1 Introduction to the Ricci calculus

1.1 The covariant derivative

A vector $\mathbf{u}$ can be expressed as a linear combination of basis vectors $\mathbf{e_j}$:

$$\mathbf{u} = u^j \mathbf{e_j} \tag{1}$$

where the Einstein summation convention is used. The derivative of $\mathbf{u}$ with respect to a coordinate x^i is obtained using the product rule:

$$\frac{\partial \mathbf{u}}{\partial x^i} = \frac{\partial u^j \mathbf{e_j}}{\partial x^i} = \frac{\partial u^j}{\partial x^i}\mathbf{e_j} + u^j \frac{\partial \mathbf{e_j}}{\partial x^i} \tag{2}$$

The derivative of $\mathbf{e_j}$ with respect to x_i is a linear combination of the basis vectors $\mathbf{e_k}$, with the coefficients given by the Christoffel symbols Γ^k_{ij}:

$$\frac{\partial \mathbf{e_j}}{\partial x^i} = \Gamma^k_{ij}\mathbf{e_k} \tag{3}$$

Substituting this expression into the previous equation yields

$$\frac{\partial \mathbf{u}}{\partial x^i} = \frac{\partial u^j}{\partial x^i}\mathbf{e_j} + u^j \Gamma^k_{ij}\mathbf{e_k} \tag{4}$$

Swapping the j and k indices in the second term yields:

$$\frac{\partial \mathbf{u}}{\partial x^i} = \frac{\partial u^j}{\partial x^i}\mathbf{e_j} + u^k \Gamma^j_{ik}\mathbf{e_j} = \left(\frac{\partial u^j}{\partial x^i} + u^k \Gamma^j_{ik}\right)\mathbf{e_j} \tag{5}$$

A shorthand notation for this expression is

$$\nabla_i u^j = \partial_i u^j + u^k \Gamma^j_{ik} \tag{6}$$

where ∇_i is called the covariant derivative (as opposed to the partial derivative, which is denoted by ∂_i). The covariant derivative introduces a correction term that accounts for changes in the basis vectors themselves.

The covariant derivative of a covector can be derived as follows:

$$\begin{aligned}\nabla_a(u^b u_b) &= (\nabla_a u^b)u_b + u^b(\nabla_a u_b)\\ &= (\partial_a u^b + \Gamma^b_{ac}u^c)u_b + u^b(\nabla_a u_b)\end{aligned} \tag{7}$$

Therefore

$$
\begin{aligned}
u^b(\nabla_a u_b) &= \nabla_a(u^b u_b) - (\partial_a u^b + \Gamma^b_{ac} u^c) u_b \\
&= \partial_a(u^b u_b) - (\partial_a u^b + \Gamma^b_{ac} u^c) u_b \\
&= (\partial_a u^b) u_b + u^b(\partial_a u_b) - (\partial_a u^b + \Gamma^b_{ac} u^c) u_b \\
&= (\partial_a u^b) u_b + u^b(\partial_a u_b) - (\partial_a u^b) u_b - \Gamma^b_{ac} u^c u_b \\
&= u^b(\partial_a u_b) - \Gamma^b_{ac} u^c u_b \\
&= u^b(\partial_a u_b) - \Gamma^c_{ab} u^b u_c \\
&= u^b(\partial_a u_b - \Gamma^c_{ab} u_c)
\end{aligned} \tag{8}
$$

Hence

$$\nabla_a u_b = \partial_a u_b - \Gamma^c_{ab} u_c \tag{9}$$

1.2 Parallel transport

Let $x^i = x^i(\lambda)$ be a curve parametrized by some parameter λ. The derivative of $\mathbf{u}$ with respect to λ can be obtained using the chain rule:

$$\frac{\partial \mathbf{u}}{\partial \lambda} = \frac{\partial \mathbf{u}}{\partial x^i} \frac{\partial x^i}{\partial \lambda} = \left(\frac{\partial u^j}{\partial x^i} \frac{\partial x^i}{\partial \lambda} + u^k \Gamma^j_{ik} \frac{\partial x^i}{\partial \lambda} \right) \mathbf{e_j} = \left(\frac{\partial u^j}{\partial \lambda} + u^k \Gamma^j_{ik} \frac{\partial x^i}{\partial \lambda} \right) \mathbf{e_j} \tag{10}$$

$\mathbf{u}$ is parallel transported about the curve if its derivative with respect to the parameter λ is zero (that is, it remains locally parallel to the curve):

$$\frac{\partial \mathbf{u}}{\partial \lambda} = 0 \tag{11}$$

Hence each component must also be zero:

$$\frac{\partial u^j}{\partial \lambda} + u^k \Gamma^j_{ik} \frac{\partial x^i}{\partial \lambda} = 0 \tag{12}$$

This is called the parallel transport equation. It can also be expressed as

$$\mathrm{d}u^j + u^k \Gamma^j_{ik} \mathrm{d}x^i = 0 \tag{13}$$

If the δx^i are the infinitesimal changes in coordinates for a parallel transport, the components of $\mathbf{u}$ after the parallel transport are

$$u'^j = u^j + \delta u^j = u^j - u^k \Gamma^j_{ik} \delta x^i \tag{14}$$

1.3 The geodesic equation

A geodesic is a curve whose tangent vector is parallel transported along the curve itself. If $\mathbf{u}$ is the tangent vector to the curve $x^i = x^i(\lambda)$, then

$$u^i = \frac{\partial x^i}{\partial \lambda} \tag{15}$$

Substituting this into the parallel transport equation yields

$$\frac{\partial^2 x^j}{\partial \lambda^2} + \frac{\partial x^k}{\partial \lambda} \Gamma^j_{ik} \frac{\partial x^i}{\partial \lambda} = 0 \tag{16}$$

This is called the geodesic equation. It can also be expressed as

$$\mathrm{d}^2 x^a + \Gamma^a_{bc} \mathrm{d}x^b \mathrm{d}x^c = 0 \tag{17}$$

$$\mathrm{d}u^a + \Gamma^a_{bc} u^b u^c = 0 \tag{18}$$

Hence the components of the tangent vector u^a after parallel transport along the geodesic are

$$u'^a = u^a + \delta u^a = u^a - \Gamma^a_{bc} u^b u^c \delta\lambda \tag{19}$$

The coordinates themselves are updated as follows:

$$x'^a = x^a + \delta x^a = x^a + u^a \delta\lambda \tag{20}$$

In Riemannian geometry, the exponential map $\exp_p \mathbf{u}$ is a map from a tangent vector $\mathbf{u}$ at some point p on a manifold M to another point q on M. More precisely, q is the endpoint of the geodesic of length $|\mathbf{u}|$ that starts at p in the direction of $\mathbf{u}$.

The exponential map can be used to define Riemannian normal coordinates, a local coordinate system in a neighborhood of p obtained by applying the exponential map to the tangent space at p. In a normal coordinate system, the Christoffel symbols of the connection vanish at p, simplifying calculations.

In a way, the exponential map provides the best description for what an observer would see within a non-Euclidean space, since light rays follow geodesics. This allows us to "project" a non-Euclidean space onto a Euclidean one.

1.4 Christoffel symbols in terms of the metric

The covariant derivative of the metric tensor vanishes:

$$0 = \frac{\partial \mathbf{g}}{\partial x^k} = \frac{\partial g_{ij} \mathbf{e^i e^j}}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k} \mathbf{e^i e^j} + g_{ij} \frac{\partial \mathbf{e^i}}{\partial x^k} \mathbf{e^j} + g_{ij} \mathbf{e^i} \frac{\partial \mathbf{e^j}}{\partial x^k} \tag{21}$$

Equivalently,

$$0 = \nabla_k g_{ij} = \partial_k g_{ij} - (g_{lj} \Gamma^l_{ik} + g_{li} \Gamma^l_{jk}) = \partial_k g_{ij} - 2 g_{l(i} \Gamma^l_{j)k} \tag{22}$$

Because the Christoffel symbols are symmetric in the lower indices (that is, $\Gamma^a_{bc} = \Gamma^a_{cb}$), one can permute the indices to obtain the Christoffel symbols purely in terms of the metric tensor:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right) \tag{23}$$

1.5 Example: Spherical coordinates

Let x^1 be the azimuthal angle and x^2 be the polar angle in a spherical coordinate system. The metric tensor in this coordinate system is

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} (\sin x^2)^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{24}$$

The inverse metric tensor is

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} (\sin x^2)^{-2} & 0 \\ 0 & 1 \end{pmatrix} \tag{25}$$

The Christoffel symbols are

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial g_{cd}}{\partial x^b} + \frac{\partial g_{db}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right) \tag{26}$$

Therefore

$$\begin{aligned} \Gamma^a_{11} &= -g^{a2} \sin x^2 \cos x^2 \\ \Gamma^a_{12} &= \Gamma^a_{21} = g^{a1} \sin x^2 \cos x^2 \\ \Gamma^a_{22} &= 0 \end{aligned} \tag{27}$$

The only nonzero Christoffel symbols are

$$\begin{aligned} \Gamma^1_{12} &= \Gamma^1_{21} = (\sin x^2)^{-1} \cos x^2 \\ \Gamma^2_{11} &= -\sin x^2 \cos x^2 \end{aligned} \tag{28}$$

The geodesic equation is

$$\begin{aligned} 0 &= \mathrm{d}^2 x^a + \Gamma^a_{bc} \mathrm{d}x^b \mathrm{d}x^c \\ &= \mathrm{d}^2 x^a + \Gamma^a_{11} \mathrm{d}x^1 \mathrm{d}x^1 + \Gamma^a_{12} \mathrm{d}x^1 \mathrm{d}x^2 + \Gamma^a_{21} \mathrm{d}x^2 \mathrm{d}x^1 + \Gamma^a_{22} \mathrm{d}x^2 \mathrm{d}x^2 \\ &= \mathrm{d}^2 x^a - g^{a2} \sin x^2 \cos x^2 \mathrm{d}x^1 \mathrm{d}x^1 + 2 g^{a1} \sin x^2 \cos x^2 \mathrm{d}x^1 \mathrm{d}x^2 \\ &= \mathrm{d}^2 x^a + (2 g^{a1} \mathrm{d}x^1 \mathrm{d}x^2 - g^{a2} \mathrm{d}x^1 \mathrm{d}x^1) \sin x^2 \cos x^2 \end{aligned} \tag{29}$$

Therefore

$$\begin{aligned}
\mathrm{d}^2x^1 &= (g^{12}\mathrm{d}x^1\mathrm{d}x^1 - 2g^{11}\mathrm{d}x^1\mathrm{d}x^2)\sin x^2\cos x^2 \\
&= -2(\sin x^2)^{-2}\sin x^2\cos x^2\mathrm{d}x^1\mathrm{d}x^2 \\
&= -2(\sin x^2)^{-1}\cos x^2\mathrm{d}x^1\mathrm{d}x^2
\end{aligned} \tag{30}$$

$$\begin{aligned}
\mathrm{d}^2x^2 &= (g^22\mathrm{d}x^1\mathrm{d}x^1 - 2g^{21}\mathrm{d}x^1\mathrm{d}x^2)\sin x^2\cos x^2 \\
&= \sin x^2\cos x^2\mathrm{d}x^1\mathrm{d}x^2
\end{aligned} \tag{31}$$

Geodesics therefore satisfy the following pair of coupled equations:

$$\frac{\partial^2 x^1}{\partial\lambda^2} = -2\frac{\cos x^2}{\sin x^2}\frac{\partial x^1}{\partial\lambda}\frac{\partial x^2}{\partial\lambda} \tag{32}$$

$$\frac{\partial^2 x^2}{\partial\lambda^2} = \sin x^2\cos x^2\frac{\partial x^1}{\partial\lambda}\frac{\partial x^1}{\partial\lambda} \tag{33}$$

These equations correspond to great circles on a sphere.

1.6 Example: Hyperbolic space

The Poincare ball model, also called the conformal model, is a model of hyperbolic geometry in which all of the hyperbolic space lies within a unit ball, and straight lines in the hyperbolic space consist of segments of spheres contained within the ball that are orthogonal to the spherical boundary of the ball.

The metric tensor of the Poincare ball model is given by

$$\mathrm{d}s^2 = \frac{|\mathrm{d}\mathbf{x}|^2}{(1-|\mathbf{x}|^2)^2} \tag{34}$$

This is equivalent to

$$g_{ij} = \frac{\delta_{ij}}{(1-\delta_{kl}x^kx^l)^2} \tag{35}$$

where x^k are the Cartesian coordinates of the ambient Euclidean space. The geodesics of the ball model are circles perpendicular to the bounding sphere.

The partial derivative of the metric tensor with respect to a coordinate x^m is

$$\begin{aligned}
\partial_m g_{ij} &= \partial_m\frac{\delta_{ij}}{(1-\delta_{kl}x^kx^l)^2} \\
&= \delta_{ij}\partial_m(1-\delta_{kl}x^kx^l)^{-2} \\
&= -2\delta_{ij}(1-\delta_{kl}x^kx^l)^{-3}\partial_m(1-\delta_{kl}x^kx^l) \\
&= 2\delta_{ij}(1-\delta_{kl}x^kx^l)^{-3}\delta_{kl}\partial_m x^kx^l \\
&= 4\delta_{ij}(1-\delta_{kl}x^kx^l)^{-3}x_m
\end{aligned} \tag{36}$$

Substituting the partial derivatives of the metric tensor with the appropriate indices into the definition of the Christoffel symbol yields

$$\Gamma^i_{jk} = \frac{2}{(1 - \delta_{kl} x^k x^l)} (\delta^i_k x_j + \delta^i_j x_k - \delta_{jk} x^i) \tag{37}$$

1.7 A non-Euclidean raytracer

Suppose a user specifies a metric g_{ij} describing a non-Euclidean space, and that the Christoffel symbols Γ^k_{ij} are obtained (through either symbolic, numerical, or automatic differentiation). A raytracer for rendering this non-Euclidean space from the perspective of an internal observer can be implemented as follows:

$$\Gamma^i_{jk} \leftarrow \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \tag{38}$$

$$\delta u^i \leftarrow -\Gamma^i_{jk} u^j u^k \delta\lambda \tag{39}$$

$$u^i \leftarrow u^i + \delta u^i \tag{40}$$

$$\delta x^i \leftarrow u^i \delta\lambda \tag{41}$$

$$x^i \leftarrow x^i + \delta x^i \tag{42}$$

$$\delta s \leftarrow \sqrt{g_{ij} \delta x^i \delta x^j} \tag{43}$$

$$s \leftarrow s + \delta s \tag{44}$$

or, more concisely,

$$\delta u^i \leftarrow -\frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) u^j u^k \delta\lambda \tag{45}$$

$$u^i \leftarrow u^i + \delta u^i \tag{46}$$

$$x^i \leftarrow x^i + u^i \delta\lambda \tag{47}$$

$$s \leftarrow s + \sqrt{g_{ij} \delta x^i \delta x^j} \tag{48}$$

Taking care to ensure that u^i is not updated before all components of δu^i have been calculated (using the terms u^j and u^k). This process can be wrapped into a while loop that updates x^i at every step until some desired distance s has been travelled by the ray. Smaller $\delta\lambda$ results in a more accurate geodesic.

For parallel transport of a vector $\mathbf{v}$, we use our previous expression:

$$v^j \leftarrow v^j + \delta v^j = v^j - v^k \Gamma^j_{ik} \delta x^i \tag{49}$$

This is useful for updating the vectors that together form the local reference frame from which an observer sees the non-Euclidean space.

Alternatively, one can use a second-order finite difference scheme to calculate geodesics. The second-order forward difference is given by

$$
\begin{aligned}
\mathrm{d}^2 x^\alpha &= (x^\alpha[t+2] - x^\alpha[t+1]) - (x^\alpha[t+1] - x^\alpha[t]) \\
&= x^\alpha[t+2] - 2x^\alpha[t+1] + x^\alpha[t]
\end{aligned}
\tag{50}
$$

Therefore

$$
\begin{aligned}
x^\alpha[t+2] &= 2x^\alpha[t+1] - x^\alpha[t] + \mathrm{d}^2 x^\alpha \\
&= 2x^\alpha[t+1] - x^\alpha[t] - \Gamma^\alpha_{\beta\gamma} \mathrm{d}x^\beta \mathrm{d}x^\gamma \\
&= 2x^\alpha[t+1] - x^\alpha[t] - \Gamma^\alpha_{\beta\gamma} (x^\beta[t+1] - x^\beta[t])(x^\gamma[t+1] - x^\gamma[t])
\end{aligned}
\tag{51}
$$

where $\Gamma^\alpha_{\beta\gamma}$ is evaluated at $x[t+1]$.