The Great Beauty of VEMs

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Abstract. In this paper I review the main features of the (newborn) Virtual Element Method, and of its application to the approximation of boundary value problems for Partial Differential Equations of particular relevance for applications. I will mostly concentrate on the definition of the Virtual Element spaces, that, roughly, consist of (vector valued) functions that are solution of (systems of) partial differential equations in each subdomain of a decomposition of the computational domain into polygons or polyhedra of quite general shape. Then I will give some hint on the use of these spaces for the discretization of some classical toy-problems like Heat conduction, Darcy flows, and Magnetostatic problems.

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1. Introduction

The aim of this paper is to give some hints on a (brand new) technique, recently introduced in Scientific Computing, with the name of *Virtual Element Methods*. It is one of the many possible applications of the so-called Galerkin Method to approximate the solution of boundary value problems for Partial Differential Equations in variational form.

To give an idea of the Galerkin method in one of the simplest possible examples, assume that one wants to compute the approximate solution of the PDE $-\Delta u = f$ in a given (say, polygonal, for hyper-simplicity) domain Ω, with the boundary conditions $u = 0$ on $\partial\Omega$. The *variational form* of this problem consists in looking for a function $u \in V$ such that

$$\int_\Omega \mathbf{grad} u \cdot \mathbf{grad} v \mathrm{d}x = \int_\Omega f\, v \mathrm{d}x \qquad \forall\, v \in V \tag{1.1}$$

where the space V is chosen as $H^1_0(\Omega)$, that is, the space of square integrable (classes of Lebesgue measurable) functions with square integrable derivatives (in Ω) that vanish on $\partial\Omega$.

The Galerkin method consists in choosing a finite dimensional subspace $V_h \subset V$ and looking for $u_h \in V_h$ such that

$$\int_\Omega \mathbf{grad} u_h \cdot \mathbf{grad} v_h \mathrm{d}x = \int_\Omega f\, v_h \mathrm{d}x \qquad \forall\, v_h \in V_h. \tag{1.2}$$

It is then (in this toy-case) an easy exercise to show that such a u_h exists and is unique in V_h, together with the estimate

$$\int_\Omega |\mathbf{grad}(u - u_h)|^2 \mathrm{d}x \le \inf_{v_h \in V_h} \int_\Omega |\mathbf{grad}(u - v_h)|^2 \mathrm{d}x \tag{1.3}$$

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that connects the *error* $\|u - u_h\|$ with the best approximation that could be given of the solution u within the subspace V_h.

More generally, the *mathematical analysis* of this type of procedures assumes that we are given a *sequence* of subspaces $\{V_h\}_h$, indexed by the parameter h (positive, and tending to zero). The target is to prove, under suitable assumptions on the sequence of decompositions, that the sequence of solutions $\{u_h\}_h$ converges to the exact solution u when h tends to 0. As far as possible, one also tries to connect the *speed* of such a convergence in terms of suitable properties of the sequence $\{V_h\}_h$. See e.g. [39].

Many choices are available for the construction of such subspaces. One of the most common and most successful ones is that of Finite Elements: one decomposes the domain Ω in small pieces and takes V_h as the space of functions that are piece-wise polynomials. The most classical case is that of decompositions in *triangles* (see two examples in Figure 1.1), in which one takes functions that are polynomials of degree ≤ 1 in each triangle. It is easy to see that each function of V_h, in this case, is characterized by its values at the vertices of the triangles, that will therefore become *the unknowns* of our approximate problems.

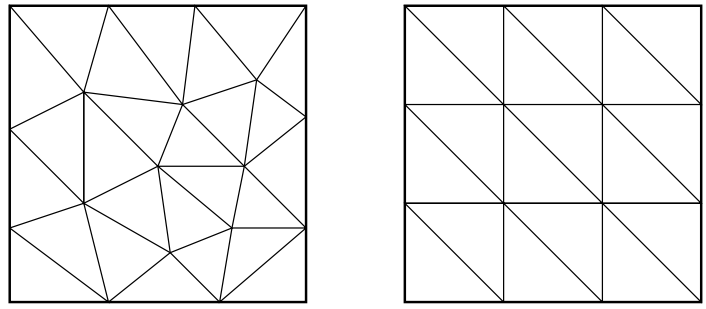

Figure 1.1. Triangulations of a rectangle: non-uniform or uniform

The most obvious generalization is obtained by taking, instead, polynomials of degree ≤ 2 in each triangle (and the unknowns will then be the values at the vertices and the values at the midpoint of each edge). And so on, using piecewise polynomials of degree $\leq k$ with $k = 1, 2, 3, ...$ etc.

For the mathematical analysis one will then consider a sequence of decompositions $\{\mathcal{T}_h\}_h$, and, for a fixed k, connect the speed of convergence of u_h to u in terms of properties of the sequence. Typically, the parameter h will be connected to the biggest among the diameters of all the elements of the decomposition $\mathcal{T}_h$. Clearly, to let $h \to 0$ will mean to consider finer and finer decompositions, and to measure *the speed of convergence* we look for estimates of the error $\|u - u_h\|$ in terms of the *powers of* h (and of the degree k). See again [39].

In three dimensions one uses, for instance, tetrahedra instead of triangles, and life is a bit more complicated. But already in two dimensions, as soon as we abandon the use of *triangles*, life becomes decidedly more complicated. Quadrilaterals (when we do not restrict ourselves to parallelograms) can already be a source of some practical (meaning: when we have to write the computer code!) headaches, and hexahedra are much worse. See for instance [6, 8, 22, 36, 44, 60, 63, 66], and the references therein.

Luckily, in the majority of applications the use of triangles/tetrahedra and or quadrilaterals/hexahedra is sufficient to give very effective practical methods.

There are however several types of problems where the use of much more general polygonal or polyhedral elements becomes highly desirable. The most relevant, so far, are the analysis of fractured materials and crack propagation (see, e.g. [1, 26, 40, 73, 74, 77, 78, 84], and the references therein), topology optimization (see, e.g. [3, 24, 25, 50, 56, 79, 87, 89, 98], and the references therein), computer graphics (see, e.g. [43, 45–47, 57, 61, 69, 72, 97]) and several other applications including fluid-structure interaction or two phase flows (see for instance [37, 38, 52, 64, 71], and the references therein). But their use for structured materials (see, e.g. [76, 79, 80, 83]) is also a promising direction, as well as for many other applications (see, e.g. [48, 76, 88, 90, 95] and the references therein).

The literature on these types of decompositions is quite wide, both from the Mathematical and the Engineering point of view. Here I just quote, in addition to the ones already mentioned: [5, 9–12, 23, 27, 29, 42, 53–55, 59, 62, 70, 75, 81, 82, 86, 91–94, 96], and the references therein.

In the last decade the use of Mimetic Finite Differences (a sort of finite differences, allowing very general decompositions, but not within the framework of Galerkin methods) underwent an impressive growth. I just mention, among the more recent papers, [4, 13, 17–20, 28, 30, 32–34, 41, 65].

The Virtual Element Methods (VEMs, in the title of the present paper) could be seen as an evolution of Mimetic Finite Differences, keeping their tremendous generality for the type of usable decompositions, but falling back into the simpler and more elegant realm of Galerkin approximations. See [2, 14–16, 21, 35, 49, 51, 68].

Here I want to describe, mostly for non-experts, the very basic features of the method, concentrating on a few very simple cases, and just giving hints and references to the more sophisticated (and practically much more interesting) developments of the last two years.

Here and there, I will do a certain amount of hand-waving, trying to trade precision for clarity. I apologize for that in advance. However, in these cases, I will always warn the readers, and address those that are interested in precise details to some papers already published or at least available on my web page.

An outline of the paper is as follows. In the next section, I will introduce some of the most commonly used functional spaces in the approximation of PDE's. In doing so, I will take, as toy-examples, some super-simplified problems in variational formulation (namely: Darcy flows, both in the primal and in the mixed formulation, and the magnetostatic problem). In the subsequent section I will try to give an idea on the classical Finite Element spaces used in the practice of Scientific Computing. Then, in Section 4 I will present the basic ideas on the construction of Virtual Element Spaces. Their main properties will be presented in the subsequent section, and their use in the approximation of PDE's will be briefly illustrated in Section 6. Some conclusions will be drawn in the final section, and a quite ample set of references will be in charge of (partly) heal the lack of details of the whole paper.

2. Typical model problems and functional spaces

In this section I will recall a few model problems of interest in applications, together with their variational formulations. To start with, I recall some of the most used functional spaces.

2.1. The spaces most used in variational formulations. Let Ω be a Lipschitz continuous polyhedral domain. The following spaces are the common **bricks** used to deal with **PDEs**.

$$
\begin{array}{lcl}
L^2(\Omega) \text{ and } (L^2(\Omega))^3 & := & \text{square integrable (vector valued) functions on } \Omega. \\
H(\mathrm{div};\Omega) & := & \{\boldsymbol{\tau} \in (L^2(\Omega))^3 \text{ s.t. } \mathrm{div}\,\boldsymbol{\tau} \in L^2(\Omega)\} \\
H(\mathbf{curl};\Omega) & := & \{\boldsymbol{\varphi} \in (L^2(\Omega))^3 \text{ s.t. } \mathbf{curl}\,\boldsymbol{\varphi} \in (L^2(\Omega))^3\} \\
H(\mathbf{grad};\Omega) & := & \{v \in L^2(\Omega) \text{ s.t. } \mathbf{grad}\, v \in (L^2(\Omega))^3\} \equiv H^1(\Omega)
\end{array}
$$

2.2. Primal formulation of Darcy problem. We consider now the classical model problem of *Darcy flow* (fluid flow through a porous medium). We denote by p the pressure, by $\mathbf{u}$ the velocities (actually, the *volumetric flow per unit area*), by f the source and by $\mathbb{K}$ a material-depending tensor (representing the ratio between the permeability tensor and the viscosity coefficient). For the sake of simplicity, we also take the (totally) unrealistic choices: $\mathbb{K} = \mathbb{I}$ (= identity) and $p = 0$ at the boundary $\partial\Omega$. Taking also into account the physical laws: $\mathbf{u} = -\mathbb{K}\nabla p = -\nabla p$ (Constitutive Equation), and $\mathrm{div}\,\mathbf{u} = f$ (Conservation Equation) we end up with the model problem already considered in the introduction: *Find* $p \in H_0^1(\Omega)$ *such that* $-\Delta p = f$ *in* Ω. As we already saw in the introduction, we can consider the variational formulation: *find* $p \in H_0^1(\Omega)$ *such that:*

$$
\int_\Omega \nabla p \cdot \nabla q \,\mathrm{d}x = \int_\Omega f\, q \mathrm{d}x \qquad \forall q \in H_0^1(\Omega). \tag{2.1}
$$

2.3. Mixed formulation of Darcy problem. There is however another variational formulation of the same problem, that in many practical cases is even more convenient than (2.1), and goes under the name of *mixed formulation*. It amounts to keep *both* unknowns $\mathbf{u}$ and p, looking for $p \in L^2(\Omega)$ and $\mathbf{u} \in H(\mathrm{div};\Omega)$ such that

$$
\int_\Omega \mathbf{u} \cdot \mathbf{v}\,\mathrm{d}\Omega = \int_\Omega p\ \mathrm{div}\,\mathbf{v}\,\mathrm{d}\Omega \quad \forall\, \mathbf{v} \in H(\mathrm{div};\Omega) \tag{2.2}
$$

and

$$
\int_\Omega \mathrm{div}\,\mathbf{u}\, q\,\mathrm{d}\Omega = \int_\Omega f\, q\,\mathrm{d}\Omega \qquad \forall\, q \in L^2(\Omega), \tag{2.3}
$$

where we see the spaces $H(\mathrm{div};\Omega)$ and $L^2(\Omega)$ coming into the game (as spaces where we look for the solution, that therefore need to be discretized).

2.4. Magnetostatic equations. Another very simple model problem is given by the *magnetostatic equations*. Here, given a polyhedral domain Ω, and given $\mathbf{j}$ = (*divergence free*) current density vector and μ = magnetic permeability constant, we consider the unknowns $\mathbf{u}$ = vector potential with the gauge $\mathrm{div}\,\mathbf{u} = 0$, $\mathbf{H} = \mu^{-1}\mathbf{curl}\,\mathbf{u}$ = magnetic field, and $\mathbf{B}$ = magnetic induction, together with the physical laws: $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{curl}\mathbf{H} = \mathbf{j}$, and $\mathrm{div}\,\mathbf{B} = 0$ (that however has already been taken into account with the use of the vector potential $\mathbf{u}$, since $\mathrm{div}\,\mathbf{B} = \mathrm{div}\,\mu\mathbf{H} = \mathrm{div}\,\mathbf{curl}\,\mathbf{u} = 0$). We supplement these equations with the (moderately realistic) boundary conditions $\mathbf{u} \wedge \mathbf{n} = 0$ on $\partial\Omega$.

The classical magnetostatic equations can therefore be written now

$$
\mathbf{curl}\,\mu^{-1}\mathbf{curl}\,\mathbf{u} = \mathbf{j} \quad \text{ and } \quad \mathrm{div}\,\mathbf{u} = 0 \qquad \text{in } \Omega \tag{2.4}
$$

and we supplement them with the boundary conditions $\mathbf{u} \wedge \mathbf{n} = 0$ on $\partial\Omega$. In order to reach a variational formulation of the problem, we define first

$$
H_0(\mathbf{curl};\Omega) := \{\boldsymbol{\varphi} \in H(\mathbf{curl};\Omega) \text{ such that } \boldsymbol{\varphi} \wedge \mathbf{n} = 0 \text{ on } \partial\Omega\} \tag{2.5}
$$

and we introduce a Lagrange multiplier $p \in H_0^1(\Omega)$ to take into account the gauge $\mathrm{div}\,\mathbf{u} = 0$. Hence we can write the variational formulation as:

$$\begin{cases} \text{Find } \mathbf{u} \in H_0(\mathbf{curl}, \Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that :} \\ (\mu^{-1}\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) - (\nabla p, \mathbf{v}) = (\mathbf{j}, \mathbf{v}) & \forall\, \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \\ (\mathbf{u}, \nabla q) = 0 & \forall\, q \in H_0^1(\Omega), \end{cases} \tag{2.6}$$

showing an example of use for $H(\mathbf{curl}; \Omega)$ and $H_0^1(\Omega)$.

2.5. Continuity requirements for the basic spaces. Before entering the details of the VEM approximations for these spaces, I will make a final consideration on the continuity requirements for each of them. Assume that we have, say, a piecewise smooth vector valued function $\mathbf{v} : \ \Omega \to \mathbb{R}^3$. Then, if you want to ensure that it belongs, globally, to $(H^1(\Omega))^3$ you must require that *all the components* of $\mathbf{v}$ are continuous at the inter-element boundaries. If instead you want to ensure that $\mathbf{v}$ belongs, globally, to $H(\mathbf{curl}; \Omega)$, you must require that *its tangential components* are continuous at the inter-element boundaries, while for having $\mathbf{v} \in H(\mathrm{div}; \Omega)$ you must require the continuity, at the inter-element boundaries, of *its normal component*. Finally, as natural, no continuity is required to ensure $\mathbf{v} \in (L^3(\Omega))^3$.

The knowledge of these continuity requirements is crucial in building approximations: roughly speaking, the quantities that are required to be continuous must be single-valued al the inter-element boundaries, and in practice one needs to prescribe them as degrees of freedom in the approximations.

3. Classical F.E. approximations

3.1. Basic polynomial spaces. To give the flavor of typical Finite Element approximations, let us see to simplest possible choices of polynomial spaces on a tetrahedron:

$$\begin{aligned} &\mathbb{P}_0 := \{\text{constants}\}\ (1\, d.o.f.) \\ &RT_0 := \{\boldsymbol{\tau} = \mathbf{a} + c\mathbf{x}\} \text{ with } \mathbf{a} \in \mathbb{R}^3 \text{ and } c \in \mathbb{R}\ (4\, d.o.f.) \\ &N_0 := \{\varphi = \mathbf{a} + \mathbf{c} \wedge \mathbf{x}\} \text{ with } \mathbf{a} \in \mathbb{R}^3 \text{ and } \mathbf{c} \in \mathbb{R}^3\ (6\, d.o.f.) \\ &\mathbb{P}_1 := \{v = a + \mathbf{c} \cdot \mathbf{x}\} \text{ with } a \in \mathbb{R} \text{ and } \mathbf{c} \in \mathbb{R}^3\ (4\, d.o.f.) \end{aligned}$$

A function in $\mathbb{P}_1$ can obviously be individuated by its value at the four vertices of the tetrahedron, and a vector in $(\mathbb{P}_1)^3$ will be individuated by the three values of its three components at each vertex. A vector valued function in N_0 will be individuated by the (constant!) values of its tangential components along each of the six edges. Instead, a vector valued function in RT_0 will be individuated by the values of its normal components on each of the four faces. It is an easy exercise to check that the normal component of an element of RT_0, on any plane, is always constant. Finally, a function in $\mathbb{P}_0$ can obviously be individuated by its value, say, at the barycenter.

3.2. Lowest order finite element spasces. Let now $\mathcal{T}_h$ be a decomposition of Ω in tetrahedra. We consider the following finite element approximations.

$$L^2(\Omega) \sim \mathcal{L}_0^0 := \{q \in L^2(\Omega) \text{ such that } q_{|T} \in \mathbb{P}_0 \quad \forall T \in \mathcal{T}_h\},$$

$$H(\mathrm{div};\Omega) \sim \mathcal{R}T_0 := \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \text{ s.t. } \boldsymbol{\tau}_{|T} \in RT_0 \quad \forall T \in \mathcal{T}_h\},$$
$$H(\mathbf{curl};\Omega) \sim \mathcal{N}_0 := \{\boldsymbol{\varphi} \in H(\mathbf{curl};\Omega) \text{ s.t. } \boldsymbol{\varphi}_{|T} \in N_0 \quad \forall T \in \mathcal{T}_h\},$$
$$H(\mathbf{grad};\Omega) \sim \mathcal{L}_1^1 := \{v \in H(\mathbf{grad};\Omega) \text{ s.t. } v_{|T} \in \mathbb{P}_1 \quad \forall T \in \mathcal{T}_h\}.$$

It is easy to see, from the previous discussion, that: i) a function in $\mathcal{L}_0^0$ is individuated by its values at the barycenter of each tetrahedron of the decomposition, ii) a function in $\mathcal{R}T_0$ is individuated by the values of its normal component at each face of the decomposition, iii) a function in $\mathcal{N}_0$ is individuated by the values of its tangential component at each edge of the decomposition, and iv) a function in $\mathcal{L}_1^1$ is individuated by its values at each vertex of the decomposition.

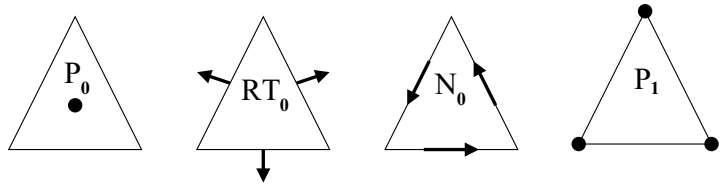

Figure 3.1. Degrees of freedom for the four polynomial spaces

All this is very elegant and, at the same time, very practical. This is not always the case. For instance, the most elegant available form for polynomial approximations (of degree k) of $H(\mathbf{curl})$ in a cube like $(-1,1)^3$ is given by

$$span\Big\{yz(w_2(x,z) - w_3(x,y)),\ zx(w_3(x,y) - w_1(y,z)),\ xy(w_1(y,z) - w_2(x,z))\Big\} + (\mathbb{P}_k)^3 + \mathbf{grad}\, s(x,y,z)$$

where each w_i $(i = 1,2,3)$ ranges over all polynomials (of 2 variables) of degree $\leq k$ and s ranges over all polynomials of *superlinear degree* $\leq k+1$, where the *superlinear degree* of a monomial is defined as "ordinary degree ignoring variables that appear linearly", [7].

Clearly *nobody ever tried* to do something similar on a dodecahedron....

4. Virtual element spaces

4.1. Polygonal and polyhedral elements. There is a wide literature on Polygonal and Polyhedral Elements, with applications to several important fields in Engineering and Computer Sciences. See for instance [5, 27, 45, 58, 61, 67, 85, 93, 94], and the references therein.

In general, these methods present the members of the discrete subspace as the solutions of suitable problems within each element. These problems are then solved in an approximate way, to obtain their values at the nodes of a suitable numerical integration formula (that, in turn, is used in order to compute the integrals that appear in the variational formulation).

The Virtual Element Methods follow this path insofar as to use solutions of (systems) of PDE equations. However, they *do not* attempt an approximate solution of these equations

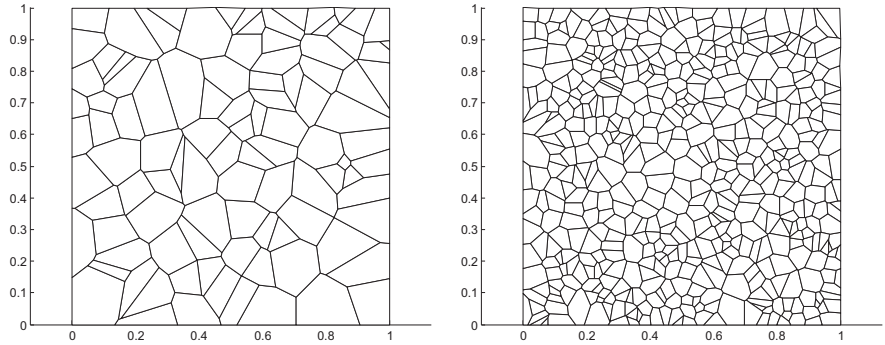

Figure 4.1. Voronoi tassellations: coarser and finer

(a most expensive step) and use instead suitable integrations by parts in order to compute the integrals (appearing in the variational formulations) *exactly*, at least when one of the two terms is a polynomial of a degree up to k, where k denotes the accuracy that has been chosen by the user (the higher is k, the most expensive is the computation). This ensures the full satisfaction of the so-called *patch-test of order* k, that roughly requires that: if the solution of the original problem is, globally, a polynomial of degree $\leq k$, then the solution of the discretized problem coincides with the exact solution. A property that is considered as very important in the Engineering literature, and that is lost when using numerical integration.

Let us see how this can be done, on some toy problem.

Assume that we are given a sequence of decompositions $\{\mathcal{T}_h\}_h$ of the computational domain Ω into polygons or polyhedra. To fix the ideas, we just assume that the decomposition satisfies the following assumption

- **H0** For the 2-dimensional case, we assume that: $\mathrm{H0}_2$ - there exists a fixed real number ρ such that each polygon E is starshaped with respect to all the points of a ball of diameter ρh_E and all its edges have a length $\geq h_E$ (where h_E is the diameter of E). In three dimensions, we assume that: $\mathrm{H0}_3$ - there exists a fixed real number ρ such that each polyhedron E is starshaped with respect to all the points of a ball of diameter ρh_E and all its faces satisfy the two dimensional assumption $\mathrm{H0}_2$ with constant ρ.

Note that **H0** easily implies, among other things, that there exists an integer number N, depending only on ρ, such that the number of edges of each element is bounded by N.

4.2. General features of VEM Spaces. As for other methods, the trial and test functions inside each element are rather complicated (e.g. solutions of suitable PDE's or systems of PDE's).

However, contrary to other methods,

i) they **do not** require the approximate evaluation of trial and test functions at the integration points.

ii) In most cases they satisfy the *patch test* **exactly** (up to the computer accuracy).

iii) We have a *whole family* of spaces (conforming and nonconforming approximations of all the main functional spaces)

4.3. Approximations of $H^1(\Omega) \equiv H(\mathbf{grad};\Omega)$. We consider first the two-dimensional case. For each element E that satisfies **H0**, and for each integer $k \geq 1$ we consider the *local spaces*:

$$\mathcal{B}_{k,2}^{nodal}(\partial E) := \{g \mid g \in C^0(\partial E) \text{ and } g_{|e} \in \mathbb{P}_k(e) \text{ for all edge } e \in \partial E\}, \tag{4.1}$$

and

$$V_{k,2}^{nodal}(E) := \{v \mid v_{|\partial E} \in \mathcal{B}_{k,2}^{nodal}(\partial E) \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}. \tag{4.2}$$

Then we define, in a very natural manner:

$$V_{k,2}^{nodal}(\Omega) := \left\{ v \in H^1(\Omega) \,\middle|\, v_{|E} \in V_{k,2}^{nodal}(E) \text{ for all } E \in \mathcal{T}_h \right\}. \tag{4.3}$$

We then consider the three dimensional case. For each element E that satisfies **H0**, and for each integer $k \geq 1$ we consider first the *local spaces*:

$$\mathcal{B}_{k,3}^{nodal}(\partial E) = \{g \mid g \in C^0(\partial E) \text{ and } g_{|f} \in V_{k,2}^{nodal}(f) \text{ for all face } f \in \partial E\} \tag{4.4}$$

and

$$V_{k,3}^{nodal}(E) := \{v \mid v_{|\partial E} \in \mathcal{B}_{k,3}^{nodal}(\partial E) \text{ and } \Delta v \in \mathbb{P}_{k-2}(E)\}, \tag{4.5}$$

and then we define:

$$V_{k,3}^{nodal}(\Omega) := \left\{ v \in H^1(\Omega) \,\middle|\, v_{|E} \in V_{k,3}^{nodal}(E) \text{ for all } E \in \mathcal{T}_h \right\}. \tag{4.6}$$

We can now consider the global degrees of freedom (say, in three dimensions):

- The values of v at the vertices of $\mathcal{T}_h$,
- $\int_e v\, q_{k-2} \mathrm{d}s$ for all edge $e \in \mathcal{T}_h,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(e)$,
- $\int_f v\, q_{k-2} \mathrm{d}f$ for all face $f \in \mathcal{T}_h,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(f)$,
- $\int_E v\, q_{k-2} \mathrm{d}E$ for all element $E \in \mathcal{T}_h,\ \forall q_{k-2} \in \mathbb{P}_{k-2}(E)$,

4.4. Approximations of $H(\mathbf{div};\Omega)$. In each element E, and for each integer k, we define

$$\mathcal{B}_{k,2}^{face}(\partial E) := \{g \mid g_{|e} \in \mathbb{P}_k \,\forall \text{ edge } e \in \partial E\} \text{ in 2d},$$

$$\mathcal{B}_{k,3}^{face}(\partial E) := \{g \mid g_{|f} \in \mathbb{P}_k \,\forall \text{ face } f \in \partial E\} \text{ in 3d}.$$

The local spaces, in two dimensions, will then be

$$V_{k,2}^{face}(E) := \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_{k,2}^{face}(\partial E),\ \mathrm{div}\, \boldsymbol{\tau} \in \mathbb{P}_{k-1},\ \mathrm{rot}\, \boldsymbol{\tau} \in \mathbb{P}_{k-1}\},$$

and in 3 dimensions

$$V_{k,3}^{face}(E) := \{\boldsymbol{\tau} \mid \boldsymbol{\tau} \cdot \mathbf{n} \in \mathcal{B}_{k,3}^{face}(\partial E),\ \mathrm{div}\, \boldsymbol{\tau} \in \mathbb{P}_{k-1},\ \mathbf{curl}\, \boldsymbol{\tau} \in (\mathbb{P}_{k-1})^3\}.$$

Finally, in all cases , the global spaces will be written as

$$V_{k,d}^{face}(\Omega) := \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \mid \boldsymbol{\tau} \in V_{k,d}(E) \text{ for all } E \in \mathcal{T}_h\}. \tag{4.7}$$

Before describing the degrees of freedom, we define, on a generic domain $\mathcal{O}$, the space $\mathcal{G}_k^{\perp}(\mathcal{O})$ as the subset of the $\mathbf{g} \in (\mathbb{P}_k(\mathcal{O}))^3$ such that

$$\int_{\mathcal{O}} \mathbf{g} \cdot \mathbf{grad}\, q_{k+1}\, \mathrm{d}\mathcal{O} = 0 \quad \forall\, q_{k+1} \in \mathbb{P}_{k+1}(\mathcal{O}).$$

Then we can choose the degrees of freedom in $V_{k,d}^{face}(\Omega)$ as

- $\displaystyle\int_e \boldsymbol{\tau}\cdot\mathbf{n}\, q_k \mathrm{d}e$ $\qquad\forall q_k \in \mathbb{P}_k(e)\ \ \forall$ edge e
- $\displaystyle\int_E \boldsymbol{\tau}\cdot\mathbf{grad}\, q_{k-1} \mathrm{d}E$ $\qquad\forall q_{k-1} \in \mathbb{P}_{k-1}(E)\ \ \forall$ element E
- $\displaystyle\int_E \boldsymbol{\tau}\cdot\mathbf{g}_k^{\perp} \mathrm{d}E$ $\qquad\forall \mathbf{g}_k^{\perp} \in \mathcal{G}_k^{\perp}(E)\ \ \forall$ element E

in two dimensions, and

- $\displaystyle\int_f \boldsymbol{\tau}\cdot\mathbf{n}\, q_k \mathrm{d}f$ $\qquad\forall q_k \in \mathbb{P}_k(f)\ \ \forall$ face f
- $\displaystyle\int_E \boldsymbol{\tau}\cdot\mathbf{grad}\, q_{k-1} \mathrm{d}E$ $\qquad\forall q_{k-1} \in \mathbb{P}_{k-1}(E)\ \ \forall$ element E
- $\displaystyle\int_E \boldsymbol{\tau}\cdot\mathbf{g}_k^{\perp} \mathrm{d}E$ $\qquad\forall \mathbf{g}_k^{\perp} \in \mathcal{G}_k^{\perp}(E)\ \ \forall$ element E

in three dimensions.

4.5. Approximations of $H(\mathbf{curl};\Omega)$. For the 2-dimensional case, we can think that $H(\mathbf{curl};\Omega)$ is obtained from $H(\mathrm{div};\Omega)$ by a simple rotation of $\pi/2$. With this, we can just think that also its discretization

$$V_{k,2}^{edge}(\Omega) \text{ is obtained by rotating } V_{k,2}^{face} \text{ of } \pi/2.$$

Namely, we can consider vector fields that on each edge have a tangential component in $\mathbb{P}_k(e)$, and whose divergence and rotation are in $\mathbb{P}_{k-1}(e)$ for each element E. The corresponding degrees of freedom can also be easily obtained by rotating the corresponding ones for $V_{k,2}^{face}(\Omega)$.

We can therefore turn to the (more complex) discretizations of $H(\mathbf{curl};\Omega)$ in three dimensions.

In each element E, and for each integer k, we therefore set

$$\mathcal{B}_{k,3}^{edge}(\partial E) := \{\boldsymbol{\varphi}|\ \boldsymbol{\varphi}_{|f} \in V_{k,2}^{edge}(f)\,\forall \text{ face } f \in \partial E \text{ and } \boldsymbol{\varphi}\cdot\mathbf{t}_e \text{ is single valued at each edge } e \in \partial E\}$$

where we denoted by $\mathbf{t}_e$ the unit tangent vector to an edge e. Now we can set

$$V_{k,3}^{edge}(E) = \{\boldsymbol{\varphi}|\ \boldsymbol{\varphi}_{|\mathbf{t}} \in \mathcal{B}_{k,3}^{edge}(\partial E),\ \mathrm{div}\boldsymbol{\varphi} \in \mathbb{P}_{k-1},\ \mathbf{curlcurl}\boldsymbol{\varphi} \in (\mathbb{P}_{k-2})^3\}$$

where $\boldsymbol{\varphi}_{|\mathbf{t}}$ is, on each face, *the tangential part* of $\boldsymbol{\varphi}$. We can therefore define the global space as:

$$V_{k,3}^{edge}(\Omega) := \{\boldsymbol{\varphi} \in H(\mathbf{curl};\Omega)|\ \boldsymbol{\varphi} \in V_{k,3}^{edge}(E) \text{ for all } E \in \mathcal{T}_h\}.$$

In $V_{k,3}^{edge}(\Omega)$ we can take the following degrees of freedom:

- for every edge e: $\displaystyle\int_e \boldsymbol{\varphi}\cdot\mathbf{t}_e\, q_k \mathrm{d}e$ $\qquad\forall q_k \in \mathbb{P}_k(e)$

- for every face f:

$$\int_f \boldsymbol{\varphi} \cdot \mathbf{rot}\, q_{k-1} \mathrm{d}f \qquad \forall q_{k-1} \in \mathbb{P}_{k-1}(f)$$

$$\int_f \boldsymbol{\varphi} \cdot \mathbf{r}^{\perp}_{k,2} \mathrm{d}f \qquad \forall \mathbf{r}^{\perp}_{k,2} \in \mathcal{R}^{\perp}_{k,2}(f)$$

where $\mathcal{R}^{\perp}_{k,2}$ is the subset of the $\mathbf{r} \in (\mathbb{P}_k(f))^3$ such that

$$\int_f \mathbf{r} \cdot \mathbf{rot}\, q_{k+1}\, \mathrm{d}f = 0 \qquad \forall q_{k+1} \in \mathbb{P}_{k+1}(f)$$

- and for every element E:

$$\int_E \boldsymbol{\varphi} \cdot \mathbf{rot}\, q_{k-1} \mathrm{d}E \qquad \forall q_{k-1} \in (\mathbb{P}_{k-1}(E))^3$$

$$\int_E \boldsymbol{\varphi} \cdot \mathbf{r}^{\perp}_{k,3} \mathrm{d}E \qquad \forall \mathbf{r}^{\perp}_{k,3} \in \mathcal{R}^{\perp}_{k,3}(E)$$

where $\mathcal{R}^{\perp}_{k,3}(E)$ is the subset of the $\mathbf{r} \in (\mathbb{P}_k(E))^3$ such that

$$\int_E \mathbf{r} \cdot \mathbf{curl}\, \mathbf{q}_{k+1} \mathrm{d}E = 0 \qquad \forall \mathbf{q}_{k+1} \in (\mathbb{P}_{k+1}(E))^3$$

4.6. Approximations of $L^2(\Omega)$. The approximation of spaces as $L^2(\Omega)$ or $(L^2(\Omega))^d$does not present any difficulties. As the space has no continuity requirements, we can just take piecewise polynomials discontinuous (vector valued) functions:

$$V^{volume}_{k,d}(\Omega) = \{q |\, q_{|E} \in \mathbb{P}_{k,d}(E) \text{ for all } E \in \mathcal{T}_h\}.$$

5. Useful properties

We observe that the classical differential operators $grad$, $curl$, and div send these VEM spaces one into the other (up to the obvious adjustments for the polynomial degree). Indeed:

$$\mathbf{grad}\,(V^{nodal}_{k,d}) \subseteq V^{edge}_{k-1,d}; \quad \mathbf{curl}\,(V^{edge}_{k,d}) \subseteq V^{face}_{k-1,d}; \quad \mathrm{div}\,(V^{face}_{k,d}) \subseteq V^{volume}_{k-1,d}. \quad (5.1)$$

But possibly the most crucial feature common to all these choices is the possibility to construct (starting from the degrees of freedom, and without solving approximate problems in the element) an *approximate L^2-type scalar product*

$$[\mathbf{u}, \mathbf{v}]_h = \sum_{E \in \mathcal{T}_h} [\mathbf{u}, \mathbf{v}]_{h,E}, \quad (5.2)$$

with the following properties:

P1 $\quad [\mathbf{p}_k, \mathbf{v}]_{h,E} = (\mathbf{p}_k, \mathbf{v})_{0,E} \qquad \forall\, \mathbf{p}_k \in (\mathbb{P}_k(E))^d,\ \forall\, \mathbf{v}$ in the VEM space

(where $(\mathbf{p}_k, \mathbf{v})_{0,E}$ represents the $L^2(E)$ inner product, or the $(L^2(E))^d$ inner product for vector valued functions), and

P2 $\quad \exists\, \alpha^* \geq \alpha_* > 0$ independent of h such that

$$\alpha_* \|\mathbf{v}\|^2_{0,E} \leq [\mathbf{v}, \mathbf{v}]_{h,E} \leq \alpha^* \|\mathbf{v}\|^2_{0,E}, \quad \forall\, \mathbf{v} \text{ in the VEM space},$$

where obviously $\|\mathbf{v}\|^2_{0,E} := (\mathbf{v}, \mathbf{v})_{0,E}$. In turn, properties $\mathbf{P1}$ and $\mathbf{P2}$ can be easily obtained, if we are able to compute the L^2-projections onto $\mathbb{P}_k$ of the elements of the VEM spaces. Indeed, assume that for every v in the VEM space and for every polynomial p_k you can compute (up to computer precision) an element $\Pi^0_k v$ in $\mathbb{P}_k$ such that

$$(v - \Pi^0_k v, p_k)_{0,E} = 0 \qquad \forall\, p_k \in \mathbb{P}_k\ \forall\, v \text{ in the VEM space}. \tag{5.3}$$

Then you can set

$$[u, v]_{h,E} := (\Pi^0_k u, \Pi^0_k v) + S(u - \Pi^0_k u, v - \Pi^0_k v) \tag{5.4}$$

where S is "any" symmetric bilinear form that, roughly speaking, scales like the true L^2 inner product (see [14], [35], or [16] for a precise definition, more details and examples).

Needless to say, these approximate L^2-type inner products depend on the type of Virtual Elements that we are dealing with. Hence, in what follows, we are going to use a different name for each of them. With obvious notation we will, therefore, have scalar products $[u, v]_{VEM,nodal}$ and $[u, v]_{VEM,volume}$ for scalar functions, together with $[\mathbf{u}, \mathbf{v}]_{VEM,edge}$ and $[\mathbf{u}, \mathbf{v}]_{VEM,face}$ for vector-valued functions.

6. VEM approximations of PDE's

Using the L^2-type projection operators, and, if needed, the properties (5.1) one can find an easy and systematic way to discretize PDE's by means of Virtual Element spaces. It should be pointed out, however, that on specific occasions alternative solutions could be more effective. Moreover, the discretization of the forcing terms requires some (minor) additional care that I do not discuss here. See for instance [14] or [31].

6.1. VEM's for primal Darcy. Remembering equation (2.1) we can now formulate the approximate problem as: *find* $p_h \in V^{nodal}_{k,2}$ *such that*:

$$[\mathbf{grad} p_h, \mathbf{grad} q_h]_{VEM,edge} = [f,\, q_h]_{VEM,nodal}$$

for all $q_h \in VEM^{nodal}_{k,2}$.

6.2. VEM's for mixed Darcy. The approximate version of the mixed formulation (2.2)–(2.3) can now be written as: *find* $p_h \in V^{volume}_{k-1,d}$ *and* $\mathbf{u}_h \in V^{face}_{k,d}$ *such that*:

$$[\mathbf{u}_h, \mathbf{v}_h]_{VEM,face} = [p_h, \operatorname{div} \mathbf{v}_h]_{VEM,volume}$$

for all $\mathbf{v}_h \in V^{face}_{k,d}$, and

$$[\operatorname{div} \mathbf{u}_h, q_h]_{VEM,volume} = [f,\, q_h]_{VEM,volume}$$

for all $q_h \in V^{volume}_{k-1,d}$.

6.3. VEM's for electromagnetic problems. The VEM approximation of the magnetostatic problem (2.6), in turn, can be chosen as: *find* $\mathbf{u}_h$ *in* $V_{k,3}^{edge}$ *and* p_h *in* $V_{k,3}^{nodal}$ *such that*:

$$[\mu^{-1}\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h]_{VEM,face} - [\nabla p_h, \mathbf{v}_h]_{VEM,edge} = [\mathbf{j}, \mathbf{v}_h]_{VEM,edge}\ \forall\,\mathbf{v}_h \in V_{k,3}^{edge}$$

and

$$[\mathbf{u}, \nabla q_h]_{VEM,edge} = 0 \quad \forall\, q_h \in V_{k,3}^{nodal}.$$

Remark 6.1. To tell the truth, in order to set up the proof, one has to *think* that the Virtual Element space has been *tilted*, or, as we say (cfr. [2]), *enhanced*. This does not correspond to a change in the code, but it simplifies the proofs that, without it, would become more cumbersome. I decided not to enter these aspects, and to refer the interested readers to [2] and [16].

It has to be pointed out that these methods are extremely robust with respect to the choice of the geometry of the decomposition. To give the flavor of their capability, I report the results made on a *totally crazy* sequence of meshes going from 4×4 to 16×16 *winged horses*, clearly inspired by Escher. The results have been obtained with the primal and mixed formulation of Darcy problem, having $p = \sin(2x)\cos(3y)$ as exact solution (courtesy of Alessandro Russo and Donatella Marini).

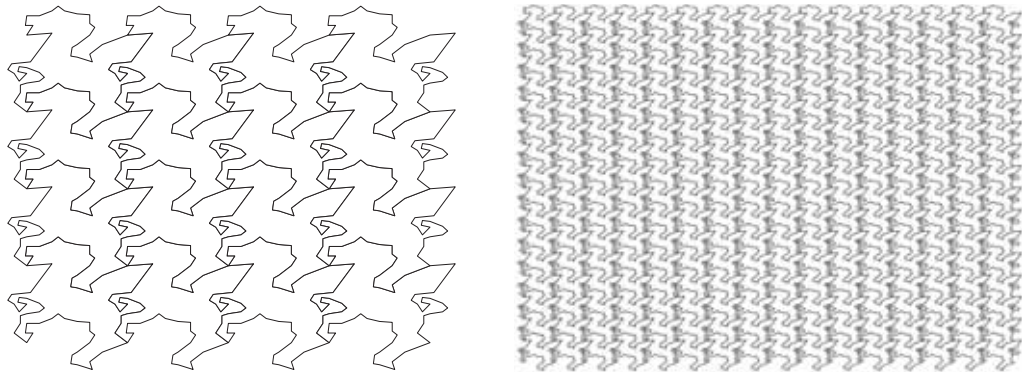

Figure 6.1. Winged horses: 4×4 and 16×16

7. Conclusions

Virtual Elements is a new method, and a lot of work is needed to assess its *pros* and *cons*. Its major interest is on polygonal and polyhedral elements, but its use on distorted quads, hexahedra, and the like, is also quite promising. For triangles and tetrahedra the interest seems to be concentrated in higher order continuity (e.g. [35]). The use of VEM mixed methods seems to be quite interesting, in particular for their connections with Finite Volumes and Mimetic Finite Differences.

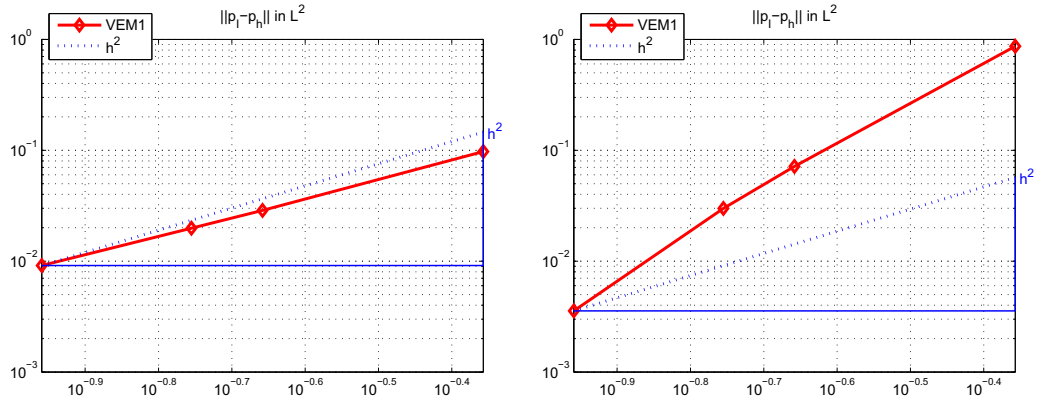

Figure 6.2. L^2 error for primal (left) and mixed (right) formulations

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