Ref.TH.2341-CERN

SUPERSYMMETRIES AND THEIR REPRESENTATIONS

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A B S T R A C T

We determine all manifest supersymmetries in more than 1+1 dimensions, including those with conformal or de Sitter space-time symmetry. For the supersymmetries in flat space we determine the structure of all representations and give formulae for an effective computation. In particular we show that at least for masses $m^2 = 0,1,2$ the states of the spinning string form supersymmetry multiplets.

Ref.TH.2341-CERN
7 July 1977

1. - INTRODUCTION

All supersymmetries of the S matrix in 3+1 dimensions are known [1]. However, there are further interesting possibilities, e.g., supersymmetries in de Sitter space-time or in higher dimensions. Particularly important is the conjecture that a suitably restricted version of the Neveu-Schwarz-Ramond string yields a renormalizable supersymmetric Yang-Mills and gravity theory in 9+1 dimensions [2]. Such theories may be reduced to 3+1 dimensions by compactifying some directions [3].

In Section 2 we shall classify all manifest supersymmetries in more than 1+1 dimensions. For a flat space-time we determine the structure of the corresponding little groups in Section 3. In Section 4 we determine their representations and derive formulae to calculate them explicitly. In Section 5 we consider as examples the theories which admit multiplets with spins at most 1. In particular we shall see that the lowest mass levels of the spinning string indeed can be regarded as supersymmetry representations, thus confirming the conjecture of Ref. 2).

The notations are those of Ref. 4). In particular, the bracket $< \ell, \ell' >$ will denote the anticommutator, if both ℓ, ℓ' are odd, and the commutator, if at least one of them is even. We shall always work with the supersymmetry algebra, not with groups.

2. - CLASSIFICATION OF SUPERSYMMETRIES

Let $L = G \oplus U$ be a finite dimensional supersymmetry algebra, where G, U denote the even and odd subspaces, resp. We assume that the generators exhibit the usual relation between spin and statistics (in fact it is sufficient to assume that U contains no Lorentz scalars). Furthermore, L must admit an adjoint operation +. This is true, if L commutes with some unitary S matrix, but we shall also consider theories with massless particles, or in de Sitter space, where the usual S matrix formalism runs into difficulties. However, we restrict ourselves to manifest supersymmetries, acting on some Hilbert space of particle states.

Taking the subspace of L generated by the elements which obey

$$\begin{aligned} g^+ &= -g \quad \text{for } g \in G, \\ u^+ &= iu \quad \text{for } u \in \mathcal{U}, \end{aligned} \tag{1}$$

we obtain a real form of L.

For the even part we write

$$G = \mathcal{S} \oplus \mathcal{J}, \tag{2}$$

where S is the space-time symmetry, and J a compact internal symmetry of the form

$$\mathcal{J} = \mathcal{T} \oplus \mathcal{A}, \tag{3}$$

with T semi-simple and A Abelian.

Consider first the case where S is simple, i.e., a conformal or a de Sitter algebra. Let $C(X)$ denote the centre of X.

Proposition 2.1 :

$L/C(L)$ is the direct sum of an internal symmetry J_c of type (3) plus a supersymmetry which is simple up to a possible extension by an algebra of outer automorphisms (of course J_c may be zero).

Proof

$C(L) \subset G$, as

$$\langle u u^+ \rangle > 0 \tag{4}$$

for all non-zero $u \in U$. Let $V \subset U$ be invariant under $(S,+)$, i.e., under S and the adjoint operation. Then $< S < VV \gg$ is an S invariant subspace of S, thus either equal to S or zero. In the latter case, $< VV^+ >$ defines a positive definite S invariant Hermitian form on V, thus that part of S which is faithfully represented on V is contained in some compact unitary algebra. As S is simple and non-compact, $< SV >$ has to vanish. But because of the spin-statistics relation, U contains no scalars. Thus for any $V \neq 0$ we have

$$\langle S\langle VV\rangle\rangle = S \tag{5}$$

Therefore no ideal of L which contains an odd element can be soluble. Now let C be the maximal soluble ideal of L. Because of $C \subset G$ one has

$$\langle CU\rangle = \sigma. \tag{6}$$

In addition because of Eqs. (2) and (3)

$$C \subset A. \tag{7}$$

Thus

$$C = C(L) \tag{8}$$

and $L/C(L)$ is semi-simple. All semi-simple graded Lie algebras are described in Ref. 5). Because G is of type (2) with simple S, and U contains no scalars, $L/C(L)$ has to be a direct sum. Its summands must be simple modulo extensions by outer automorphisms. Because of Eq. (5) all odd elements of $L/C(L)$ belong to that direct summand which contains S. Q.E.D. Apart from the outer automorphisms the direct summand which contains U can be written as (9)

$$U \oplus \langle UU\rangle$$

and this supersymmetry algebra is simple modulo "central charges" as defined in Ref. 1). Here

$$\langle UU\rangle = S \oplus J' \tag{10}$$

where J' is a direct summand of J, thus again of type (3).

All real simple graded Lie algebras have been classified [5),6)]. Thus we just have to select the algebras which are compatible with our assumptions. We use the notation (G, U as representation space of G).

Proposition 2.2 :

The simple supersymmetry algebras are :

$$(o(2,1) \oplus u(N),\ (2,N) + (2,\bar{N})),\ N \neq 2 \qquad (\mathrm{I})$$
$$(o(2,1) \oplus su(2),\ (2,2) + (2,2)) \qquad (\mathrm{I}_1)$$
$$(o(2,1) \oplus o(N),\ (2,N)),\ N = 1,2,\ldots \qquad (\mathrm{II})$$
$$(o(2,1) \oplus o(4),\ (2,4))_\alpha \qquad (\mathrm{II}_\alpha)$$
$$(o(2,1) \oplus o(3) \oplus su(N,H),\ (2,2,2N)),\ N = 1,2,\ldots \qquad (\mathrm{III})$$
$$(o(2,1) \oplus o(7),\ (2,8)) \qquad (\mathrm{IV})$$
$$(o(2,1) \oplus g_2,\ (2,7)) \qquad (\mathrm{V})$$
$$(o(3,1),\ (2,1) + (1,2)) \qquad (\mathrm{VI})$$
$$(o(3,2) \oplus o(N),\ (4,N)),\ N = 1,2,\ldots \qquad (\mathrm{VII})$$
$$(o(4,1) \oplus u(1),\ 4 + \bar{4}) \qquad (\mathrm{VII}_1)$$
$$(o(4,2) \oplus u(N),\ (4,N) + (\bar{4},\bar{N})),\ N \neq 4 \qquad (\mathrm{VIII})$$
$$(o(4,2) \oplus su(4),\ (4,4) + (\bar{4},\bar{4})) \qquad (\mathrm{VIII}_1)$$
$$(o(6,1) \oplus su(2),\ (8,2)) \qquad (\mathrm{IX}_1)$$
$$(o(5,2) \oplus su(2),\ (8,2)) \qquad (\mathrm{IX}_2)$$
$$(o(6,2) \oplus su(N,H),\ (8,2N)),\ N = 1,2,\ldots \qquad (\mathrm{X})$$

The Lie algebras are denoted by lower case letters, capitals are reserved for the groups. $SU(N,H)$ is the group of unitary quaternionic $N \times N$ matrices, it is the compact real form of $Sp(2N,C)$. In particular $su(1,H) \sim su(2)$, $su(2,H) \sim o(5)$. In (II_α), α is a real constant which enters only into the structure constants for $< UU >$. The algebras involving $o(4,2)$ have been classified in Ref. 1).

Recently, Euclidean supersymmetries have been studied [7]. The algebras $o(R+1,1)$ may be interpreted as conformal algebras of an R dimensional Euclidean space. For compact de Sitter spaces we obtain the additional possibilities

$$(o(3) \oplus u(N),\ (2,N) + (2,\bar{N})),\ N \neq 2 \qquad (\mathrm{I}')$$
$$(o(3) \oplus su(2),\ (2,2) \rightarrow (2,2)) \qquad (\mathrm{I}'_1)$$
$$(o(5) \oplus u(1),\ 4 + \bar{4}) \qquad (\mathrm{VII}'_1)$$
$$(o(6) \oplus u(N),\ (4,N) + (\bar{4},\bar{N})),\ N \neq 4 \qquad (\mathrm{VIII}')$$
$$(o(6) \oplus su(4),\ (4,4) + (\bar{4},\bar{4})). \qquad (\mathrm{VIII}'_1)$$

Note that most of the orthosymplectic algebras are unsuitable, as the $sp(2N)$ are non-compact. Thus we had to use the isomorphisms [8]

$o(2,1) \sim su(1,1)$
$o(3) \sim su(2)$
$o(2,1) \oplus o(3) \sim s\alpha u(2,H)$
$o(3,1) \sim sp(2,C)$
$o(3,2) \sim sp(4)$

$o(4,1) \sim su(1,1,H)$
$o(5) \sim su(2,H)$
$o(4,2) \sim su(2,2)$
$o(6) \sim su(4)$
$o(6,2) \sim s\alpha u(4,H)$

$S\alpha U(N,H)$ denotes the group of anti-unitary quaternionic $N\times N$ matrices.

The representations of S in U are always spinor representations.

To find central charges, one has to look at the decomposition of the symmetric part of the tensor product of U with itself. For algebras involving a $u(1)$, any G scalar can be absorbed into it. Most other algebras yield no G scalars, with the exception of (I_1), $(I_1^{!})$, $(VIII_1)$ and $(VIII_1^{!})$. These algebras admit one central charge. In addition, only they admit outer automorphisms $U(1)$, and (I_1) even $SU(2)$. This extension of (I_1) by $SU(2)$ can be obtained from (II_α) in the limit $\alpha = 0$. In this latter case no central charge is allowed.

The algebras for de Sitter spaces may be contracted to algebras of flat spaces. Here any direct summand of J may either be left unchanged or contracted to a vector space of central charges.

For supersymmetries where S is a conformal algebra we have a natural grading over the integers

$$L = L^{(-2)} \oplus L^{(-1)} \oplus L^{(0)} \oplus L^{(1)} \oplus L^{(2)}, \tag{11}$$

$$\langle L^{(m)} L^{(n)} \rangle \subset L^{(m+n)}, \tag{12}$$

defined by

$$L^{(n)} = \{ \ell \in L \,/\, \langle d\ell \rangle = n\ell \} \tag{13}$$

with a suitably normalized dilation generator d. Here

$$L^{(-2)} = P \tag{14}$$

is the subspace of translation generators, and

$$L^{(-1)} \oplus L^{(1)} = \mathcal{U}. \tag{15}$$

Let us now consider the supersymmetries with

$$S = io(R,1), \tag{16}$$

i.e., the Poincaré algebra in R+1 dimensions. As we have seen, these supersymmetries cannot be simple. A typical example is the subalgebra

$$L^{(-2)} \oplus L^{(-1)} \oplus L_d^{(0)} \tag{17}$$

of (11), where

$$L_d^{(0)} \oplus \text{dilations} = L^{(0)}. \tag{18}$$

In fact, we shall show that such a grading by dimension can always be constructed. Provisionally we define recursively a filtration

$$P \oplus C(L) = \tilde{L}^{(-2)} \subset \tilde{L}^{(0)} \subset \ldots \subset \tilde{L}^{(2k)} = L \tag{19}$$

by

$$\tilde{L}^{(m)} = \left\{ \ell \in L \,\middle|\, \langle P \ell \rangle \subset \tilde{L}^{(m-2)} \right\} \text{ for } m \geq 0. \tag{20}$$

The proof of Ref. 9) that k is finite applies also to supersymmetries. Obviously

$$\langle \tilde{L}^{(m)} \tilde{L}^{(n)} \rangle \subset \tilde{L}^{(m+n)}, \tag{21}$$

$$\tilde{L}^{(m)+} = \tilde{L}^{(m)}, \tag{22}$$

$$G \subset \tilde{L}^{(0)}. \tag{23}$$

Proposition 2.3 :

In (19), $L = \tilde{L}^{(o)}$.

Proof

Put

$$U^{(i)} = U \cap \tilde{L}^{(i+1)}, \quad i = -1, 1, \ldots \tag{24}$$

We have

$$\langle P U^{(-1)} \rangle = \mathcal{O} \tag{25}$$

and therefore

$$\langle P \langle U^{(-1)} U^{(-1)} \rangle\rangle = \mathcal{O}. \tag{26}$$

Thus

$$\langle U^{(-1)} U^{(-1)} \rangle \subset P \oplus J. \tag{27}$$

From

$$\langle P \langle P U^{(1)} \rangle\rangle = \mathcal{O} \tag{28}$$

we obtain for any $u \in U^{(1)}$

$$\begin{aligned} \langle\langle P u \rangle \langle P u \rangle\rangle &= \langle P \langle\langle P u \rangle u \rangle\rangle = \\ &= \langle P \langle P \langle u u \rangle\rangle\rangle \subset \langle P \langle P G \rangle\rangle = \mathcal{O}. \end{aligned} \tag{29}$$

For $u \in U^{(1)}$ with $u^+ = u$ this means

$$\langle P u \rangle = \mathcal{O}. \tag{30}$$

But these elements span $U^{(1)}$. Thus

$$U^{(1)} = U^{(-1)} = U. \tag{31}$$

Q.E.D.

Proposition 2.4 :

U consists of spinor representations of $o(R,1)$.

Proof

Consider a Cartan subalgebra of the Lorentz algebra spanned by the Hermitian generators M_{01}, $M_{23}, \dots$. Decompose U into eigenspaces of this Cartan algebra. For any element u of one of these spaces

$$\langle M_{2i,2i+1}\, u \rangle = \alpha_i(u)\, u \,. \tag{32}$$

Because of the compactness properties of $O(R,1)$, $\alpha_0(u)$ must be purely imaginary, and all the other $\alpha_i(u)$ real. This yields

$$\langle M_{2i,2i+1} \langle u u^+ \rangle\rangle = 2\alpha_0(u)\, \delta_{0i} \langle u u^+ \rangle . \tag{33}$$

But there is no element with this property in G, unless

$$\alpha_0(u) = \pm \tfrac{1}{2} \text{ or } 0. \tag{34}$$

This must be true for all eigenspaces of the Cartan algebra. As scalars have been excluded, U must consist of spinors. Q.E.D.

From (4) and Eq. (33) one obtains that the coefficient of P^0 in $< uv^+ >$ defines a positive definite $o(R) \oplus J$ invariant Hermitian form on U. We write

$$\langle u v^+ \rangle = (uv)\, P^0 + \dots \tag{35}$$

From the existence of this form it follows that even the representation of A in U is completely reducible.

Proposition 2.5 :

For $R > 2$ the filtration (19) can be refined to a grading

$$L = L^{(-2)} \oplus L^{(-1)} \oplus L^{(0)} , \tag{36}$$

where

$$L^{(-2)} = P \oplus C(L), \tag{37}$$

$$L^{(-1)} = U, \tag{38}$$

$$L^{(0)} = o(R,1) \oplus T \oplus A_c \tag{39}$$

with

$$A_c \oplus C(L) = A. \tag{40}$$

Proof

Because of (4), C(L) contains only even elements. Taking into account Eqs. (25), (27) and (31), we only have to prove that $< UU > \cap J$ lies in C(L). Put

$$M = U \oplus \langle UU \rangle, \tag{41}$$

$$M_c = M / (L^{(-2)} \cap \langle UU \rangle) = V \oplus \langle VV \rangle. \tag{42}$$

We have to prove that M_c contains only odd elements. Let $W \oplus B$ be an Abelian idea of M_c with B even, W odd. As the representation of B in V is completely reducible,

$$\langle BV \rangle = \langle B \langle BV \rangle \rangle \subset \langle BW \rangle = \sigma. \tag{43}$$

Thus

$$B \subset \langle VV \rangle \cap C(M_c) = \sigma \tag{44}$$

and

$$\langle VW \rangle \subset B = \sigma. \tag{45}$$

This yields

$$W \oplus B = W \subset C(M_c) \subset V. \tag{46}$$

Moreover, $C(M_c)$ is even a direct summand of M_c, as the complete reducibility of the representation of $< VV >$ in V yields

$$\langle\langle VV\rangle V\rangle \cap C(M_c) \subset \langle\langle VV\rangle C(M_c)\rangle = 0. \tag{47}$$

Thus

$$M_c = M_s \oplus C(M_c) \tag{48}$$

with semi-simple or vanishing M_s.

But M_s admits $o(R,1)$ as outer automorphism. Thus for $R > 2$ it has to vanish. Q.E.D.

For $R = 2$, M_s has to be a direct sum whose summands are all of the form (I_1'), i.e., $(su(2) \oplus su(2), (2,2) + (2,2))$. This algebra has the outer automorphism algebra $o(2,1)$. As M_s admits no further outer automorphisms, all direct summands of J which act non-trivially on M_s are contained in $< UU >$. Furthermore, M_s admits no central charges. Thus $(U \oplus P \oplus J)/P$ is a direct sum of an algebra isomorphic to M_s and one of type (36). This yields all L with $S = io(2,1)$.

In the simplest case, where M_s is just (I_1'), L can be obtained from (II_α) in the limit $\alpha = -1$. In this limit, $o(2,1)$ may become the outer automorphism, or it may be scaled down to a three dimensional centre. If one does both, the centre transforms as the adjoint, i.e., the vector representation of the outer automorphism $o(2,1)$. This doubling of $o(2,1)$ thus yields the Poincaré algebra $io(2,1)$.

The supersymmetry algebra just described apparently has not been discussed before.

For $R = 1$, no new possibilities for M_s appear, as (I_1') is the only real form of a simple graded Lie algebra which admits $o(1,1)$ as outer automorphism. For example, $(su(N) \oplus su(N), (N,N) + (\bar{N},\bar{N}))$ with $N > 2$ has the outer automorphism algebra $o(2)$, which prevents the introduction of momenta.

However, the extension of $(I_1^!)$ by $o(1,1)$ admits now one central charge which may be obtained from one momentum component in $2+1$ dimensions by reduction to $1+1$ dimensions.

3. - LITTLE GROUPS

We shall only determine the representations of the supersymmetries graded according to Eqs. (36)-(39). For the conformal supersymmetries this means that we represent only a subalgebra. For the de Sitter case we are anyhow only interested in representations which have a limit for the contraction to flat space-time.

The representations can be induced in the usual way from the representations of the little group. Thus we fix some subspace H of the Hilbert space on which P is constant and which is irreducible with respect to

$$L' = G' \oplus \mathcal{U}, \tag{49}$$

where

$$G' = S' \oplus J \tag{50}$$

and

$$\begin{aligned} S' &= o(R) \text{ for the massive case,} \\ S' &= io(R-1) \text{ for the massless case.} \end{aligned} \tag{51}$$

In the latter case the "Galilei-transformations" of $io(R-1)$ have to be represented by zero, as otherwise the representations become infinite dimensional. Thus S' can be restricted to $o(R-1)$.

On H, A is represented by constants. Thus $U|_H$ (U restricted to H) yields a Clifford algebra with bilinear form $< UU >|_H$. This bilinear form is not necessarily positive definite, though by (4) it is non-negative definite.

Proposition 3.1 :

U can be decomposed into $(G',+)$ invariant subspaces

$$\mathcal{U} = \mathcal{U}^\circ \oplus \mathcal{U}', \tag{52}$$

such that

$$\mathcal{U}^\circ|_H = \sigma \tag{53}$$

and $< U'U' >|_H$ is positive definite.

Proof

We may choose H such that $p^0 = p^R$ in the massless case and $p = m,\vec{0})$ for massive particles. Write

$$\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_- \tag{54}$$

where

$$\langle M_{0R}\, u \rangle = \pm u/2 \quad \text{for } u \in \mathcal{U}_\pm . \tag{55}$$

Eq. (33) yields

$$\langle u^+ u \rangle = c\,(P^0 \pm P^R) \quad \text{for } u \in \mathcal{U}_\pm \tag{56}$$

with $c > 0$ for any non-trivial u. In the massless case we obtain

$$\begin{aligned} \mathcal{U}^\circ &= \mathcal{U}_-, \\ \mathcal{U}' &= \mathcal{U}_+ . \end{aligned} \tag{57}$$

In the massive case we use the positive definite Hermitian form on U defined by Eq. (35). Note that Eq. (33) yields

$$(U_+ U_-) = 0. \tag{58}$$

In general, let U^0 be the subspace of U which annihilates H and take its orthogonal complement U' with respect to this form. As the form is $(G',+)$ invariant, this is also true for the decomposition. Q.E.D.

Note that in the massless case all central charges have to vanish, as

$$C(L)|_H \subset \langle U_+ U_- \rangle|_H = 0. \tag{59}$$

In the massive case without central charges one obtains

$$\langle U_+ U_- \rangle|_H = 0, \tag{60}$$

such that $< UU >|_H$ is positive definite and U^0 vanishes. Even with central charges according to Eq. (56)

$$\dim U' \geqslant \dim U_+ = \frac{1}{2} \dim U. \tag{61}$$

However, there may be linear relations between $U_+|_H$ and $U_-|_H$

<u>Proposition 3.2</u> :

$C(L)|_H$ forms a compact, convex set. At its boundary and only at its boundary $U_0 \neq 0$.

Proof

Choose a basis u^k of $U|_H$, c^i of $C(L)|_H$. We have

$$\langle \lambda_j u^j, (\lambda'_k u^k)^+ \rangle = m a_0(\lambda\lambda') + c^i a_i(\lambda\lambda'), \tag{62}$$

where the a_0, a_i are Hermitian forms. We have seen that a_0 has to be positive definite. In contrast, no linear combination of the a_i can be positive or negative definite, as the non-compact algebra G has no invariant finite-dimensional positive definite Hermitian forms.

The allowed values for $C(L)|_H$ are those for which

$$\mathcal{P}(\lambda) = m a_0(\lambda\lambda) + c^i a_i(\lambda\lambda) \geq 0 \quad \text{for all } \lambda . \tag{63}$$

U^o is non-vanishing, if in addition

$$\mathcal{P}(\lambda) = 0 \tag{64}$$

for some non-zero λ. We may restrict λ to the compact space

$$\Lambda = \{\lambda \mid \sum |\lambda_i|^2 = 1\} . \tag{65}$$

Then

$$\min_{\Lambda} \left(\mathcal{P}(\lambda) / a_0(\lambda\lambda) \right)$$

is a continuous function of the c^i. Along any ray in $C(L)|_H$ from zero to infinity it is a linear function which will take at first positive, then negative values, with one zero in between. The convexity follows from

$$\min_{\Lambda} \left((c^i + d^i) a_i(\lambda\lambda) \right) \geqslant \min_{\Lambda} \left(c^i a_i(\lambda\lambda) \right) + \min_{\Lambda} \left(d^i a_i(\lambda\lambda) \right) . \tag{66}$$

Q.E.D.

An important special case arises, if $S = io(R,1)$ is reduced to $io(R',1)$ with $R' < R$. Here the superfluous momentum components become central charges, and the boundary of $C(L)|_H$ corresponds to zero mass in $R+1$ dimensions.

Now we shall show that as far as the representations of L' are concerned, the central charges enter only via the determination of U^o.

<u>Proposition 3.3</u> :

Let the representation $U'|_H$ of $(G',+)$ be given. Then $< U'U' >|_H$ is fixed up to an isomorphism.

Proof

Let

$$\mathcal{U}' = \mathcal{U}^{(1)} \oplus \mathcal{U}^{(2)} \oplus \ldots \tag{67}$$

be the decomposition of U' into inequivalent representations of (G',+). As $\langle U'U' \rangle|_H$ contains only G' scalars,

$$\langle \mathcal{U}^{(m)} \mathcal{U}^{(n)} \rangle|_H = \sigma \quad \text{for} \quad m \neq n. \tag{68}$$

Let u_{ai} be a basis of some $U^{(m)}$, where (G',+) acts irreducibly on the first index, whereas i counts the multiplicity of the representation. We may write

$$\langle u_{ai}\, u^+_{bj} \rangle = K_{ab}\, X_{ij}\ , \tag{69}$$

where K is the uniquely defined positive definite Hermitian invariant form of the corresponding representation of (G',+). $K \otimes X$ has to be positive definite Hermitian, thus also X. In particular, we may choose a basis such that

$$\begin{aligned} K_{ab} &= \delta_{ab}\,, \\ X_{ij} &= \delta_{ij}. \end{aligned} \tag{70}$$

Q.E.D.

Now let us classify the supersymmetries with regard to the representations. As we have seen, this requires the classification of all possible U'.

Proposition 3.4 :

Let $G' = S' \oplus J$ be an algebra of type (3), (51) and U' a spinorial representation of (G',+). Then one can always find a supersymmetry L which yields $G' \oplus U'$ as algebra of the little group, both in the massless and in the massive case.

Proof

It is sufficient to consider an irreducible U', otherwise one just takes a direct sum with components orthogonal under the Lie bracket. For any irreducible spinorial representation of some $(G,+)$ with $G = S \oplus J$ one can define a supersymmetry by

$$\langle Q_{\alpha i} Q^{+}_{\beta j} \rangle = (\gamma^0 \gamma_\mu)_{\alpha\beta} P^\mu X_{ij} \tag{71}$$

Here S acts on the first and J on the second index of Q. X_{ij} is the positive definite Hermitian form on the representation of J.

Let

$$S' = o(r), \tag{72}$$

where $r = R$ for the massive and $r = R-1$ for the massless case. Take

$$G = io(r+1,1) \oplus J \tag{73}$$

and choose a U which transforms under J according to the given representation. Furthermore let its transformation properties under $o(r+1,1)$ be given by the embedding of the representation of $o(r)$ into the spinorial representation of $o(r+1,1)$ of twice its dimension. As the real, quaternionic, or non-self-conjugated nature of the spinorial representations of $o(r+n,n)$ is independent of n [8], the representation of $(G',+)$ is embedded into a representation of $(G,+)$ of twice its dimension. Now consider the corresponding algebra (71). For an H with $p^0 = p^{r+1}$ it yields the wanted algebra of the little group. In the massless case, we have finished. In the massive case we just have to take the subalgebra $io(R,1)$ of $io(R+1,1)$, and to interpret p^{R+1} as central charge. Q.E.D.

4. - REPRESENTATIONS

In this section we always take the restriction to H, without noting it explicitly.

It remains to determine the representations of $(L',+)$. As the undecomposable representations of Clifford algebras with non-degenerate bilinear form are fixed up to isomorphisms, this problem is completely solved by

Proposition 4.1 :

The universal associative enveloping algebra $U(L')$ of L' decomposes as

$$\mathcal{U}(L') = \bar{\mathcal{U}}(G') \otimes \mathcal{U}(\mathcal{U}'), \tag{74}$$

where $\bar{U}(G')$ is isomorphic to $U(G')$.

Proof

Take a basis g^i of G', Q_α of U', such that

$$\langle Q_\alpha Q_\beta^+ \rangle = \delta_{\alpha\beta} \tag{75}$$

$$\langle g^i Q_\alpha \rangle = Q_\beta \sigma^i_{\beta\alpha} . \tag{76}$$

Let

$$\langle Q_\alpha Q_\gamma \rangle = T_{\alpha\gamma} - T_{\gamma\alpha} . \tag{77}$$

Then

$$Q_\alpha = T_{\alpha\beta} Q_\beta^+ . \tag{78}$$

The Jacobi identity yields

$$T_{\beta\gamma} \sigma^i_{\beta\alpha} + T_{\alpha\beta} \sigma^i_{\beta\gamma} = 0 \tag{79}$$

and

$$\langle g^i - \tfrac{1}{2} Q_\beta \sigma^i_{\beta\gamma} Q^+_\gamma , Q_\alpha \rangle = 0 \quad \text{for all } \alpha, i . \tag{80}$$

Thus we obtain a set of elements

$$\bar{g}^i = g^i - \tfrac{1}{2} Q_\beta \sigma^i_{\beta\gamma} Q^+_\gamma \tag{81}$$

of $U(L')$ which commute with all elements of $U(U')$ and form a Lie algebra isomorphic to G'. The enveloping algebra $\bar{U}(G')$ of this Lie algebra fulfils Eq. (74). Q.E.D.

Thus all representations of L' are products of a representation of G' with the irreducible representation F of U'. Taking the trivial representation of $\bar{U}(G')$ we obtain the fundamental representation $1 \otimes F$ of L', for which the generators g^i are represented according to

$$g^i = \tfrac{1}{2} Q_\beta \sigma^i_{\beta\gamma} Q^+_\gamma . \tag{82}$$

Its dimension is

$$\dim F = 2^{\dim U'/2} . \tag{83}$$

Proposition 4.2 :

The representations of L' contain the same number of fermion as of boson states.

Proof

Let f be the fermion number. Because of

$$(-)^f Q = - Q (-)^f \quad \text{for } Q \in U' \tag{84}$$

one has

$$\mathrm{Tr}\left((-)^f \langle Q Q' \rangle \right) = 0 \quad \text{for } Q, Q' \in U' . \tag{85}$$

The number $\langle QQ' \rangle$ is in general not zero. Q.E.D.

To tackle the calculation of the representations it is convenient to use characters, i.e., the traces of elements of the group generated by the g^i. We may restrict ourselves to a maximal Abelian subgroup, because this determines already all the weights. Let $\bar{A}$ be any representation of $\bar{U}(G')$. It corresponds, via the isomorphism to $\dot{U}(G')$, to a representation A of G' and vice versa. Thus we obtain

$$\chi_{\bar{A}\otimes F}(\exp(\rho_i g^i)) = \chi_{\bar{A}\otimes F}(\exp(\rho_i \bar{g}^i)\exp(\tfrac{1}{2}\rho_i Q_\alpha \sigma^i_{\alpha\beta} Q^+_\beta)) =$$
$$= \chi_{\bar{A}}(\exp(\rho_i \bar{g}^i))\chi_F(\tfrac{1}{2} Q_\alpha \sigma^i_{\alpha\beta} Q^+_\beta) =$$
$$= \chi_A(\exp(\rho_i g^i))\,\chi_{1\otimes F}(\exp(\rho_i g^i)). \tag{86}$$

Furthermore let us use Eq. (67) Let $U^{(m)}$ contain c_m irreducible representations of type m, which by themselves yield fundamental representations $1 \otimes Fm$. Then

$$\chi_{1\otimes F}(g) = \prod_m \chi_{1\otimes Fm}(g)^{c_m}. \tag{87}$$

Thus we can restrict ourselves to irreducible U'. We can even reduce the calculation of $X_{1\otimes Fm}$, using the same formula, to the corresponding one for irreducible representations of a maximum Abelian subgroup of G'.

Now take an r dimensional Abelian group with generators $g^1,\ldots,g^r$ and a 2^r dimensional real representation U' for which no g^i is represented trivially. We may assume that the eigenvalues of all g^i are $\pm i/2$. For convenience we define

$$\chi_r(\rho_1,\ldots,\rho_r) = \chi_{1\otimes F}(\exp(\rho_i g^i)). \tag{88}$$

A change of base from g^1, g^2 to $(g^1 \pm g^2)/2$ yields the recursion relation

$$\chi_r(\rho_1,\rho_2,\rho_3,\ldots) = \chi_{r-1}(\rho_1+\rho_2,\rho_3,\ldots)\chi_{r-1}(\rho_1-\rho_2,\rho_3,\ldots). \tag{89}$$

χ_1 can easily be calculated directly. We take a base $Q_\pm$ of U', where Q_- is the adjoint of Q_+. The σ of Eq. (76) has the form

$$\sigma^1 = \frac{1}{2} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{90}$$

Thus

$$g^1 = \frac{1}{4}(Q_+Q_- - Q_-Q_+). \tag{91}$$

Then $Q_\pm$ may be represented by the Pauli matrices $(\sigma_y \pm i\sigma_y)/2$. Thus

$$\chi_1(\zeta) = 2\cos(\zeta/4). \tag{92}$$

Equation (89) then yields

$$\chi_2(\zeta_1,\zeta_2) = 2\cos(\zeta_1/2) + 2\cos(\zeta_2/2), \tag{93}$$

$$\chi_3(\zeta_1,\zeta_2,\zeta_3) = 2 + 2\sum_{i=1}^{3}\cos\zeta_i + \sum_{\pm}\exp\big((\pm\zeta_1\pm\zeta_2\pm\zeta_3)i/2\big). \tag{94}$$

Equation (92) yields in general

$$\chi_{1\otimes F}\big(\exp(i\zeta M_{12})\big) = \big(2\cos(\zeta/4)\big)^{\dim U'/2}. \tag{95}$$

Equation (93) shows that the state of $1 \otimes F$ which yields the highest eigenvalue of M_{12} has zero eigenvalue for all generators commuting with M_{12}. Thus for any $g \in J$

$$\chi_{1\otimes F}\Big(\exp\big(i\sum_i \zeta_i M_{2i-1,2i} + g\big)\Big) = \sum_i 2\cos(\zeta_i \dim U'/8) + \ldots \tag{96}$$

One sees that for $\dim U' \equiv 0 \bmod 8$, the fundamental representation contains a totally symmetric $d/8$ tensor. For $\dim U' = 4$, one has a spinor, more generally for $\dim U' \equiv 4 \bmod 8$ some spinor-tensor. For $\dim U' \equiv 2 \bmod 4$, which may happen for massless particles in 3+1 dimensions, $\chi_A = 1$ is obviously impossible. $\chi_A(\exp(i\zeta M_{12}))$ has to be a sum of terms of the form $2\cos((2n+1)\zeta/4)$.

For $o(10,1)$, the spin representation has dimension 32, such that for massless particles $\dim U'$ is at least 16. Thus any representation in more than 9+1 dimensions contains at least a symmetric tensor field. For $o(11,1)$, no Majorana-Weil-spinor exists [8], thus $\dim U'$ is at least 32 and higher spins have to occur. Consequently, supergravity theories are impossible in more than 10+1 dimensions, supersymmetric Yang-Mills theories in more than 9+1 dimensions.

Note that the minimal value of $\dim U'$ grows exponentially with the dimension. According to Eq. (83), $\dim F$ grows like an iterated exponential.

5. - EXAMPLES

At first we shall list the fundamental representations of the supersymmetries which allow multiplets with highest spin one. This requires $\dim U' \leq 8$.

A) - $S' = o(2)$

For $J = 0$ we obtain the character of Eq. (92). With

$$\chi_A(\exp(i\varphi M_{12})) = 2\cos((2n+1)\varphi/4),\ n = 0,1,\ldots \tag{97}$$

we obtain

$$\chi_{\bar{A}\otimes F}(\exp(i\varphi M_{12})) = 2\cos((n+1)\varphi/2) + 2\cos(n\varphi/2). \tag{98}$$

These are the well-known massless multiplets of the standard supersymmetry in 3+1 dimensions.

For $J = su(2)$ and isospin $\frac{1}{2}$ we may embed the representation U' into the corresponding one for $S' = o(3)$ without changing $\dim U'$. Therefore we need not treat this case separately. From now on we omit most representations for which such an embedding is possible.

For $J = su(3)$ and representation $3+\bar{3}$ a slightly more complicated embedding into the vector representation of $o(6)$ is possible. As fundamental multiplet, one obtains $su(3)$ singlets for "spin" $\pm 3/4$ and triplet, resp., antitriplet for "spin" $\pm 1/4$. Multiplying by the χ_A of Eq. (97) with $n = 0$ one obtains singlets with spins ± 1, $\pm\frac{1}{2}$, triplet, resp., antitriplet for spin $\pm\frac{1}{2}$ and both triplet and antitriplet for spin 0. Multiplying by the octet of $su(3)$ one obtains the particles of a possible supersymmetric Yang-Mills theory.

However, here a general difficulty of those theories becomes apparent. In Eq. (96) we have seen that for the fundamental representation the particles with highest spin are J singlets. Thus either one has to except multiplets with spin larger than one, or one has to multiply by the adjoint representation of some gauge group. But this procedure yields unreasonably high representations of the gauge group for the fermions. If one takes supercharges which commute with the gauge group, one obtains only adjoint representations of this group, otherwise higher representations have to occur. But, of course, one has to keep in mind that our investigation concerns only manifest symmetries.

For $J = su(4)$ and representation $4+\bar{4}$ compare the case $S' = o(6)$, $J = o(2)$, which may be embedded into the case $G' = o(8)$ discussed below. The fundamental representation has a singlet for spin ± 1, quartet, resp, antiquartet for spin $\pm\frac{1}{2}$ and an antisymmetric tensor for spin zero. Note that the simplest multiplet for $J = su(3)$ discussed above admits the larger symmetry $J = su(4)$.

B) - $\underline{S' = o(3)}$

Even for $J = 0$ the invariance under the adjoint operation requires that U' contains an even number of spinors. As smallest multiplet one obtains

$$\chi_{1\otimes F}(\zeta) = \chi_1(\zeta)^2 = 2\cos(\zeta/2) + 2. \qquad (99)$$

Taking the spinor representation for $\bar{A}$ one obtains in $\bar{A} \otimes F$ a vector, a scalar, and two Majorana spinors. These multiplets are well known [10].

Here an embedding into the $(2,2)$ representation of $o(3) \oplus o(3)$ is possible. The character of the fundamental representation has already been given in Eq. (93).

If one chooses the isospin $\frac{3}{2}$ representation of $J = su(2)$, the representation of the maximum Abelian subgroup in U' is reducible. One obtains

$$\chi_{1\otimes F}(\varphi_1, \varphi_2) = \chi_2(\varphi_1, \varphi_2)\,\chi_2(\varphi_1, 3\varphi_2). \tag{100}$$

i.e., $(3,1) + (1,5) + (2,4)$.

$o(4) = o(3) \oplus o(3)$ need not be considered separately, as for its spinors one $o(3)$ is represented trivially.

From $S' = o(5)$ on representations with $\dim U' < 8$ no longer occur and those with $\dim U' = 8$ can be embedded into a Majorana-Weyl-spinor of $o(8)$. This representation occurs for the massless particles of the supersymmetric spinning string. Let us consider this system in detail.

For all supersymmetries in more than 5+1 dimensions, $\dim U$ is at least 16. All supersymmetries with $R > 5$, $\dim U = 16$ can be considered as subsymmetries of the supersymmetry $L = G \oplus U$ with $G = io(9,1)$, $U =$ Majorana-Weyl-spinor. Central charges can be interpreted as components of the momentum in 9+1 dimensions. There are at most 9-R of them. The maximal internal symmetry for $S = io(R,1)$ is just $o(9-R)$. This can easily be checked case by case.

For the massless multiplet in 9+1 dimensions, the fundamental representation is essentially determined by $\dim F = 16$ and proposition 4.2. Alternatively it can be read of from Eq. (94). According to the chirality of U one finds

$$\chi_\pm(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \sum_\varepsilon \exp\Big(i \sum_{i=1}^{4} \varepsilon_i \varphi_i / 2\Big) + \sum_{i=1}^{4} 2\cos\varphi_i \,, \tag{101}$$

where all $\varepsilon_i \in \{1,-1\}$ and the sum goes over all quadruples $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with

$$\prod_i \varepsilon_i = \pm 1 \quad \text{for } \chi_\pm \text{ resp.} \tag{102}$$

Equation (101) just yields the $m = 0$ states of the spinning string of Ref. 2). The supersymmetry admits no central charges. Thus according to Eq. (60), one finds, for the fundamental representation of the massive case,

$$\chi_4(\varrho_1,\varrho_2,\varrho_3,\varrho_4) = \chi_+(\varrho_1,\varrho_2,\varrho_3,\varrho_4)\,\chi_-(\varrho_1,\varrho_2,\varrho_3,\varrho_4). \tag{103}$$

As can easily be read off from the helicity partition function [11] this yields exactly the multiplet which occurs at the $m^2 = 1$ level of the Neveu-Schwarz-Ramond string as considered in Ref. 2). For $m^2 = 2$ one finds the $\bar{V} \otimes F$ representation, where V is the $o(9)$ vector with

$$\chi_V(\varrho_1,\varrho_2,\varrho_3,\varrho_4) = 2\sum_{i=1}^{4}\cos\varrho_i + 1. \tag{104}$$

This is a strong confirmation for the conjecture that this model is supersymmetric.

For the closed string the representations considered in Ref. 2) are just the tensor product of the open string representations with the corresponding representation of the boson sector alone. Thus for a supersymmetric open string, the closed string has to be supersymmetric too. This yields one possible supergravity theory in 9+1 dimensions, which by reduction yields the $o(4)$ supergravity [12]. The representation is given by

$$\chi(\varrho_1,\varrho_2,\varrho_3,\varrho_4) = \chi_+(\varrho_1,\varrho_2,\varrho_3,\varrho_4)\cdot 2\sum_{i=1}^{4}\cos\varrho_i\;. \tag{105}$$

To restrict the representations of $io(9,1)$ to those of $io(3,1) + su(4)$ one just has to interpret M_{45}, M_{67} and M_{89} in the character formulae as generators of $SU(4)$. For the fundamental massless representation, Eq. (101) yields a $su(4)$ singlet with spin ± 1, a $su(4)$ quartet, resp., antiquartet with spin $\pm\frac{1}{2}$ and a $su(4)$ sextet with spin 0.

To obtain a Yang-Mills theory in 9+1 dimensions, one has to multiply the multiplet of Eq. (101) with the ajoint representation of the gauge group. As noted above, the fermions then transform according to the adjoint representation, too.

Supergravity theories are possible in at most $10+1$ dimensions, as we have seen. For $G = io(10,1)$, U = Majorana spinor, one obtains the fundamental representation (103).

If seven dimensions are compactified, one finds $G = io(3,1) \oplus \oplus\, o(7)$, while U transforms as Majorana spinor both under $io(3,1)$ and $o(7)$. Now, one can enlarge $o(7)$ to $o(8)$ without changing the representation space U. As the Majorana-Weil-spinor and the vector representations of $o(8)$ are connected by outer automorphisms of $o(8)$, the embedding of $o(7)$ into $o(8)$ may be done in such a way that U transforms as a vector under $o(8)$. Thus one should obtain the $o(8)$ supergravity by dimensional reduction, if the supergravity in $10+1$ dimensions can be constructed.

In $9+1$ dimensions, there is one further supergravity, which arises, if one takes all tensor products of the open string with itself, including the Fermion-fermion sector. This yields an internal symmetry $J = o(2)$. Taking into account only the space-time symmetry, one obtains for the fundamental representation

$$\chi(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \chi_+(\zeta_1, \zeta_2, \zeta_3, \zeta_4)^2. \tag{106}$$

Scherk has discovered that dimensional reduction of this theory probably yields the $o(8)$ supergravity [13)]. Indeed, as far as the little group $G' = o(8) \oplus o(2)$ is concerned, an exchange of $S' = o(8)$ and $J = o(2)$ would yield the representations of this supergravity. This exchange may arise automatically by dimensional reduction of G' to $o(2) \oplus o(6) \oplus o(2)$. Now, the $o(6)$ counts as part of the internal symmetry $J = o(6) \oplus o(2)$. As discussed for $o(7)$ above, the representation of J in U admits an extension to the vector representation of $o(8)$.

Thus in $9+1$ dimensions three supergravity theories may exist, with multiplets given by the Eqs. (105), (103) and (106), resp. Dimensional reduction of the first should yield the $o(4)$ supergravity, whereas from the other two one might obtain the $o(8)$ supergravity.

ACKNOWLEDGEMENTS

I wish to thank D. Olive, B. Zumino and J. Scherk for stimulating discussions.

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