)^2 A^2(\tau)}{\phi^2\psi^2(\beta-\alpha)}\right) V. \quad (24)$$

This variance depends on the maturity of the bond as well as the state variables r and V. This explicit dependence on both factors is important because it implies a greater variety of possible term structures of volatility than can be generated from single-factor models. Holding r fixed, the variance of returns converges to zero as $\tau \to 0$. As $\tau \to 0$, the variance of returns converges to a fixed value. The partial derivative of the variance with respect to τ is always positive, which implies that the volatility is an increasing function of the maturity or duration of the bond.

III. Discount Bond Options

The general equilibrium framework also allows us to derive closed-form expressions for other contingent claims such as bond options. In recent years, many single-factor models of discount bond option prices have appeared in the literature. These include Brennan and Schwartz (1977), Courtadon (1982), Ball and Torous (1983), CIR (1985b), Schaefer and Schwartz (1987), and Jamshidian (1989). In this section, we use the general equilibrium framework to obtain an explicit two-factor model of discount bond option prices. The resulting model has the advantage of allowing option prices to depend explicitly on the current level of interest rate volatility. This is important because volatility is a fundamental determinant of option values. This model can be used to value a number of common types of interest-rate-sensitive contingent claims such caps and floors.

Let K denote the strike price of a European call option on a discount bond. Let τ denote the time until expiration of the option. If the option is in the money at expiration, the callholder exercises the call by paying K and receives a discount bond with a maturity date T periods later. Thus, the payoff function for the call option at maturity is

$$\max(0, F(r, V, T) - K). \tag{25}$$

We require that $F(0, 0, T) > K$. Otherwise, $F(r, V, T) < K$ for all r and V, and the option is worthless since it will never be exercised. Denote the value of the call option $C(r, V, \tau; K, T)$. Since the payoff function does not depend on the level of aggregate wealth, the call price satisfies the partial differential equation (8) subject to the maturity condition (25). A factorization technique, in conjunction with multiple separations of variables, leads to the following solution for $C(r, V, \tau; K, T)$:

$$\begin{aligned} C(r, V, \tau; K, T) = {} & F(r, V, \tau + T)\Psi(\theta_1, \theta_2; 4\gamma, 4\eta, \omega_1, \omega_2) \\ & - KF(r, V, \tau)\Psi(\theta_3, \theta_4; 4\gamma, 4\eta, \omega_3, \omega_4), \end{aligned} \tag{26}$$

where

$$\theta_1 = \frac{4\zeta\phi^2}{\alpha(\exp(\phi\tau) - 1)^2 A(\tau + T)},$$

$$\theta_2 = \frac{4\zeta\psi^2}{\beta(\exp(\psi\tau) - 1)^2 B(\tau + T)},$$

$$\theta_3 = \frac{4\zeta\phi^2}{\alpha(\exp(\phi\tau) - 1)^2 A(\tau) A(T)},$$

$$\theta_4 = \frac{4\zeta\psi^2}{\beta(\exp(\psi\tau) - 1)^2 B(\tau) B(T)},$$

and

$$\omega_1 = \frac{4\phi \exp(\phi\tau) A(\tau + T)(\beta r - V)}{\alpha(\beta - \alpha)(\exp(\phi\tau) - 1) A(T)},$$

$$\omega_2 = \frac{4\psi \exp(\psi\tau) B(\tau + T)(V - \alpha r)}{\beta(\beta - \alpha)(\exp(\psi\tau) - 1) B(T)},$$

$$\omega_3 = \frac{4\phi \exp(\phi\tau) A(\tau)(\beta r - V)}{\alpha(\beta - \alpha)(\exp(\phi\tau) - 1)},$$

$$\omega_4 = \frac{4\psi \exp(\psi\tau) B(\tau)(V - \alpha r)}{\beta(\beta - \alpha)(\exp(\psi\tau) - 1)},$$

$$\zeta = \kappa T + 2\gamma \ln A(T) + 2\eta \ln B(T) - \ln K.$$

The function $\Psi(\theta_1, \theta_2; 4\gamma, 4\eta, \omega_1, \omega_2)$ is the bivariate noncentral chi-square distribution function

$$\int_0^{\theta_1} \int_0^{\theta_2 - \theta_2 u/\theta_1} \chi^2(u; 4\gamma, \omega_1)\chi^2(v; 4\eta, \omega_2)\, dv\, du, \tag{27}$$

where $\chi^2(\cdot\,; p, q)$ is the noncentral chi-square density with p degrees of freedom and noncentrality parameter q. The density function of the noncentral chi-square distribution is given in Johnson and Kotz (1970), Chapter 28, p. 133. Note that the product of the densities for the noncentral chi-square variates u and v in (27) is actually the joint density since the two variates are independent of each other. This feature dramatically reduces the computations that are required to calculate the value of the distribution function. The integral is evaluated over the triangular region in the $u \times v$ plane defined by the points $(0, 0)$, $(\theta_1, 0)$, and $(0, \theta_2)$. The function $\Psi(\theta_3, \theta_4; 4\gamma, 4\eta, \omega_3, \omega_4)$ is defined similarly. Johnson and Kotz ((1970), Chapter 28) also discuss a number of transformations of noncentral chi-square variates which are approximately normally distributed. Applying one of these transformations simplifies the problem of estimating the integral in (27) to one of estimating a cumulative bivariate normal distribution function. The value of a European discount bond put can be obtained directly from (26) by the put-call parity relation.

Since the variance of the riskless rate is stochastic in this general equilibrium framework, this bond option valuation model provides an important extension to the recent literature on option pricing with stochastic volatilities. Examples of stochastic-volatility option pricing models include Scott (1987), Wiggins (1987), Hull and White (1987), Johnson and Shanno (1987), and Bailey and Stulz (1989). It is important to observe, however, that none of these stochastic-volatility models provides a closed-form expression for option prices. In this respect, the closed-form expression for discount bond calls given in (26) is unique.[15]

As is the case for discount bond prices, many of the comparative statistics results for discount bond call values are indeterminate. For example, an increase in r can either increase or decrease the call value. This contrasts with the Black-Scholes option pricing formula in which calls are increasing functions of the riskless rate. Intuitively, the reason for the indeterminate sign is that while an increase in r reduces the present value of the strike price and thus increases the value of the call, it can also reduce the value of the underlying discount bond. The net effect depends on which effect dominates. This feature also explains why the properties of option prices implied by this model will differ from those implied by the Black-Scholes formula—the Black-Scholes model assumes that changes in the value of the underlying

[15] The expression in (26) is a closed-form solution in the same sense as the Black-Scholes formula, although computation of the cumulative distribution functions can be more cumbersome. In some circumstances, however, we have found that numerically solving the partial differential equation for the discount bond option price is as easy as evaluating (26) directly.

asset are unrelated to changes in the interest rate. The partial derivative of the call price with respect to V is also indeterminate. The intuition for this result is that while an increase in volatility tends to increase the value of the call, it can either increase or decrease the value of the underlying discount bond. Again, the actual sign of the partial derivative depends on which effect dominates. Finally, it is straightforward to show that the call price is a decreasing function of the strike price K.

IV. The Empirical Tests

Since contingent claim prices depend explicitly on r and V, the two-factor model is able to capture both the level of interest rates and the level of interest rate volatility. The two-factor model, however, goes beyond this and places strong cross-sectional restrictions on the evolution of the term structure. Specifically, (21) implies that changes in yields are known functions of changes in r and V. In this section, we propose a simple way of estimating the volatility of the riskless interest using the generalized autoregressive conditional heteroskedasticity (GARCH) framework introduced by Bollerslev (1986). We then use these estimates to test the cross-sectional restrictions imposed by the model using the generalized method of moments (GMM) of Hansen (1982). The results indicate that the two-factor model not only captures the level and volatility of the term structure, but also the cross-sectional structure of yield changes.

A. The Data

The model developed in this paper deals only with the real economy since it is beyond the scope of this research to introduce the additional elements required to provide a valid role for money.[16] Consistent with other empirical work in this area, however, we proceed to test the model using nominal yields.[17] We note that if nominal yields are inconsistent with the implications of the model, the cross-sectional restrictions are more likely to be rejected by the data.

In estimating the volatility of the short-term riskless rate we use data for one-month U.S. Treasury bill yields.[18] The tests of the cross-sectional restrictions are then conducted using the volatility estimates and yields for U.S. Treasury bills and notes with maturities ranging from three months to five years. The T-bill yield data is obtained from the data set originally con-

[16] As examples of the additional structure required to introduce money into the analysis in a meaningful way, see Gibbons and Ramaswamy (1987) and Foresi (1990).

[17] For example, see Brown and Dybvig (1986) and Pearson and Sun (1989).

[18] The one-month data is the shortest-maturity data available to us. Ideally, we would like to use a measure of the instantaneous riskless rate in estimating V. This is because, if the instantaneous riskless rate is strongly mean-reverting, estimates of V from one-month yields could differ slightly from estimates obtained from the instantaneous riskless rate. The effect of this, however, would probably be to bias the tests against the two-factor model.

structed by Fama (1984) and subsequently updated by the Center for Research in Security Prices (CRSP). These yields are based on the average of bid and ask prices for Treasury bills and are normalized to reflect a standard month of 30.4 days. The yield data for one- to five-year maturity bonds are obtained from the Fama and Bliss (1987) data set which is also updated by CRSP. These yields are based on the term structure for taxable, non-callable U.S. Treasury bonds with maturities up to five years. Fama and Bliss compute these yields by first constructing a step-function term structure of forward rates in which forward rates are assumed to be the same between successive-maturity Treasury bonds, and then interpolating to compute implied discount bond prices and yields. The data is monthly and covers the period from 6/64 to 12/89. All yields are expressed in annualized form. Table I presents summary statistics for the variables used in the tests.

B. Estimating Interest Rate Volatility

There are many approaches that could be used in practice to obtain estimates of the volatility of the short-term riskless rate. For example, actual bond or bond option prices can be set equal to their theoretical values and then solved as a system of nonlinear equations for the parameter values and the "implied volatility" V—similar to the procedure used for stock options. Since our focus is on testing the model, however, we use an estimation approach that is independent of the functional form of prices implied by the model to avoid the possibility of biasing the results in favor of the model.

In estimating V, we use the well-known GARCH framework of Bollerslev (1986). This framework is an extension of the ARCH class of processes. ARCH

Table I

Summary Statistics for the One-Month U.S. Treasury-Bill Yield and for Monthly Changes in the One-, Three-, Six-, Nine-Month U.S. Treasury-Bill and One-, Two-, Three-, Four-, and Five-Year Fama and Bliss (1987) U.S. Treasury Bond Yields

ρ_i denotes the ith order autocorrelation. The data are monthly and the sample period is 6/64 to 12/89. (306 observations)

Variable	Mean	Std. Dev.	ρ_1	ρ_2	ρ_3
One-month yield	0.06717	0.02679	0.951	0.909	0.860
One-month yield change	0.00008766	0.008211	−0.080	0.073	−0.124
Three-month yield change	0.00013965	0.006950	0.108	−0.087	−0.055
Six-month yield change	0.00013572	0.006629	0.152	−0.089	−0.105
Nine-month yield change	0.00012654	0.006563	0.160	−0.100	−0.125
One-year yield change	0.00013153	0.006568	0.103	−0.112	−0.079
Two-year yield change	0.00012922	0.005541	0.143	−0.108	−0.110
Three-year yield change	0.00012646	0.005042	0.103	−0.108	−0.115
Four-year yield change	0.00012395	0.004863	0.046	−0.119	−0.028
Five-year yield change	0.00012431	0.004451	0.058	−0.089	−0.069

and GARCH processes have been used successfully to model a wide variety of economic time series including inflation, foreign exchange rates, stock returns, and interest rates.[19] Specifically, we model discrete changes in the riskless rate by the following econometric specification:

$$r_{t+1} - r_t = \alpha_0 + \alpha_1 r_t + \alpha_2 V_t + \varepsilon_{t+1}, \tag{28}$$

$$\varepsilon_{t+1} \sim N(0, V_t), \quad \text{and} \tag{29}$$

$$V_t = \beta_0 + \beta_1 r_t + \beta_2 V_{t-1} + \beta_3 \varepsilon_t^2. \tag{30}$$

This specification allows unexpected changes in r to be conditionally heteroskedastic through their dependence on the value of V. In turn, V follows an autoregressive process since its current value depends on its lagged value. This discrete-time specification parallels the continuous-time dynamics given in (14) and (15) since (28) implies that changes in r depend on r and V and are conditionally heteroskedastic. On the other hand, it is important to acknowledge that this discretized specification is only an approximation of the continuous-time model.[20] If the GARCH model is mis-specified, however, the likely effect will be to bias the tests against the two-factor model in the tests of the cross-sectional restrictions.

We estimate the GARCH system in (28), (29), and (30) using the Berndt-Hall-Hall-Hausman (1974) numerical algorithm to find the maximum likelihood parameter estimates. In doing this, we used a variety of different starting values to ensure that the volatility estimates were robust and that the algorithm converged to the global maximum. Fig. 1 plots the estimated standard deviations of changes in the riskless rate (the square root of V) during the 1964–1989 study period. For comparison, Fig. 1 also plots the absolute changes in r which serves as an ex post measure of volatility. As shown, the GARCH estimates are positive and appear to be successful in capturing the patterns of volatility evident in the ex post estimates.

C. Tests of the Two-Factor Model

The two-factor model developed in this paper places a number of strong cross-sectional restrictions on the term structure. To see this, note that (21) implies that changes in observed values of Y_τ can be expressed as linear functions of changes in r and V

$$\Delta Y_\tau = a_\tau + b_\tau \, \Delta r + c_\tau \, \Delta V, \tag{31}$$

[19] For example, see Engle (1982, 1983), Bollerslev (1987), French, Schwert, and Stambaugh (1987), Engle, Lilien, and Robbins (1987), Hsieh (1988), Baillie and Bollerslev (1989), Hsieh (1989), and Engle, Ng, and Rothschild (1990). Nelson (1991) studies the relation between GARCH processes and diffusion processes.

[20] Because of this, the parameters of the GARCH model need not map directly into the parameters of the continuous-time process. A more sophisticated approach might involve estimating V and its dynamics jointly with the tests of the cross-sectional restrictions of the two-factor model. This approach, however, would be comptutationally very difficult to implement and would require methods that would take us far beyond the scope of this paper.

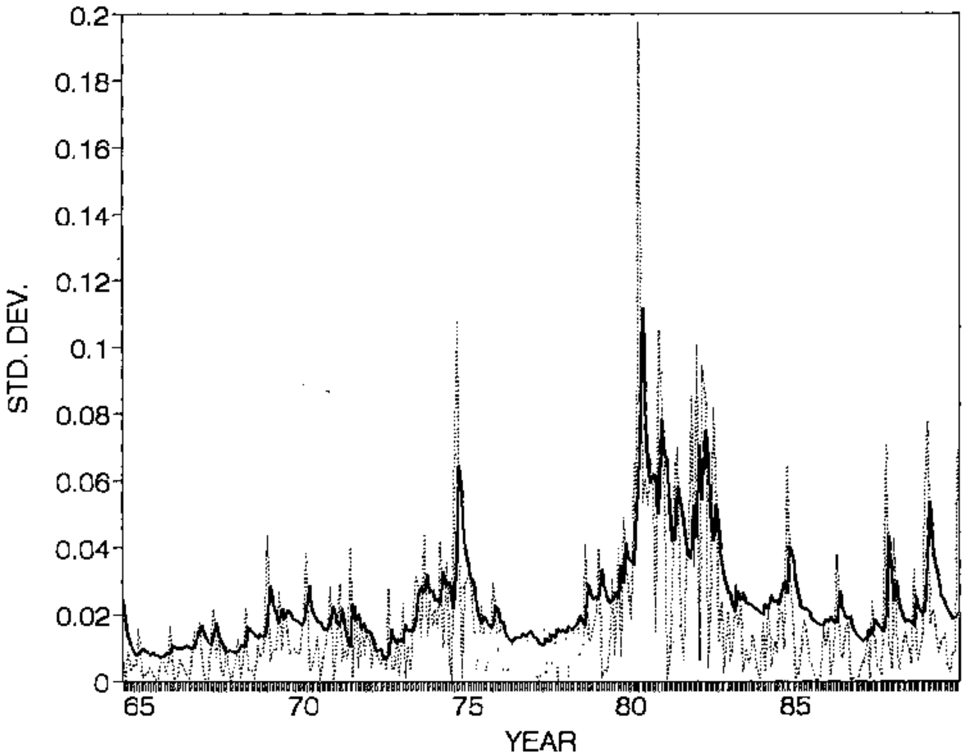

Figure 1. The monthly changes in the one-month Treasury bill yield. The solid line is the yield implied by the GARCH model and the dotted line is the absolute value of monthly changes in the one-month Treasury bill yield.

where a_τ, b_τ, and c_τ are maturity-specific constants, and Δ represents the difference operator. Next, consider a system of equations of the form (31) for n different maturities. In the absence of any cross-sectional restrictions, this system would include $3n$ different constants or parameters. The cross-sectional restrictions of the two-factor model, however, dramatically reduce the dimensionality of the parameter space. From (21), the constants a_τ, b_τ, and c_τ equal

$$\begin{aligned} a_\tau &= 0, \\ b_\tau &= -C(\tau)/\tau, \\ c_\tau &= -D(\tau)/\tau, \end{aligned} \tag{32}$$

where $C(\tau)$ and $D(\tau)$ are as in (20). For a given τ, however, $C(\tau)$ and $D(\tau)$ are completely determined by the four constants α, β, δ, and ν. Thus, the two-factor model places $3n - 4$ restrictions on the $3n$ parameters of the system. Note that these restrictions are much more stringent than simply requiring changes in r and V to have explanatory power for yield changes—we require that changes in r and V have explanatory power in a very specific way which depends on only four parameters.[21]

Our econometric approach is to test these cross-sectional restrictions as a set of over-identifying restrictions on a system of moment equations using the GMM approach of Hansen (1982). This approach has a number of important advantages which make it an intuitive and logical choice for testing these restrictions. First, the GMM approach does not require that yield changes be normally distributed. The asymptotic justification for the GMM procedure requires only stationarity, ergodicity, and existence of relevant expectations. Second, the GMM estimators and their standard errors are consistent even if error terms in the moment equations are conditionally heteroskedastic, serially correlated, or correlated across maturities. Finally, the GMM technique has been used in other empirical tests of term-structure models by Gibbons and Ramaswamy (1987), Harvey (1988), Longstaff (1989), Chan, Karolyi, Longstaff, and Sanders (1991), and Flesaker (1991).

Let ε_t denote the deviation of the observed value of ΔY_τ from the theoretical value implied by (31)[22]

$$\varepsilon_t = \Delta Y_\tau - a_\tau - b_\tau \, \Delta r - c_\tau \, \Delta V. \tag{33}$$

In a regression framework, the three constants a_τ, b_τ, and c_τ would be chosen so that the expected values of ϵ_t, $\epsilon_t \, \Delta r$, and $\epsilon_t \, \Delta V$ equal zero. This follows since regression residuals are constrained to be mean zero and orthogonal to the independent variables. In implementing the GMM tests, we choose moment conditions that parallel those implied by the regression framework. This makes the results more intuitive and easier to interpret since the GMM approach can be viewed as testing nonlinear restrictions on the parameters of a system of regression equations. In this sense, our estimation approach closely parallels generalized least squares.

Define θ to be the parameter vector with elements α, β, δ, and ν and let the vector $h_t(\theta)$ be

$$h_t(\theta) = \begin{bmatrix} \varepsilon_t \\ \varepsilon_t \, \Delta r \\ \varepsilon_t \, \Delta V \end{bmatrix}. \tag{34}$$

Let $f_t(\theta)$ be the $3n$-vector formed by stacking the $h_t(\theta)$ vectors for n different maturities. Under the null hypothesis that the restrictions implied by (32) are true, $E[\, f_t(\theta)] = 0$. The GMM procedure consists of replacing $E[\, f_t(\theta)]$ by

[21] This approach has the potential to provide powerful tests for the specification of the model. For example, if the underlying state variables, preferences, or production dynamics are not correctly specified, it is likely that the model's strong cross-sectional restrictions will be rejected by the data.

[22] Stambaugh (1988) argues that quotation errors, the averaging of bid and ask prices, and other imperfections in the data make it important to use statistical tests which allow for the possibility of measurement error in the data.

its sample counterpart, $g_T(\theta)$, using the T observations where

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} f_t(\theta), \tag{35}$$

and then choosing parameter estimates that minimize the quadratic form

$$J_T(\theta) = g_T'(\theta) W_T(\theta) g_T(\theta), \tag{36}$$

where $W_T(\theta)$ is a positive-definite symmetric weighting matrix. Hansen (1982) shows that choosing $W_T(\theta) = S^{-1}(\theta)$, where

$$S(\theta) = E[f_t(\theta) f_t'(\theta)], \tag{37}$$

results in the GMM estimator of θ with the smallest asymptotic covariance matrix.

The minimized value of the quadratic form in (36) is distributed as a χ^2 variate under the null hypothesis that the cross-sectional restrictions are true. In our tests, we use eight different maturities: the three-month, six-month, nine-month, one-year, two-year, three-year, four-year, and five-year maturities. Thus, we estimate the four-parameter vector θ from a system of 24 moment equations, resulting in 20 over-identifying restrictions—the test statistic is χ^2_{20} under the null hypothesis.[23] A high value for the test statistic means that the cross-sectional restrictions of the two-factor model are rejected. The estimation technique uses a Newton-Raphson algorithm to find the parameter vector that minimizes the quadratic form in (36).[24]

Table II reports the GMM minimized criterion (χ^2) value, and its associated p-value. As shown, the GMM test statistic is 17.02 with a p-value of 0.651. Hence, the cross-sectional restrictions imposed by the two-factor model cannot be rejected by the data at conventional significance levels. These results are particularly striking given that the cross-sectional restrictions are imposed on yields with maturities up to five years. Thus, the model holds for both short-term and intermediate-term maturities. Previous empirical work often finds that the explanatory power of equilibrium term-structure models drops rapidly for maturities in excess of one year (see Longstaff (1989)).[25]

These results indicate that changes in r and V not only have explanatory power for yield changes, but that they have explanatory power in the way predicted by the model. Although these results support our two-factor model, it is important to point out that they do not necessarily provide evidence

[23] Since θ is a vector of four parameters, it is not possible to estimate θ for individual maturities. Hence, we cannot compare parameter estimates across maturities.

[24] Simulation evidence by Flesaker (1991) suggests that the minimized value of the GMM quadratic form in a term-structure application similar to ours conforms well to its asymptotic distribution.

[25] The point estimate of α in Table II is negative. This is consistent with the model if μ in equation (1) is allowed to be negative. If μ is negative, however, it is possible that r can take on negative values. This is clearly an undesirable property for a model of nominal interest rates (although not necessarily for a model of real rates).

Table II

GMM Tests of the Cross-sectional Restrictions on Monthly Yield Changes Implied by the Two-Factor Model

The yields included in the tests are the three-, six-, and nine-month, and the one-, two-, three-, four-, and five-year U.S. Treasury yields. The four parameters α, β, δ, and ν are estimated from a system of 24 moment conditions, resulting in 20 overidentifying restrictions. The data are monthly and are expressed in annualized form. The sample period is 6/64 to 12/89 (306 observations).

	α	β	δ	ν	Test Statistic
Value	−0.0439	0.0814	0.3299	14.4227	17.02
t-statistic	−5.65	6.00	15.08	8.18	—
p-value	0.000	0.000	0.000	0.000	0.652

against other two-factor models such as Brennan and Schwartz (1979), Schaefer and Schwartz (1984), CIR (1985b), and Heath, Jarrow and Morton (1988). This is because our approach focuses on the over-identifying conditions of the model rather than the relative explanatory power of alternative two-factor models. In order to provide diagnostics for the model, we examine how well the model fits the term structure of interest rate volatility. This is done by comparing the unconditional standard deviations of yield changes implied by the model—evaluated at the point estimates of the parameters—to the actual unconditional standard deviations reported in Table I. Note that the GMM estimation does not impose the restriction that these moments match. Despite this, however, the fit between the actual and model standard deviations is very close. For example, Table I shows that the actual standard deviations range from 82 basis points for monthly changes in the three-month yield to 45 basis points for monthly changes in the five-year yield. The standard deviations implied by the model range from 76 basis points for changes in the three-month yield to 45 basis points for changes in the five-year yield. The mean difference between the actual and model standard deviations is only five basis points.

Finally, since only four of the six parameters of the model are estimated in our tests, the point estimates given in Table II may differ from those obtained by procedures that estimate all six parameters simultaneously. The reason for this is simply that the parameters are highly correlated—estimating a subset of the parameters ignores the correlation between these parameters and the remaining parameters.[26] This poses no problem to the tests of the cross-sectional restrictions which are our primary focus in this section. In

[26] The high correlation of the parameters and the existence of local minima make the point estimates of the parameters somewhat dependent on the choice of starting values. The minimized value of the GMM criterion function, however, is not sensitive to the choice of starting values. A referee pointed out that the test of the over-identifying conditions may lack power in small samples because of the correlation of the orthogonality conditions across maturities.

applied work, however, we recommend that parameters be estimated using techniques that take the correlations of all six parameters into account.

D. Tests of the CIR Single-Factor Model

Recall from Section I that the single-factor CIR (1985b) model can be nested within the two-factor model. This is done by imposing the restrictions $\alpha = \delta = \gamma = 0$ in (8) and (10) and rederiving equilibrium discount bond prices.[27] As with the two-factor model, the single-factor CIR model also places testable cross-sectional restrictions on the coefficients of the expression in (31). Using our notation, these restrictions are

$$\begin{aligned} a_\tau &= 0, \\ b_\tau &= \frac{2(\exp(\psi\tau) - 1)/\tau}{(\nu + \psi)(\exp(\psi\tau) - 1) = 2\psi}, \\ c_\tau &= 0. \end{aligned} \tag{38}$$

As before, we test these cross-sectional implications as over-identifying restrictions using the GMM methodology. Since α and δ equal zero, the parameter vector θ consists of just β and ν. Using the same set of 24 moment equations to estimate the parameter vector θ results in 22 over-identifying restrictions. Hence, the GMM test statistic is χ^2_{22} under the null hypothesis.

The test results for the single-factor model are reported in Table III. As shown, the data provide evidence against the single factor model. In particular, the cross-sectional restrictions imposed by the model can be rejected at the 10 percent level, although not at the five percent level. The GMM test statistic is 32.86 with a p-value of 0.064. To examine the reasons for the poor performance of the single-factor model, we regressed changes in yields on changes in r and V. Although the single-factor model implies that changes in V should have no explanatory power for yield changes, we found that changes in V were significant for all maturities. In particular, the t-statistics for changes in V from simple OLS regressions range from -2.5 for three-month yield changes to -3.1 for five-year yield changes. These results indicate that one important reason for the rejection of the single-factor model is that it does not allow yield changes to depend on changes in V.

V. Conclusion

In this paper, we have developed a two-factor general equilibrium model of the term structure of interest rates. We use this model to derive closed-form expressions for discount bond prices and discount bond option prices. The factors used—the short-term interest rate and the volatility of the short-term

[27] Imposing the parameter restrictions directly on the discount bond pricing function (20) is not feasible since the change of variables employed results in terms that involve division by zero.

Table III

GMM Tests of the Cross-sectional Restrictions on Monthly Yield Changes Implied by the Cox, Ingersoll, and Ross (1985b) Single-Factor Model

The yields included in the tests are the three-, six-, and nine-month, and one-, two-, three-, four-, and five-year U.S. Treasury yields. The four parameters α, β, δ, and ν are estimated from a system of 24 moment conditions, resulting in 20 overidentifying restrictions. The data are monthly and are expressed in annualized form. The sample period is 6/64 to 12/89 (306 observations).

	β	ν	Test Statistic
Value	−0.1350	0.5234	32.86
t-statistic	−2.39	8.07	—
p-value	0.000	0.000	0.064

interest rate—are intuitively appealing and readily obtainable. The cross-sectional restrictions imposed by the model on the evolution of the term structure appear consistent with the data.

In addition to its theoretical appeal, the model has the potential of becoming a useful practical tool for the valuation and hedging of interest rate-contingent claims. Its advantage over the two-factor arbitrage models of the term structure is that the functional form of the market price of interest rate risk is endogenous to the model and consistent with general equilibrium. Also, the flexibility obtained by having one interest rate factor and one volatility factor might be important in the practical applications of the model.

However, more extensive empirical work on the use of the model in the valuation of contingent claims will be necessary to ascertain its superiority relative to alternative models. In many of the more complex applications, closed-form solutions may not exist and numerical solutions may be required.

REFERENCES

Bailey, Warren and René M. Stulz, 1989, The pricing of stock index options in a general equilibrium model, *Journal of Financial and Quantitative Analysis* 24, 1–12.

Baillie, Richard T. and Tim Bollerslev, 1989, The message in daily exchange rates: A conditional variance tale, *Journal of Business and Economic Statistics* 7, 297–305.

Ball, Clifford B. and Walter N. Torous, 1983, Bond-price dynamics and options, *Journal of Financial and Quantitative Analysis* 18, 517–531.

Berndt, E. R., B. H. Hall, R. E. Hall, and J. A. Hausman, 1974, Estimation and inference in nonlinear structural models, *Annals of Economic and Social Measurement* 3, 653–665.

Bollerslev, Tim, 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics* 31, 307–327.

———, 1987, A conditionally heteroscedastic time series model of security prices and rates of return data, *Review of Economics and Statistics* 69, 542–547.

Breeden, Douglas T., 1986, Consumption, production, inflation, and interest rates: A synthesis, *Journal of Financial Economics* 18, 373–399.

Brennan, Michael J. and Eduardo S. Schwartz, 1977, Savings bonds, rectractable bonds, and callable bonds, *Journal of Financial Economics* 3, 133–155.

———, 1979, A continuous time approach to the pricing of bonds, *Journal of Banking and Finance* 3, 133–155.

Brown, Stephen J. and Philip H. Dybvig, 1986, The empirical implications of the Cox, Ingersoll, Ross theory of the term structure of interest rates, *Journal of Finance* 41, 617–630.

Campbell, John Y., 1986, A defense of traditional hypotheses about the term structure of interest rates, *Journal of Finance* 41, 183–193.

Chan, K. C., G. Andrew Karolyi, Francis A. Longstaff, and Anthony B. Sanders, 1991, The volatility of short-term interest rates: an empirical comparison of alternative models of the term structure of interest rates, Working paper, The Ohio State University, Columbus, Ohio.

Courtadon, Georges, 1982, The pricing of options on default-free bonds, *Journal of Financial and Quantitative Analysis* 17, 75–100.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross, 1981, A reexamination of traditional hypotheses about the term structure of interest rates, *Journal of Finance* 36, 769–799.

———, Jonathan E. Ingersoll, and Stephen A. Ross, 1985a, An intertemporal general equilibrium model of asset prices, *Econometrica* 53, 363–384.

———, Jonathan E. Ingersoll, and Stephen A. Ross, 1985b, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Dothan, Uri L., 1978, On the term structure of interest rates, *Journal of Financial Economics* 6, 59–69.

Dybvig, Philip H., 1989, Bond and bond option pricing based on the current term structure, Working paper, Washington University, St. Louis, Missouri.

Engle, Robert F., 1982, Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation, *Econometrica* 50, 987–1008.

———, 1983, Estimates of the variance of U.S. inflation based on the ARCH model, *Journal of Money, Credit, and Banking* 15, 286–301.

———, David M. Lilien, and Russell P. Robins, 1987, Estimating time varying risk premia in the term structure: The ARCH-M model, *Econometrica* 55, 391–407.

———, Victor Ng, and Michael Rothschild, 1990, Asset pricing with a factor ARCH covariance structure: Empirical estimates for Treasury bills, *Journal of Econometrics*, forthcoming.

Fama, Eugene F., 1984, The information in the term structure, *Journal of Financial Economics* 13, 509–528.

——— and Robert Bliss, 1987, The information in long-maturity forward rates, *American Economic Review* 77, 680–692.

Feller, William, 1951, Two singular diffusion problems, *Annals of Mathematics* 54, 173–182.

Flesaker, Bjorn, 1991, Estimation and testing of the constant volatility Heath, Jarrow, Morton model of interest-rate contingent claims pricing, Working paper, University of Illinois at Urbana-Champaign.

Foresi, Silverio, 1990, Essays in money and asset pricing, Ph.D. Dissertation, Department of Economics, UCLA, Los Angeles, California.

French, Kenneth R., G. William Schwert, and Robert F. Stambaugh, 1987, Expected stock returns and volatility, *Journal of Financial Economics* 19, 3–30.

Froot, Kenneth A., 1989, New hope for the expectations hypothesis of the term structure of interest rates, *Journal of Finance* 44, 283–305.

Gibbons, Michael R. and Krishna Ramaswamy, 1987, The term structure of interst rates: Empirical evidence, Working paper, Stanford University, Stanford, California.

Hahsen, Lars P., 1982, Large sample properties of Generalized Method of Moments estimators, *Econometrica* 50, 1029–1054.

Harvey, Campbell, 1988, The real term structure and consumption growth, *Journal of Financial Economics* 22, 305–333.

Heath, David, Robert Jarrow, and Andrew Morton, 1988, Bond pricing and the term structure of interst rates: A new methodology, Working paper, Cornell University, Ithaca, NY.

Hemler, Michael L. and Francis A. Longstaff, 1991, General equilibrium stock index futures prices: Theory and empirical evidence, *Journal of Financial and Quantitative Analysis* 25, 287–308.

Ho, Thomas S. Y. and S. B. Lee, 1986, Term structure movements and pricing interest rate contingent claims, *Journal of Finance* 41, 1011–1029.

Hsieh, David A., 1988, The statistical properties of daily foreign exchange rates: 1974–1983, *Journal of International Economics* 24, 129–145.

———, 1989, Modeling heteroskedasticity in daily foreign exchange rates, *Journal of Business and Economic Statistics* 7, 307–317.

Hull, John and Alan White, 1987. The pricing of options on assets with stochastic volatilities, *Journal of Finance* 42, 281–301.

Jamshidian, Farshid, 1989, An exact bond option formula, *Journal of Finance* 44, 205–209.

Johnson, Herb and D. Shanno, 1987. Option pricing when the variance is changing, *Journal of Financial and Quantitative Analysis* 22, 143–153.

Johnson, Norman L. and Samuel Kotz, 1970, *Continuous Univariate Distributions*—1 and 2 (Houghton Mifflin Co., Boston, Mass.).

Lauterbach, Beni, 1989, Consumption volatility, production volatility, spot-rate volatility, and the returns on treasury bill and bonds, *Journal of Financial Economics* 24, 155–179.

Longstaff, Francis A., 1989, A nonlinear general equilibrium model of the term structure of interest rates, *Journal of Financial Economics* 23, 195–224.

Mankiw, N. G., 1986, The term structure of interest rates revisited, *Brookings Papers on Economic Activity*, 61–96.

Merton, Robert C., 1971, Optimum Consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373–413.

———, 1980, On estimating the expected return on the market: An exploratory investigation, *Journal of Financial Economics* 8, 323–361.

Nelson, Daniel B., 1991, Conditional heteroskedasticity in asset returns: A new approach, *Econometrica* 59, 347–370.

Pearson, Neil D. and Tong-Sheng Sun, 1989, A test of the Cox, Ingersoll, Ross model of the term structure of interest rates using the method of maximum likelihood, Mimeo, Massachusetts Institute of Technology.

Schaefer, Stephen M. and Eduardo S. Schwartz, 1984, A two-factor model of the term structure: An approximate analytical solution, *Journal of Financial and Quantitative Analysis* 19, 413–424.

——— and Eduardo S. Schwartz, 1987, Time dependent variance and the pricing of bonds, *Journal of Finance* 42, 1113–1128.

Scott, Louis O., 1987. Option pricing when the variance changes randomly: Theory, estimation, and applications, *Journal of Financial and Quantitative Analysis* 22, 419–438.

Shiller, Robert J., 1979, The volatility of long-term interest rates and expectations models of the term structure, *Journal of Political Economy* 87, 1190–1219.

———, 1986, Comments and discussion, *Brookings Papers on Economic Activity*, 100–107.

———, John Y. Campbell, and Kermit L. Schoenholtz, 1983, Forward rates and future policy: Interpreting the term structure of interst rates, *Brookings Papers on Economic Activity*, 173–217.

Simon, David P., 1989, Expectations and risk in the Treasury bill market: An instrumental variables approach, *Journal of Financial and Quantitative Analysis* 24, 357–365.

Stambaugh, Robert, 1988, The information in forward rates: Implications for models of the term structure, *Journal of Financial Economics* 21, 41–70.

Startz, Richard, 1982, Do forecast errors or term premia really make the difference between long and short rates?, *Journal of Financial Economics* 10, 323–329.

Vasicek, Oldrich, 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177–188.

Wiggins, James B., 1987, Stochastic volatility option valuation: Theory and empirical estimates, *Journal of Financial Economics* 19, 351–372.