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# A NON-PROBABILISTIC PROOF OF THE RELATIVE FATOU THEOREM

by J. L. DOOB

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## 1. Introduction.

Let  $R$  be a Green space, as defined by Brelot and Choquet, with Martin boundary  $R'$ . Naïm [4] has extended the Cartan fine topology on  $R$  to  $R \cup R'$ . Limits involving this topology will be called « fine limits ».

Let  $h$  be a strictly positive superharmonic function. Then  $h$  has a canonical integral representation [1], going back to Martin if  $R$  is an open subset of a Euclidean space, involving a uniquely determined measure  $\mu^h$  on  $R \cup R'$ . It has been shown [3] using probabilistic methods that, if  $u$  is a positive superharmonic function on  $R$ , then  $u/h$  has a finite fine limit at  $\mu^h$ -almost every point of  $R \cup R'$ . The purpose of this note is to prove this theorem non-probabilistically. Note that, if  $h$  is harmonic, the theorem is a boundary limit theorem, because then  $\mu^h$  is a measure of subsets of  $R'$ . In particular, if  $h$  is a constant function, the theorem states that  $u$  has a finite fine limit at  $\mu^h$ -almost every point of  $R'$ . This is the justification for calling the theorem the relative Fatou theorem.

## 2. Fine limits.

The fine topology, a refinement of the Martin topology, is defined in terms of the concept of « thinness » (« effilement »). A set is a fine neighborhood of a point if it contains the point and if its complement is thin at the point. Let  $\eta$  be a point of  $R \cup R'$ , and let  $g$  be a function defined on a set  $A$  which is

not thin at  $\eta$ , that is, for which  $\eta$  is a fine limit point. Then the fine superior limit  $b$  of  $g$  at  $\eta$  is defined as the infimum of the numbers  $c$  such that the inequality  $g(\xi) > c$  defines a set thin at  $\eta$ . We write

$$(2.1) \quad F \limsup_{\xi \rightarrow \eta} g(\xi) = b.$$

The fine inferior limit is defined and denoted correspondingly, and is also equal to the infimum of the numbers  $c$  such that the inequality  $g(\xi) \leq c$  defines a set which is not thin at  $\eta$ . The function  $g$  is said to have the fine limit  $b$  at  $\eta$  if  $b$  is both its fine superior and inferior limit. Notions involving limits on  $R$  will never involve the fine topology unless « fine » appears explicitly.

If  $g$  has the fine limit  $b$  at  $\eta$  along  $A \subset R$ , Naim [4] has shown that there is a subset  $B$  of  $A$ , thin at  $\eta$ , and such that  $g$  has the limit  $b$  at  $\eta$  along  $A - B$ . More generally, an adaptation of this proof shows that if  $g$  has the fine superior and inferior limits  $b_1$  and  $b_2$  respectively at  $\eta$  along  $A \subset R$ , then there is a subset  $B$  of  $A$ , thin at  $\eta$ , such that  $g$  has the superior and inferior limits  $b_1$  and  $b_2$  respectively at  $\eta$  along  $A - B$ . The following related theorem goes slightly deeper.

**THEOREM 2.1.** — *If  $g$  has fine superior limit  $b$  at  $\eta$  along  $A \subset R$ , there is a subset  $A_0$  of  $A$ , not thin at  $\eta$ , such that  $g$  has limit  $b$  at  $\eta$  along  $A_0$ .*

The corresponding theorem is of course valid for fine inferior limits. To prove the theorem, we can suppose that  $b$  is finite. Let  $\xi_0$  be a point of  $R$ , not  $\eta$  if  $\eta$  is in  $R$ . Let  $h$  be the minimal harmonic function corresponding to  $\eta$  if  $\eta$  is a point of  $R'$ . (Naïm showed that the set of minimal boundary points is the set of fine limit points of  $R$  on the boundary). If  $\eta$  is a point of  $R$ , let  $h$  be the Green function with pole  $\eta$ . Let  $A_{ijk}$  be the subset of  $A$  at distance  $< 1/j$  but  $\geq 1/k$  from  $\eta$ , and satisfying the inequality.

$$(2.2) \quad b - 1/i \leq g(\xi) \leq b + 1/i.$$

Then  $\bigcup_k A_{ijk}$  is not thin at  $\eta$ , so that the smoothed lower envelope  $h_{ijk}$  of the positive superharmonic functions majorizing  $h$  on a neighborhood of  $A_{ijk}$  is arbitrarily near  $h(\xi_0)$  if

$k$  is sufficiently large. Choose  $j_1 = 1$ . If  $j_1, \dots, j_n$  have been chosen, choose  $j_{n+1}$  so large that  $h_{n/j_{n+1}}(\xi_0) \geq h(\xi_0)/2$ . Define  $A_0 = \bigcup_n A_{n/j_{n+1}}$ . Then  $A_0$  satisfies the conditions of the theorem.

Naïm proved a theorem [4, Theorem 23], which can be restated in the following more perspicuous form, in view of Theorem 2.1.

**THEOREM 2.2.** — *Let  $\xi$  be a point of  $R$ ,  $\varepsilon$  a strictly positive number,  $h$  a strictly positive harmonic function on  $R$ , and let  $f$  be an extended real valued function on  $R'$ . Let  $u$  be a function on  $R$  with the following properties.*

(a)  *$u$  is subharmonic, and  $u/h$  is bounded from above.*

(b)  *$u/h$  has fine inferior limit  $\leq f(\eta)$  at each minimal boundary point  $\eta$ . Then there is a function  $u_{\xi\varepsilon}$  on  $R$ , satisfying (a), with*

$$(2.3) \quad u_{\xi\varepsilon}(\xi) \geq u(\xi) - \varepsilon,$$

*and having superior limit  $\leq f(\eta)$  at each minimal boundary point  $\eta$ .*

### 3. The first boundary value problem.

If  $h$  is strictly positive and superharmonic on  $R$ , a function  $u/h$  with  $u$  superharmonic, subharmonic or harmonic will be called  $h$ -superharmonic,  $h$ -subharmonic or  $h$ -harmonic respectively. The remarks in this section presuppose that  $h$  is harmonic but can be extended to the general case. Suppose then that  $h$  is harmonic and strictly positive. The first boundary value problem for  $h$ -harmonic functions on  $R$  can be solved using the standard Perron-Wiener-Brelot method. A few details of this method will be needed. If  $f$  is the specified boundary function on  $R'$ , consider the following classes C1, C2, C3 of functions  $\varphi/h$ . In each class  $\varphi$  is subharmonic or identically  $-\infty$ , and  $\varphi/h$  is bounded from above. The following further condition is to be satisfied in the indicated class.

C1  $\varphi/h$  has limit superior  $\leq f(\eta)$  at each point  $\eta$  of  $R'$ .

C2 The preceding condition need hold only at the minimal boundary points.

C3  $\varphi/h$  has fine limit inferior  $\leq f(\eta)$  at each minimal point  $\eta$  of  $R'$ .

Obviously  $C1 \subset C2 \subset C3$ . The lower  $h$ -solution is defined as the upper envelope of the class  $C1$ . It is then shown that this upper envelope is the same as that of  $C2$ , and Theorem 2.2 shows that this upper envelope is the same as that of  $C3$ . The upper  $h$ -solution is defined dually, and  $f$  is called  $h$ -resolutive if these two solutions are identical and  $h$ -harmonic. The  $h$ -harmonic function thereby obtained is the  $h$ -solution corresponding to  $f$ .

Brelot [1] has shown that all continuous boundary functions are  $h$ -resolutive, and has thereby defined  $h$ -harmonic measure of boundary sets, generalizing ordinary harmonic measure. The class of boundary sets of  $h$ -harmonic measure 0 is independent of the reference point and is the same as the class of boundary sets of  $\mu^h$ -measure 0. (We observe, to avoid misunderstanding, that although Brelot calls the « solution » a certain harmonic function  $u$ , we call the « solution » the  $h$ -harmonic function  $u/h$ , to conform to the spirit of the general first boundary value problem).

#### 4. The relative Fatou theorem.

The following theorem was proved by probabilistic methods in [3].

**THEOREM 4.1.** — *If  $h$  is strictly positive and harmonic, and if  $f$  is an  $h$ -resolutive boundary function, corresponding to the solution  $u/h$ ,  $u/h$  has fine limit  $f$   $\mu^h$ -almost everywhere on  $R'$ .*

We prove this theorem using Theorem 2.2. If  $u/h$  has fine limit inferior  $< f(\eta)$  at the minimal boundary point  $\eta$ , define  $f'(\eta)$  as this fine limit inferior; at other boundary points set  $f' = f$ . If  $v/h$  is in the lower class  $C3$  for  $f$ ,  $v \leq u$ , so  $v/h$  is in this same class for  $f'$ . Hence, if  $u'/h$  is the lower  $h$ -solution for  $f'$ ,  $u' \geq u$ . These two functions are identical, since  $f' \leq f$ . The upper  $h$ -solution for  $f'$  is majorized by  $u/h$ , that for  $f$ , so that both upper and lower  $h$ -solutions for  $f'$  are  $u/h$ . That is,  $f$  and  $f'$  are both  $h$ -resolutive, with the same solution. According to Brelot [1] this fact implies that  $f = f'$   $\mu^h$ -almost everywhere on  $R'$ , that is,

$$(4.1) \quad \text{F} \liminf_{\xi \rightarrow \eta} \frac{u(\xi)}{h(\xi)} \geq f(\eta)$$

$\mu^h$ -almost everywhere on  $R'$ . Applying this result to  $-f$  and combining the two we obtain Theorem 4.1.

The following theorem is due to Naïm [4].

**THEOREM 4.2.** — *If  $h$  is strictly positive and harmonic, and if  $u$  is superharmonic and the potential of a measure on  $R$ , then  $u/h$  has the fine limit 0  $\mu^h$ -almost everywhere on  $R'$ .*

**THEOREM 4.3.** — *If  $h$  is strictly positive and harmonic, and if  $u$  is positive and superharmonic,  $u/h$  has a finite fine limit  $\mu^h$ -almost everywhere on  $R'$ .*

In view of Theorem 4.2 and the Riesz decomposition, it is sufficient to consider only the case when  $u$  is harmonic. Now if  $u/h$  is positive and  $h$ -harmonic, it is [4] the sum of an  $h$ -solution  $u_1/h$  and of a function  $u_2/h$  with the property that  $u_2$  is positive and harmonic, with  $\min[u_2, h]$  the potential of a measure on  $R$ . Then  $u/h$  has the fine limit  $f$   $\mu^h$ -almost everywhere on  $R'$ , where  $f$  is a boundary function with  $h$ -solution  $u_1/h$ .

The following theorem includes the three preceding ones, aside from the identification of the limit. It was first proved in [3] by probabilistic methods, without using the decompositions necessary in the present treatment.

**THEOREM 4.4.** — *Let  $u$  and  $h$  be strictly positive and superharmonic. Then  $u/h$  has a finite fine limit at  $\mu^h$ -almost every point of  $R \cup R'$ .*

This theorem will be proved by applying Theorem 4.3. The functions  $u$  and  $h$  are continuous on  $R$  in the fine topology (with the obvious conventions at infinities). In fact one definition of the fine topology on  $R$  is that it is the least fine topology making all superharmonic functions continuous. Thus the function  $u/h$  has a finite fine limit at each point of  $R$  except possibly at an infinity of  $u$ . The set of these infinities has zero capacity, but may have strictly positive  $\mu^h$ -measure. Let  $h = h_1 + h_2$ , where  $h_1$  is the potential determined by the restriction of  $\mu^h$  to  $R$  and  $h_2$  is the harmonic function determined by the restriction of  $\mu^h$  to  $R'$ . We have already proved that  $u/h_2$  has a finite fine limit  $\mu^h$ -almost everywhere on  $R'$ , and that  $h_1/h_2$  has the fine limit 0  $\mu^h$ -almost everywhere on  $R'$ . Then  $u/h$  has the same fine limit as  $u/h_2$   $\mu^h$ -almost everywhere on  $R'$ . Let  $A$  be a compact subset of the set of infinities of  $u$ .

Then  $A$  has zero capacity. Suppose that  $\mu^h(A) > 0$ . (If there is no such set  $A$ , there is nothing further to prove). The space  $R_0 = R - A$  is (with the obvious conventions) a Green space, and the definition of the Martin boundary yields at once that (with the obvious identifications), the Martin boundary of  $R_0$  is  $R' \cup A$ . Moreover fine limits relative to  $R$  are also fine limits relative to  $R_0$ , and conversely. Each point  $\eta$  of  $A$  is a minimal boundary point of  $R_0$ , with corresponding minimal function the Green function on  $R$  with pole  $\eta$ , restricted to  $R_0$ . Let  $h'$  be the potential determined by the restriction of  $\mu^h$  to  $A$ . Then  $h'$  is harmonic on  $R_0$ , so  $u/h'$  has a finite fine limit relative to  $R_0$  and so also relative to  $R$ ,  $\mu^h$ -almost everywhere on  $A$ , according to Theorem 4.3. Similarly,  $h/h'$  has a finite fine limit, obviously  $\geq 1$ ,  $\mu^h$ -almost everywhere on  $A$ . Then  $u/h$  has a finite fine limit  $\mu^h$ -almost everywhere on  $A$ , and it follows that the same is true  $\mu^h$ -almost everywhere on the set of infinities of  $u$ , as was to be proved.

The assertion of this theorem about limits on  $R$  can be generalized as follows. *If  $h$  is strictly positive and superharmonic and if  $u$  is superharmonic,  $u/h$  has a finite fine limit at  $\mu^h$ -almost every point of  $R$ .* It is sufficient to prove that  $u/h$  has a finite fine limit at  $\mu^h$ -almost every point of every open subset  $R_0$  of  $R$  whose closure is a compact subset of  $R$ . In  $R_0$   $u$  is bounded from below by some constant  $c$ . Hence  $(u - c)/h$  has a finite fine limit at  $\mu^h$ -almost every point of  $R_0$ , according to Theorem 4.4. (We use the fact that  $\mu^h$ -measure as defined relative to  $R_0$ , and  $\mu^h$ -measure as defined relative to  $R$ , restricted to subsets of  $R_0$ , are absolutely continuous relative to each other.) Since  $1/h$  has a finite fine limit at every point of  $R$ , the stated conclusion is true.

### 5. On a theorem of Calderon.

Generalizing a theorem of Privalov, Calderon [2] proved the following. *Let  $u$  be a function harmonic on an  $N$ -dimensional halfspace  $R$ . Suppose that, at each point of a subset  $A$  of the boundary,  $u$  is bounded on the set of points in some neighborhood of the point which lie in some right circular cone with vertex at the point but otherwise in  $R$ . Then  $u$  has a finite non-*

*tangential limit at almost every (Lebesgue  $(N-1)$ -dimensional measure) point of  $A$ . The following theorem generalizes this result in several directions.*

**THEOREM 5. 1.** — *Let  $R$  be a Green space, with Martin boundary  $R'$ . Let  $h$  and  $u$  be superharmonic functions on  $R$ , with  $h > 0$ . Suppose that  $u/h$  is bounded from below on a fine neighborhood of each point of a set  $A$  of minimal boundary points. Then  $u/h$  has a finite fine limit at  $\mu^h$ -almost every point of  $A$ .*

It is easy to see that boundedness from below of  $u/h$  and of  $u$  in the stated sets are equivalent hypotheses as far as the conclusion of the theorem is concerned. It is sufficient to prove the theorem, and we shall do so, under the hypothesis that  $u$  is strictly positive in some fine neighborhood of each point of  $A$ , since, for every positive  $n$ , we can replace  $A$  by the subset of  $A$  for each point of which there is a fine neighborhood in which  $u/h > -n$ , and then replace  $u/h$  by  $(u + nh)/h$ . Finally, we can and shall suppose that  $h$  is harmonic, since the general case can be reduced to the harmonic case as in the proof of Theorem 4.4. Let  $R_0$  be the subset of  $R$  on which  $u$  is strictly positive. Then  $R_0$  is itself a fine neighborhood of every point of  $A$ . The restriction of  $h$  to subspace  $R_0$  determines a corresponding measure  $\mu_0^h$ . According to Theorem 4.4,  $u/h$  has a finite fine limit  $\mu_0^h$ -almost everywhere on the Martin boundary  $R'_0$  of  $R_0$ . According to a theorem of Naïm [4], each point of  $A$  corresponds to a point of  $R'_0$ , and a fine neighborhood of the latter point relative to  $R_0$  is a fine neighborhood of the former relative to  $R$ . Then  $u/h$  has a finite fine limit at all points of  $A$  except those corresponding to points of a subset of  $R'_0$  of  $\mu_0^h$ -measure 0. But such a set must also have  $\mu^h$ -measure 0, according to another theorem of Naïm [4], and this finishes the proof of the theorem.

Calderon actually proved a more general theorem than the theorem quoted at the beginning of this section. In fact he considered functions on the direct product of a finite number  $m$  of half-spaces, harmonic on each half-space if the remaining arguments are held fast. If  $m = 1$  this theorem reduces to the quoted one. Presumably his general theorem has an analogue in a corresponding extension of Theorem 5.1.



## BIBLIOGRAPHY

- [1] M. BRELLOT, Le problème de Dirichlet. Axiomatique et frontière de Martin (*J. Math. pures et appl.*, 35 (1956), 297-335).
  - [2] A. P. CALDERON, On the behavior of harmonic functions at the boundary (*Trans. Amer. Math. Soc.*, 68 (1950), 47-54).
  - [3] J. L. DOOB, Conditional Brownian motion and the boundary limits of harmonic functions (*Bull. Soc. Math. France*, 85 (1957), 431-458).
  - [4] L. NAÏM, Sur le rôle de la frontière de R. S. Martin dans la théorie du potentiel (*Ann. Inst. Fourier*, 7 (1957), 183-281).
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