Polar Decomposition of a Matrix

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1 Introduction

The matrix representation of systems reveals many useful and fascinating properties of linear transformations. One such representation is the polar decomposition. This paper will investigate the polar decomposition of matrices. The polar decomposition is analogous to the polar form of coordinates. We will begin with a definition of the decomposition and then a proof of its existence. We will then move on to the construction of the decomposition and some interesting properties including a guest appearance by the singular value decomposition. The final component of the paper will be a discussion of the geometric underpinnings of the polar decomposition through an example.

2 The Polar Decomposition

We will jump right in with some definitions.

Definition 2.1 (Right Polar Decomposition)**.** *The right polar decomposition of a matrix* $A \in \mathbb{C}^{m\times n}$ $m \geq n$ *has the form* $A = UP$ *where* $U \in \mathbb{C}^{m\times n}$ *is a matrix with orthonormal columns and* $P \in \mathbb{C}^{n\times n}$ *is positive semi-definite.*

Definition 2.2 (Left Polar Decomposition)**.** *The left polar decomposition of a matrix* $A \in \mathbb{C}^{n\times m}$ $m \geq n$ *has the form* $A = HU$ *where* $H \in \mathbb{C}^{n\times n}$ *is positive semi-definite and* $U \in \mathbb{C}^{n\times m}$ *has orthonormal columns.*

We know from Theorem OD in a First Course in Linear Algebra that our matrices P and H are both orthonormally diagonalizable. It is necessary however to prove that we can take powers of these matrices with nothing more than just the diagonalizing matrices S and S^*

Theorem 2.3. *If* A *is a normal matrix then there exists a positive semi-definite matrix* P *such that* $A = P^2$*.*

Proof. Suppose you have a normal matrix A of size n. Then A is orthonormally diagonalizable by Theorem OD from *A First Course in Linear Algebra*. This means that there is a unitary matrix S and a diagonal matrix B whose diagonal entries are the eigenvalues of A so that $A = SBS^*$ where $S^*S = I_n$. Since A is normal the diagonal entries of B are all positive, making B positive semi-definite as well. Because B is diagonal with real, non-negative entries all along the diagonal we can easily define a matrix C so that the diagonal entries of C are the square roots of the eigenvalues of A. This gives us the matrix equality $C^2 = B$. We now define P with the equality $P = SCS^*$ and show that

$$
\begin{aligned}
A &= SBS^* \\
&= SC^2S^* \\
&= SCCS^* \\
&= SCI_nCS^* \\
&= SCS^*SCS^* \\
&= PP \\
&= P^2
\end{aligned}
$$

□

This is not a difficult proof, but it will prove to be useful later. All we have to do is find a diagonal matrix for A^*A to diagonalize to. This is also a convient method computing the the square root, and is the method we will employ.

We can now define our matrices P and H more clearly.
Definition 2.4. *The matrix P is defined as* $\sqrt{A^*A}$ *where* $A \in \mathbb{C}^{m\times n}$.
Definition 2.5. *The matrix H is* $\sqrt{AA^*}$ *where* $A \in \mathbb{C}^{n\times m}$.

With these definitions in place we can prove our decomposition's existence.
Theorem 2.6 (Right Polar Decomposition)**.** *For any matrix $A \in \mathbb{C}^{m\times n}$, where $m \geq n$, there is a matrix $U \in \mathbb{C}^{m\times n}$ with orthonormal columns and a positive semi-definite matrix $P \in \mathbb{C}^{n\times n}$ so that $A = UP$.*

Proof. Suppose you have an $m \times n$ matrix A where $m \geq n$ and $\text{rank}A = r \leq n$. Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ be an orthonormal basis of eigenvectors for $\sqrt{A^*A}$ (the existence of which is guaranteed by Theorem 1 because A^*A is a normal matrix). This means that $\sqrt{A^*A}\mathbf{x}_i = P\mathbf{x}_i = \lambda_i\mathbf{x}_i$ where $1 \leq i \leq n$. It is important to note that $\lambda_1, \lambda_2, ..., \lambda_r > 0$ and that $\lambda_{r+1}, \lambda_{r+2}, ..., \lambda_n = 0$ because there is the possibility that we do not have a matrix with full rank.

To demonstrate this fact we grab the orthonormal set of vectors

$$\{\tfrac{1}{\lambda_1}A\mathbf{x}_1, \tfrac{1}{\lambda_2}A\mathbf{x}_2, ..., \tfrac{1}{\lambda_r}A\mathbf{x}_r\}.$$

We know that this set is orthonormal because if we take the inner product of any two vectors in the set we will get zero as we will now demonstrate. For $1 \leq j \leq r$ and $1 \leq l \leq r$ where $j \neq l$

$$\begin{aligned}\langle \frac{1}{\lambda_j} A\mathbf{x}_j, \frac{1}{\lambda_l} A\mathbf{x}_l \rangle &= \frac{1}{\lambda_j \lambda_l} \langle A\mathbf{x}_j, A\mathbf{x}_l \rangle \\ &= \frac{1}{\lambda_j \lambda_l} \langle \mathbf{x}_j, A^* A\mathbf{x}_l \rangle \\ &= \frac{1}{\lambda_j \lambda_l} \langle \mathbf{x}_j, \lambda_l^2 \mathbf{x}_l \rangle \\ &= \frac{\lambda_l^2}{\lambda_j \lambda_l} \langle \mathbf{x}_j, \mathbf{x}_l \rangle \\ &= \frac{\lambda_l}{\lambda_j} \langle \mathbf{x}_j, \mathbf{x}_l \rangle \\ &= \frac{\lambda_l}{\lambda_j} (0) \\ &= 0\end{aligned}$$

Assuming that $r < n$ we must extend our set to include $n-r$ more orthonormal vectors $\{\mathbf{y}_{r+1}, \mathbf{y}_{r+2}, ..., \mathbf{y}_n\}$. Now we make our orthonormal set into the columns of a matrix and multiply it by the adjoint of a matrix containing our original orthonormal basis of eigenvectors for $\sqrt{A^*A}$ which we call E. This is how we will define U.

$$\begin{aligned}U &= [\frac{1}{\lambda_1} A\mathbf{x}_1 | \frac{1}{\lambda_2} A\mathbf{x}_2 | ... | \frac{1}{\lambda_r} A\mathbf{x}_r | \mathbf{y}_{r+1} | \mathbf{y}_{r+2} | ... | \mathbf{y}_n] \cdot E \\ &= [\frac{1}{\lambda_1} A\mathbf{x}_1 | \frac{1}{\lambda_2} A\mathbf{x}_2 | ... | \frac{1}{\lambda_r} A\mathbf{x}_r | \mathbf{y}_{r+1} | \mathbf{y}_{r+2} | ... | \mathbf{y}_n] \cdot [\mathbf{x}_1 | \mathbf{x}_2 | ... | \mathbf{x}_n]^*\end{aligned}$$

This gives us the $m \times n$ matrix with orthonormal columns that we have been looking for. Define our standard unit vector $\mathbf{s}_i \in \mathbb{C}^m$ as $[\mathbf{s}_i]_j = 0$ whenever $j \neq i$ and $[\mathbf{s}_i]_j = 1$ whenver $j = i$. Now we should investigate the matrix vector product of E with any element of the orthonormal basis for P.

$$\begin{aligned}E\mathbf{x}_i &= [\mathbf{x}_1 | \mathbf{x}_2 | ... | \mathbf{x}_n]^* \cdot \mathbf{x}_i \\ &= \mathbf{x}_1^* [\mathbf{x}_i]_1 + \mathbf{x}_2^* [\mathbf{x}_i]_2 + ... + \mathbf{x}_i^* [\mathbf{x}_i]_i + ... + \mathbf{x}_n^* [\mathbf{x}_i]_n \\ &= \mathbf{0} + \mathbf{0} + ... + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} + ... + \mathbf{0} \\ &= \mathbf{s}_i\end{aligned}$$

This means that $U\mathbf{x}_i = \frac{1}{\lambda_i} A\mathbf{x}_i$ when $1 \leq i \leq r$ and $U\mathbf{x}_i = \mathbf{y}_i$ when $r < i \leq n$. Now we investigate what happens when $1 \leq i \leq r$ and we slip P between our orthonormal matrix U and our basis eigenvector $\mathbf{x}_i$. We get that

$$\begin{aligned} UP\mathbf{x}_i &= U\lambda_i\mathbf{x}_i \\ &= \lambda_i U\mathbf{x}_i \\ &= \lambda_i \frac{1}{\lambda_i} A\mathbf{x}_i \\ &= A\mathbf{x}_i \end{aligned}$$

Now we need to examine the result of the same procedure when $r < i \leq n$.

$$\begin{aligned} UP\mathbf{x}_i &= U\lambda_i\mathbf{x}_i \\ &= \lambda_i U\mathbf{x}_i \\ &= (0)U\mathbf{x}_i \\ &= (0)\mathbf{y}_i \\ &= 0 \end{aligned}$$

We now see that $A = UP$ for the basis eigenvectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$. □

We have not given much effort to the left decomposition because it is a simple task to show the left from the right. The U from the left decomposition is actually the adjoint of the U from the right decomposition. This might require some demonstration. Suppose we have found the right decomposition of A^* but we wish to find the left. It is fairly straight forward to show that

$$\begin{aligned} A^* &= UP \\ &= U\sqrt{(A^*A)^*} \\ &= U\sqrt{AA^*} \end{aligned}$$

Now we adjoint A^* and get A so the the equality now becomes

$$A = \sqrt{AA^*}U^*$$

It is also worth noting that if our matrix A is invertible then $\sqrt{A^*A}$ is also invertible since the product of invertible matrices is also invertible. This means that the calculation of U could be accomplished with the equality $U = AP^{-1}$ which would give us a uniquely determined, orthonormal matrix U.

Example 1

Consider the matrix

$$A = \begin{bmatrix} 3 & 8 & 2 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{bmatrix}$$

We will be doing the right polar decomposition so the first step is to find the matrix P. So we need to find the matrix A^*A.

$$A^*A = \begin{bmatrix} 3 & 2 & 1 \\ 8 & 5 & 4 \\ 2 & 7 & 6 \end{bmatrix} \begin{bmatrix} 3 & 8 & 2 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 38 & 26 \\ 38 & 105 & 75 \\ 25 & 76 & 89 \end{bmatrix}$$

That was easy enough but now we must find a couple of change-of-basis matrices to diagonalize A^*A. We do so by finding the basis vectors of our eigenspace and making them the columns of our change-of-basis matrix S. Unfortunately these things rarely turn out pretty. Doing these calculations yields the change-of-basis matrix

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -0.3868 & 2.3196 & 2.8017 \\ 0.0339 & -3.0376 & 2.4687 \end{bmatrix}$$

and its inverse

$$S^{-1} = \begin{bmatrix} 0.8690 & -0.3361 & 0.0294 \\ 0.0641 & 0.1486 & -0.1946 \\ 0.0669 & 0.1875 & 0.1652 \end{bmatrix}$$

We also get the diagonal matrix

$$B = \begin{bmatrix} 0.1833 & 0 & 0 \\ 0 & 23.1678 & 0 \\ 0 & 0 & 184.6489 \end{bmatrix}$$

Now we take the square roots of the entries in B and get the matrix

$$C = \begin{bmatrix} 0.4281 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 13.5886 \end{bmatrix}$$

Now we find the matrix product SCS^{-1} and get

$$\sqrt{A^*A} = S^*CS^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -0.3868 & 2.3196 & 2.8017 \\ 0.0339 & -3.0376 & 2.4687 \end{bmatrix} \begin{bmatrix} 0.4281 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 13.5886 \end{bmatrix} \begin{bmatrix} 0.8690 & -0.3361 & 0.0294 \\ 0.0641 & 0.1486 & -0.1946 \\ 0.0669 & 0.1875 & 0.1652 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.5897 & 3.1191 & 1.3206 \\ 3.1191 & 8.8526 & 4.1114 \\ 1.3206 & 4.1114 & 8.3876 \end{bmatrix}$$

The entries of these matrices are not exact, but that is acceptable for the purposes of an example. What is neat about this matrix is that it is nonsingular, which means we can easily compute U by taking the matrix product AP^{-1}. Doing this operation gives us

$$U = \begin{bmatrix} 0.3019 & 0.9175 & -0.2588 \\ 0.6774 & -0.0154 & 0.7355 \\ -0.6708 & 0.3974 & 0.6262 \end{bmatrix}$$

If these calculations are done in Sage we find that the matrix product UP does in fact result in the matrix A.

3 SVD and Polar Decomposition

We now discuss how the polar decomposition is related to the singular value decomposition. The singular value decomposition is everybody's favorite because it allows us to see so many properties of whatever matrix we are decomposing. What is even better is that it works on any matrix just like the polar decomposition. Here is another proof of the polar decomposition, only this time we take the SVD approach.

Theorem 3.1 (Polar Decomposition from SVD)**.** *For any matrix $A \in \mathbb{C}^{m\times n}$ there is a matrix $U \in \mathbb{C}^{m\times n}$ with orthonormal columns and the $n \times n$ matrix P from Definition 3, so that $A = UP$.*

Proof. Take any $m \times n$ matrix A where $m \geq n$. A is guaranteed by Theorem 2.13 in Section 2 of *A Second Course in Linear Algebra* to have a singular value decomposition, $U_S S V^*$. I have made the decision to subscript the U traditionally used in the singular value decomposition to differentiate it from the U of our polar decomposition. We know that the matrix V is unitary. This means that we can perform the following operation.

$$\begin{aligned} A &= U_S S V^* \\ &= U_S I_n S V^* \\ &= U_S V^* V S V^* \end{aligned}$$

Because both U_S and V are matrices with orthonormal columns there product will also have orthonormal columns. Using the definition of U_S and V provided in *A Second Course in Linear Algebra* we see that our definition of U provided in Theorem 2.6 is the product of U_S and V. The matrix S is a matrix containing only real positive values along the diagonal which means that when we multiply it by V and V^* the resulting matrix will be an $n \times n$ positive semi-definite just like P which is unique. This means that

$$\begin{aligned} A &= U_S V^* V S V^* \\ &= UP \end{aligned}$$

□

Example 2

We will use the same matrix for this example that we used for Example 1. This will make it easier to see the parallels between the decompositions. So we have the same matrix A and we want to find its singular value decomposition. We could go through the long process of finding orthonormal sets of eigenvectors, but there is limited space so I will just provide the U_S, S, and V matrices.

$$U_S = \begin{bmatrix} 0.5778 & 0.8142 & 0.0575 \\ 0.6337 & 0.4031 & 0.6602 \\ 0.5144 & 0.4179 & 0.7489 \end{bmatrix}$$

$$S = \begin{bmatrix} 13.5886 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 0.4281 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.2587 & 0.2531 & 0.9322 \\ 0.7248 & 0.5871 & 0.3605 \\ 0.6386 & 0.7689 & 0.0316 \end{bmatrix}$$

This will be a much easier computation than the one before because we do not have to find basis eigenvectors like in the raw polar decomposition. With these three matrices we can build our P and our U just as we had before. We relied on the fact that our matrix was square and non-singular to more easily compute U. Here we do not need to rely on such pretenses and can even compute U first. Appealing to Theorem 3.1 we can say

$$\begin{aligned} U &= U_S V^* \\ &= \begin{bmatrix} 0.5778 & 0.8142 & 0.0575 \\ 0.6337 & 0.4031 & 0.6602 \\ 0.5144 & 0.4179 & 0.7489 \end{bmatrix} \begin{bmatrix} -0.2587 & -0.7248 & -0.6386 \\ 0.2531 & 0.5871 & -0.7689 \\ -0.9322 & 0.3605 & -0.0316 \end{bmatrix} \\ &= \begin{bmatrix} 0.3019 & 0.9175 & -0.2588 \\ 0.6774 & -0.0154 & 0.7355 \\ -0.6708 & 0.3974 & 0.6262 \end{bmatrix} \end{aligned}$$

Voilà. We get the exact same U as before with much less headache. The computation of P is just as straight forward.

$$\begin{aligned} P &= VSV^* \\ &= \begin{bmatrix} 0.2587 & 0.2531 & 0.9322 \\ 0.7248 & 0.5871 & 0.3605 \\ 0.6386 & 0.7689 & 0.0316 \end{bmatrix} \begin{bmatrix} 13.5886 & 0 & 0 \\ 0 & 4.8132 & 0 \\ 0 & 0 & 0.4281 \end{bmatrix} \begin{bmatrix} -0.2587 & -0.7248 & -0.6386 \\ 0.2531 & 0.5871 & -0.7689 \\ -0.9322 & 0.3605 & -0.0316 \end{bmatrix} \\ &= \begin{bmatrix} 1.5897 & 3.1191 & 1.3206 \\ 3.1191 & 8.8526 & 4.1114 \\ 1.3206 & 4.1114 & 8.3876 \end{bmatrix} \end{aligned}$$

Once again the theory succeeds in practice. We should check to make sure that the product of these matrices, U and P, does once again gives us A.

$$
\begin{aligned}
A &= UP \\
&= \begin{bmatrix} 0.3019 & 0.9175 & -0.2588 \\ 0.6774 & -0.0154 & 0.7355 \\ -0.6708 & 0.3974 & 0.6262 \end{bmatrix} \begin{bmatrix} 1.5897 & 3.1191 & 1.3206 \\ 3.1191 & 8.8526 & 4.1114 \\ 1.3206 & 4.1114 & 8.3876 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 8 & 2 \\ 2 & 5 & 7 \\ 1 & 4 & 6 \end{bmatrix}
\end{aligned}
$$

We have our original matrix just as planned.

4 Geometric Interpretation

There are some interesting ideas surrounding the geometric interpretation of the polar decomposition. The analogous scalar polar form takes coordinates from the 2D Cartesian coordinate system and turns them into polar coordinates via the formulas $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\frac{y}{x}$. It expresses this information in the equation $z = re^{i\theta}$. Before digging into the parallels we will examine a motivating example.

There is very intuitive example provided by Bob McGinty at continuummechanics.org. He takes a relatively simple set of linear equations

$$x = 1.300X - .375Y$$

$$y = .750X + .650Y$$

This gives us the matrix

$$A = \begin{bmatrix} 1.300 & -.375 \\ .750 & .650 \end{bmatrix}$$

A has the right polar decomposition UP where

$$U = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.50 & 0.0 \\ 0.0 & 0.75 \end{bmatrix}.$$

If we investigate the values of the matrix U we see that we can replace the entries with trigonometric functions. By either eyeballing them or using the inverse trigonometric functions on the values of U we find that

$$U = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}$$

This is a very simplified left handed example of what a matrix U does to P. The matrix P in the example is diagonal which means that our diagonalizing matrices were just I_2. When examining the rest of the rotation matrices we will make them right handed by changing the sign on the angle. This will only affect the sin function because it is odd.

To get a better grasp on the geometric interpretation of these two matrices we should include a discussion of rotation matrices. A rotation matrix R will rotate a vector a certain direction if you are thinking about the transformation geometrically. In two dimensions R looks like this

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

This shows how the matrix rotates vectors a certain angle in a plane. Usually this rotation would occur in the xy-plane of the cartesian coordinate system. If we were operating $\mathbb{R}^3$ then we could represent a rotation around an axis with a $3x3$ that leaves the axis of rotation unchanged. Adopting the notation of Gruber we would define a rotation about the x-axis by an angle θ as

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

In higher dimensions it can be convenient to think of a rotation matrix as being composed of several simpler rotation matrices.

For example suppose we have a $3x3$ rotation matrix R_3. We can decompose R_3 as three different rotations which turn about each axis. We already defined the the x-axis rotation so we can now define the y-axis rotation denoted by the angle ψ and z-axis rotation denoted by the angle κ.

$$R_\psi = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \quad R_\kappa = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can think of the matrix U as the matrix product of these three vectors and the V^* from the SVD. This gives us the matrix product

$$\begin{aligned} U &= R_\theta R_\psi R_\kappa V^* \\ &= \begin{bmatrix} \cos\psi\cos\kappa & -\cos\psi\sin\kappa & -\sin\psi \\ -\sin\theta\sin\psi\cos\kappa - \cos\theta\sin\kappa & \sin\theta\sin\psi\sin\kappa + \cos\theta\cos\kappa & \sin\theta\cos\psi \\ \cos\theta\sin\psi\cos\kappa + \sin\theta\sin\kappa & \cos\theta\sin\psi\sin\kappa - \sin\theta\cos\kappa & \cos\theta\cos\psi \end{bmatrix} V^* \end{aligned}$$

Something lost from the decomposition

Just like the matrix U is related to the $e^{i\theta}$ component of coordinate polar factorization, the matrix P is similar in spirit to the r piece. We define $r = \sqrt{x^2 + y^2}$ where x and y are the cartesian coordinates of some point in a plane. If we thnk of r as being a vector $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ the equation from before is a little more recognizble. It is actually the norm of the vector which is the inner product of the vector with itself $\|\mathbf{r}\| = \sqrt{\mathbf{r}^*\mathbf{r}}$. Moving forward with this idea we can now see the relationship between r for the complex numbers and P for matrices with complex values.

5 Applications

The polar decomposition is used mainly for two applications. The first, which will just be mentioned briefly, is to decompose stress tensors in continuum mechanics. This makes it popular among materials engineers. Their interest lies in the fact that it is a convenient way to find the deformation of an object or transformation.

The second use found for the polar decomposition is in computer graphics. This may seem strange considering the amount of computation involved, since one of the goals of the field of computer science is speed and efficiency of computation. While we might want to rely on the SVD to compute the polar decomposition, there are faster iterative methods for computing the orthogonal matrix U often called the orthogonal polar factor. The most common iterative method for computing the polar decomposition of a matrix A is the Newton method,

$$U_{k+1} = \frac{1}{2}(U_k + U_k^{-t}), \quad U_0 = A.$$

The algorithm assumes A is non-singular, which it almost always is. The idea behind the iteration is that the matrix A will converge quadratically to U by averaging it with its inverse transpose. We then use U to find P via $A = UP$. While this method is fast, it can be accelerated by introducing one of two factors. The first is the γ_F. This factor is calculated via the equation

$$\gamma_{F_k} = \frac{\|U_k^{-1}\|_F^{\frac{1}{2}}}{\|U_k\|_F^{\frac{1}{2}}}$$

Where $\|\dot{\|}_F$ denotes the Frobenius norm of that matrix, which is the square root of the trace of that matrix multiplied by its transpose. This method is actually not very costly and is widely considered the most economical. The second involves the spectral norm γ_{S_k}, which is the largest singular value of A. The method for getting this factor is

$$\gamma_{S_k} = \frac{\|U_k^{-1}\|_S^{\frac{1}{2}}}{\|U_k\|_S^{\frac{1}{2}}}$$

Where $\|\dot{\|}_S$ denotes the spectral norm. This is also a costly computation although it does reduce the number of iterations the most according to Byers and Xu. However it would require that we calculate the SVD of A which was something we wanted to avoid in the first place by implementing the iteration.

6 Conclusion

We have seen that the polar decomposition is possible for any matrix and we can compute it with relative ease. The decomposition has a close tie to the singular value decomposition, related by their nature as rank revealing matrices. We have also discussed the decomposition's ties to the polar coordinate mapping function $z = re^{i\theta}$ and discussed a couple of the decomposition's applications. There are many other wonderful things about the decomposition beyond the scope of this paper that I encourage the reader to investigate them.

7 References

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